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ON THE LIMIT OF PLANETARY STABILITY.

BY PROFESSOR DANIEL KIRKWOOD.

LAPLACE, in his *Syste'me du Monde*, pointed out the limit at which, according to his estimate, the moon's attraction could have retained an elastic atmosphere.* The question of a satellite's stability was also considered by the late Professor Vaughan, of Cincinnati.† I have seen no attempt, however, to obtain for the different members of our system any definite numerical results. In the present paper it is proposed to find the approximate limits of stability in the cases of the eight major planets and certain of the satellites, on the hypothesis that their primitive condition was either liquid or gaseous.

Let $M =$ the mass of the larger or central body,

$m =$ that of the dependent planet or satellite,

$x =$ the distance from the centre of the former to the limit of stability of the latter,

$a =$ the distance between their centres; then, since the disturbing or separating force of the larger upon the smaller mass is the difference between the attraction of the former on the nearest point of the surface of the latter and that on its centre of gravity, we have

$$\frac{M}{x^2} - \frac{M}{a^2} = \frac{m}{(a-x)^2}, \qquad (1)‡$$

or, putting $a = 1$ and reducing,

$$x^4 - 2x^3 + \frac{m}{M}x^2 + 2x = 1. \qquad (2)$$

*Syst. du Monde, B. IV., Ch. X.

†Pop. Sci. Monthly for Sept, 1878. See also the Proc. of the A. A. A. S. for 1856.

‡We neglect the centrifugal force due to the planet's rotation, as the modification would be slight and we propose to obtain merely approximate results.

If we adopt the masses and distances given in Newcomb's Popular Astronomy and solve equation (2) for each of the eight principal planets we shall obtain the distance from the centre of each to its limit of stability, as given in the second column of the following table. If, moreover, the planets, with their present masses, be reduced to the sun's mean density their radii as stated in the third column are found by the formula

$$r_n = 430{,}000\left(\frac{m_n}{M}\right)^{\frac{1}{3}},$$

and the respective ratios of the limits of stability to these radii are seen in column fourth.

TABLE.

Planet.	R_n	r_n	$\frac{R_n}{r_n}$
Mercury	165,165 ms	2,514.6 ms	65.7
Venus	701,746	5,719.2	122.7
Earth	1,059,386	6,242.7	169.7
Mars	764,900	2,951.1	259.2
Jupiter	37,354,287	42,335	882.35
Saturn	45,859,381	28,317	1619.48
Uranus	49,512,900	15,209	3255.51
Neptune	81,663,510	16,009	5101.10

On the assumption that in each case the mean density of the separated mass was equal to that of the central body, the sun's present radius multiplied by the respective numbers in column fourth will give the radii of the solar nebula when the planets extended to their respective limits of stability. These radii are less than the mean distances of the planets in the ratio of 1 to 1.265. This fact may have some significance in regard to the former oblateness of the solar nebula or the law of its density.

The Earth and the Moon.—For the moon, which in perigee approaches within 221500 miles of the earth, the limit of stability is about 38000 miles. Were the moon's density reduced to that of the earth its radius would be 916 miles, the ratio of which to the limit of stability is 1 : 41.6. The moon's least distance diminished by 38000 miles is 183500 miles. If our satellite originally extended to the limit, and if the moon and the earth had the same form and density, the radius of the latter was 165000 miles.

The Martian System.—The diameter of Phobos, according to Prof. Pickering, is 5.57 miles. If its density, therefore, be equal to that of Mars the limit of stability is about two miles exterior to the surface; or, if the density be to that of the primary in the same ratio as the density of the moon to that of the earth, the limit is less than a mile from the surface of the satellite;

and finally if the density were no greater than that of water the satellite, if fluid, would be unstable, the limit being actually within the surface. Since, therefore, the satellite could never have existed at its present distance in a nebular state, it must follow, if any form of the nebular hypothesis is to be accepted, that its original distance was much greater than the present. Can we find a probable cause for this ancient disturbance?

If we suppose the former period of Mars to have been very nearly one-sixth that of Jupiter the close commensurability would render the orbit of Mars more and more eccentric. The planet in perihelion would thus pass through the sun's atmosphere, or rather through the outermost equatorial zone of the solar nebula. This resisting medium would not only accelerate the motion of Mars but also in a much greater degree that of his extremely small satellite. The solar mass contracting more rapidly than the orbit of Mars would finally leave the latter moving in an excentric path without sensible resistance.

Other Secondary Systems.—For the first satellite of Jupiter the limit is 5250 miles, or $4\frac{1}{2}$ times the radius of the satellite. For Mimas, the innermost satellite of Saturn, it is less than twice the radius. The rings of Saturn, in all probability, could not exist as three satellites, the limits of stability being interior to the surface.*

The effect of perturbation in the dismemberment of comets is known to all astronomers. The nucleus of the great comet of 1880, which approached within less than 100,000 miles of the sun's surface, must have had a density greater than that of granite, as well as a strong cohesive force betw'n its parts, in order to withstand the tendency to disintegration during its perihelion passage. Had the nucleus been either liquid or gaseous, or even a cluster of solid meteorites, the difference between the sun's attraction on the central and the superficial parts would have pulled the comet asunder, spreading out the fragments into somewhat different orbits, like the meteoric streams of August and November.

LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST, WATERTOWN, CONN.

In a recent paper (ANALYST, Nov., 1880) I gave some account of what is known as Lagrange's theorem in probability, and showed how it can be

*It has been recently shown that Bessel's mass of the ring is much greater than the true value.

applied to the determination, in the most general manner, of the law of facility of errors in one dimension, such as errors in the measured length of a line for example. The elementary errors to which any measurement is liable, give rise by their combinations to all the errors possible in the sum or the mean of several measurements. The sum of k observations is always k times their arithmetical mean, so that the sum and the mean follow the same law of facility of deviation from their most probable values. The probability of a deviation x in the sum is equal to the probability of a deviation $x \div k$ in the mean. Denoting the quadratic mean error of a single observation by r, it was noticed (p. 172) that the quadratic mean error of the sum of k observations is $r\sqrt{k}$. This is true whether k be finite or infinite, being a consequence of my theorem respecting the radius of gyration of the coefficients in the expansion of a polynomial to the k power. (ANALYST, May, 1880, p. 75.) Hence the quadratic mean error of the arithmetical mean, for any value of k, is strictly equal to

$$r\sqrt{k} \div k = r \div \sqrt{k}.$$

It will be borne in mind that r here is in strictness the square root of the mean of the squares of the deviations of all the possible values of a single observation, from their arithmetical mean, and that in forming this arithmetical mean, and the mean of the squares of the deviations, each possible value, and its squared deviation, is taken with a weight proportional to the probability of its occurrence.

If the number of observations is large, the determination of the law of facility of error reduces itself to finding the limiting form of the series of coefficients in the expansion of a polynomial to an infinitely high power. This, as I had before shown, is the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}. \tag{1}$$

We will now consider the somewhat analogous case of errors in two dimensions, or errors in the position of a point in a plane. Taking the true position of the point as an origin, and drawing through it any rect. axes, we refer to these axes the positions of the other points corresponding to the several elementary errors to which the observations are subject. Let Δx and Δy denote the units of measure in the x and y directions. If one elementary error acting alone would place the observed point at the position $x = 2\Delta x$, $y = 3\Delta y$, for example, we will say that $\lambda_{2,3}$ is the probability of the occurrence of that error; and so on for the other errors. Now write the probabilities λ as coefficients, and their sub-indices as exponents of ξ and η, in the polynomial,

$$\begin{array}{c|c} \lambda_{-m,m}\xi^{-m}\eta^{m} & \lambda_{m,m}\xi^{m}\eta^{m} \\ \hline \lambda_{-m,-m}\xi^{-m}\eta^{-m} & \lambda_{m,-m}\xi^{m}\eta^{-m} \end{array} \tag{2}$$

For convenience, only the four corner terms are written here, as in a previous article of mine (Vol. VII, p. 41). The omitted terms can be readily constructed from these, since all sub-indices, exponents and coeffic'ts which vary from term to term, are understood to vary by differences equal to unity. The greatest elementary error, positive or negative, which can occur, is supposed not to exceed m units in either the x or y directions. The polynomial, if written in full, would form a square, with $2m+1$ rows and $2m+1$ columns, containing $(2m+1)^2$ terms in all.

If any particular elementary error cannot occur, its probability λ is of course zero; and any polynomial can be put in the square form by adding terms, if necessary, with zero coefficients. By an extension of the principle of Lagrange's theorem, which a little consideration will render obvious, it appears that when k observations are made, the probability that the sum of the errors in the x direction will be $t\Delta x$, and that at the same time the sum of those in the y direction will be $v\Delta y$, is the coefficient of $\xi^t\eta^v$ in the expansion of the polynomial (2) to the k power. The same coefficient is also the probability that the mean of the x-errors will be $t\div k$ units and the mean of the y-errors $v\div k$ units. The units of measure Δx and Δy can be taken as small as we please, and the exponents in the polynomial will be regarded as whole numbers.

In k observations, the most probable total effect, in the x and y directions, of the elementary error $2\Delta x$ and $3\Delta y$ for example, will be

$$k\lambda_{2,3}(2\Delta x), \qquad k\lambda_{2,3}(3\Delta y),$$

and so on for other errors. The most probable algebraic sums of the x and y errors then will be

$$k\Delta x\left(\begin{array}{c|c} -m\lambda_{-m,m} & m\lambda_{m,m} \\ \hline -m\lambda_{-m,-m} & m\lambda_{m,-m} \end{array}\right), \qquad k\Delta y\left(\begin{array}{c|c} m\lambda_{-m,m} & m\lambda_{m,m} \\ \hline -m\lambda_{-m,-m} & -m\lambda_{m,-m} \end{array}\right), \qquad (3)$$

so that these are approximately the exponents of ξ and η in that term in the expansion of (2) whose coefficient is a maximum. Hence the most probable error in the mean of the k observations, in the x and y directions respectively, is found by dividing the expressions (3) by k. It will be seen that the quotients thus obtained are the statical moments of the coefficients λ about the axes of Y and X, these coefficients being regarded as masses in a system of material points. The same quotients also represent the lever arms of the system about those axes, since the sum of all the masses λ is unity. From the principle established in my article of March '80, p. 46, it follows that the lever arms of the system of coefficients in the expansion of (2) to the k power, about the axes of X and Y respectively, are k times what they are in the first power. According to (3) therefore, the maximum coefficient in the expansion will coincide approximately with the centre of forces, or cen-

tre of gravity, of the whole expanded series of coefficients, whether k be finite or infinite. We shall hereafter get in another way a proof that this is true at the limit, when k becomes infinite.

Now to find the probability of a given error in the mean result of k observations, we must be able to find, approximately at least, the coefficient of any term of given position in the expansion of the polynomial (2) to the k power. To this end, we shall proceed to demonstrate a relation which exists between the coefficients in any square group of $(2m+1)^2$ terms in the expansion, and the co-ordinates which fix the position of the group. This is equivalent to finding a "multinomial formula" for two variables, analogous to the known formula for one variable which was employed by me in ANALYST, Sept., 1879, p. 141. That formula is demonstrated most easily by the method of indeterminate coefficients. (See for instance Bertrand's *Calcul Différentiel*, p. 330.) A similar process will be followed here, and we shall then pass to the differential equations of the surface which is the limiting form of the coefficients in the expansion of (2), in a manner somewhat analogous to that by which, in my own article just cited, the differential equation of the limiting curve was reached.

The coefficients in the expansion of a polynomial such as (2) to the k power are not altered by adding a constant number to all the exponents of ξ, or to all those of η, or to both together. If then we write

$$u = \frac{a_{0,n}\xi^0\eta^n \mid a_{n,n}\xi^n\eta^n}{a_{0,0}\xi^0\eta^0 \mid a_{n,0}\xi^n\eta^0}, \tag{4}$$

and make $n = 2m$, the whole number of terms will be the same as in (2), and if we also make all the coefficients a equal to the coefficients λ which are in the corresponding positions, the coefficients in the expansion of (4) to the k power will be the same as those in the expansion of (2). Denoting them by B, the expanded polynomial may be written

$$U = \frac{B_{0,kn}\xi^0\eta^{kn} \mid B_{kn,kn}\xi^{kn}\eta^{kn}}{B_{0,0}\xi^0\eta^0 \mid B_{kn,0}\xi^{kn}\eta^0}, \tag{5}$$

the number of terms in it being $(kn+1)^2$. We now have

$$u^k = U. \tag{6}$$

Differentiation with respect to ξ and η gives

$$ku^{k-1}\left(\frac{du}{d\xi}\right) = \frac{dU}{d\xi}, \qquad ku^{k-1}\left(\frac{du}{d\eta}\right) = \frac{dU}{d\eta}, \tag{7}$$

and dividing these by (6) and clearing of fractions, we get

$$kU\left(\frac{du}{d\xi}\right) = u\left(\frac{dU}{d\xi}\right), \qquad kU\left(\frac{du}{d\eta}\right) = u\left(\frac{dU}{d\eta}\right). \tag{8}$$

By differentiation of (4) and (5) with respect to ξ, we find

$$\frac{du}{d\xi}=\frac{1a_{1,n}\xi^0\eta^n \mid na_{n,n}\xi^{n-1}\eta^n}{1a_{1,0}\xi^0\eta^0 \mid na_{n,0}\xi^{n-1}\eta^0},\qquad \frac{dU}{d\xi}=\frac{1B_{1,kn}\xi^0\eta^{kn} \mid knB_{kn,kn}\xi^{kn-1}\eta^{kn}}{1B_{1,0}\xi^0\eta^0 \mid knB_{kn,0}\xi^{kn-1}\eta^0}, \tag{9}$$

and by substitution in the first of the two equations (8),

$$\left.\begin{aligned} &k\left(\frac{B_{0,kn}\zeta^0\mu^{kn} \mid B_{kn,kn}\zeta^{kn}\mu^{kn}}{B_{0,0}\zeta^0\mu^0 \mid B_{kn,0}\zeta^{kn}\mu^0}\right)\left(\frac{1a_{1,n}\zeta^0\mu^n \mid na_{n,n}\zeta^{n-1}\mu^n}{1a_{1,0}\zeta^0\mu^0 \mid na_{n,0}\zeta^{n-1}\mu^0}\right)^*\\ &=\left(\frac{a_{0,n}\zeta^0\mu^n \mid a_{n,n}\zeta^n\mu^n}{a_{0,0}\zeta^0\mu^0 \mid a_{n,0}\zeta^n\mu^0}\right)\left(\frac{1B_{1,kn}\zeta^0\mu^{kn} \mid knB_{kn,kn}\zeta^{kn-1}\mu^{kn}}{1B_{1,0}\zeta^0\mu^0 \mid knB_{kn,0}\zeta^{kn-1}\mu^0}\right)\end{aligned}\right\} \tag{10}$$

Also differentiating (4) and (5) with respect to μ, and substituting in the second of the equations (8), we get a second equation analogous to (10), which I omit, to save space. The reader can easily construct it for himself. Each member of (10) contains the product of two rectangular polynomials.

If we multiply together any two such polynomials, for instance

$$\left(\frac{a_{0,u}\zeta^0\mu^u \mid a_{t,u}\zeta^t\mu^u}{a_{0,0}\zeta^0\mu^0 \mid a_{t,0}\zeta^t\mu^0}\right)\left(\frac{c_{0,w}\zeta^0\mu^w \mid c_{v,w}\zeta^v\mu^w}{c_{0,0}\zeta^0\mu^0 \mid c_{v,0}\zeta^v\mu^0}\right), \tag{11}$$

their product will also be a rectangular polynomial, and any term in it will have for its coefficient the sum of products of a and c, such that this sum can be expressed in the rectangular form. For example, the coefficient of $\zeta^p\mu^q$ in the product of (11) is

$$\frac{a_{0,q}c_{p,0} \mid a_{p,q}c_{0,0}}{a_{0,0}c_{p,q} \mid a_{p,0}c_{0,q}}. \tag{12}$$

Now forming in this way the coefficient of $\zeta^{r-1}\mu^s$ in the product in each member of (10), and equating the two to each other by the principle of indeterminate coefficients, we get

$$k\left(\frac{ra_{r,0}B_{0,s} \mid 1a_{1,0}B_{(r-1),s}}{ra_{r,s}B_{0,0} \mid 1a_{1,s}B_{(r-1),0}}\right)=\frac{ra_{0,s}B_{r,0} \mid 1a_{(r-1),s}B_{1,0}}{ra_{0,0}B_{r,s} \mid 1a_{(r-1),0}B_{1,s}}. \tag{13}$$

Likewise forming the coefficients of $\zeta^r\mu^{s-1}$ in the products in the omitted equation similar to (10), and equating them to each other, we have

$$k\left(\frac{1a_{r,1}B_{0,(s-1)} \mid 1a_{0,1}B_{r,(s-1)}}{sa_{r,s}B_{0,0} \mid sa_{0,s}B_{r,0}}\right)=\frac{1a_{0,(s-1)}B_{r,1} \mid 1a_{r,(s-1)}B_{0,1}}{sa_{0,0}B_{r,s} \mid sa_{r,0}B_{0,s}}. \tag{14}$$

These two equations express a relation between the coefficients a and B in the rectangular groups

$$\begin{matrix} a_{0,s} & \vdots & a_{r,s} \\ a_{0,0} & \vdots & a_{r,0}\end{matrix}\,,\qquad \begin{matrix} B_{0,s} & \vdots & B_{r,s} \\ B_{0,0} & \vdots & B_{r,0}\end{matrix}\,, \tag{15}$$

the one group containing coefficients in the first power of the given polynomial, while the other group contains an equal number of coefficients in its expansion to the k power. Since r and s are arbitrary, suppose each of them to be greater than n. This will not alter the value of the given polynomial, nor that of its expansion, provided we suppose the coefficients a to be equal to zero whenever either of the sub-indices of a is greater than n.

*For want of sorts ζ and μ are here written to represent *Xi* and *Eta*, respectively.

Omitting therefore in (13) and (14) all terms which contain such values of a as factors, we shall have remaining

$$\left.\begin{array}{l} k\left(\dfrac{na_{n,0}B_{(r-n),s} \;\Big|\; 1a_{1,0}B_{(r-1),s}\ \cdots}{na_{n,n}B_{(r-n),(s-n)} \;\Big|\; 1a_{1,n}B_{(r-1),(s-n)}}\right) = \dfrac{ra_{0,n}B_{r,(s-n)} \;\Big|\; (r-n)a_{n,n}B_{(r-n),(s-n)}}{ra_{0,0}B_{r,s} \;\Big|\; (r-n)a_{n,0}B_{(r-n),s}} \\ k\left(\dfrac{1a_{n,1}B_{(r-n),(s-1)} \;\Big|\; 1a_{0,1}B_{r,(s-1)}}{na_{n,n}B_{(r-n),(s-n)} \;\Big|\; na_{0,n}B_{r,(s-n)}}\right) = \dfrac{\ddagger a_{0,n}B_{r,(s-n)} \;\Big|\; (s-n)a_{n,n}B_{(r-n),(s-n)}}{s\,a_{0,0}B_{r,s} \;\Big|\; s\,a_{n,0}\;B_{(r-n),s}} \end{array}\right\} \qquad (\ddagger = s-n). \quad (16)$$

These two equations express a relation between the coefficients a and B in the square groups

$$\begin{array}{c:c} a_{0,n} & a_{n,n} \\ \hdashline a_{0,0} & a_{n,0} \end{array}, \qquad\qquad \begin{array}{c:c} B_{(r-n),s} & B_{r,s} \\ \hdashline B_{(r-n),(s-n)} & B_{r,(s-n)} \end{array}, \qquad (17)$$

the first group containing the $(n + 1)^2$ coefficients of the given polynomial (4), while the second contains an equal number of coefficients in its expansion to the k power, these last being situated so as to form any square group, such that the highest sub-index of B in the x direction is r, and the highest in the y direction is s. The equations (16) can be advantageously modified as follows; in the first members add a column or a row, with coeffic'ts zero; and reverse the positions of the terms in the second members, as if by turning the two rectangular tables round through 180° in their own plane. We thus get

$$k\left(\frac{na_{n,0}B_{(r-n),s} \;\Big|\; 0a_{0,0}B_{r,s}}{na_{n,n}B_{(r-n),(s-n)} \;\Big|\; 0a_{0,n}B_{r,(s-n)}}\right) = \frac{(r-n)a_{n,0}B_{(r-n),s} \;\Big|\; (r-0)a_{0,0}B_{r,s}}{(r-n)a_{n,n}B_{(r-n),(s-n)} \;\Big|\; (r-0)a_{0,n}B_{r,(s-n)}}, \quad (18)$$

while the second equation, being formed in a similar way, may be omitted here. In the second member of (18), let that portion which does not have the coefficient r be transferred to the first member, and we get

$$(k+1)\left(\frac{na_{n,0}B_{(r-n),s} \;\Big|\; 0a_{0,0}B_{r,s}}{na_{n,n}B_{(r-n),(s-n)} \;\Big|\; 0a_{0,n}B_{r,(s-n)}}\right) = r\left(\frac{a_{n,0}B_{(r-n),s} \;\Big|\; a_{0,0}B_{r,s}}{a_{n,n}B_{(r-n),(s-n)} \;\Big|\; a_{0,n}B_{r,(s-n)}},\right) \quad (19)$$

the second equation being omitted as before. Since the coefficients in the polynomial (2) are represented by those in (4), and we have $n=2m$, the number of terms in each row or column is $2m + 1$ in the first power of (4), and $2km + 1$ in the expansion to the k power, so that one coefficient in the expansion will be in the middle of the square. This construction does not affect the generality of our proof of the limiting form of the expansion of a polynomial, for any given polynomial can be made to take this square form with one term in the centre, by adding terms on one or more sides, if need be, with coefficients zero. Now let us fix the position of the square group (17) of coefficients B, not by the extreme ranks r and s reckoned from the place of the first coefficient $B_{0,0}$, but by the ranks i and j of the middle term of the group, reckoned from the place of the middle coefficient in the whole expansion, in the x and y directions respectively. This gives

$$r = m(k+1)+i, \qquad\qquad s = m(k+1)+j \tag{20}$$

Substituting these values for r and s in (19) and its companion equation, and transposing from the second members to the first ones those portions which have the coefficient $m(k+1)$, the former equation becomes

$$(k+1)\left(\frac{ma_{n,0}B_{(r-n),s} \;\Big|\; -ma_{0,n}B_{r,s}}{ma_{n,n}B_{(r-n),(s-n)} \;\Big|\; -ma_{0,n}B_{r,(s-n)}}\right) = i\left(\frac{a_{n,0}B_{(r-n)} \;\Big|\; a_{0,0}B_{r,s}}{a_{n,n}B_{(r-n),(s-n)} \;\Big|\; a_{0,n}B_{r,(s-n)}}\right), \tag{21}$$

and I omit the second one as before. Let us now change the notation so that instead of the coeff's a and B as in (17), we shall write λ and l, as follows,

$$\frac{\lambda_{-m,m} \;\vdots\; \lambda_{m,m}}{\lambda_{-m,-m} \;\vdots\; \lambda_{m,-m}}, \qquad\qquad \frac{l_{(i-m),(j+m)} \;\vdots\; l_{(i+m),(j+m)}}{l_{(i-m),(j-m)} \;\vdots\; l_{(i+m),(j-m)}}. \tag{22}$$

Substituting these for the a and B in (21) and its companion equation, and using V as an auxiliary letter, we get the following results.

$$\left.\begin{aligned}
V &= \frac{\lambda_{m,-m}l_{(i-m),(j+m)} \;\Big|\; \lambda_{-m,-m}l_{(i+m),(j+m)}}{\lambda_{m,m}\,l_{(i-m),(j-m)} \;\Big|\; \lambda_{-m,m}\,l_{(i+m),(j-m)}} \\
&\frac{-m\lambda_{m,-m}l_{(i-m),(j+m)} \;\Big|\; m\lambda_{-m,-m}l_{(i+m),(j+m)}}{-m\lambda_{m,m}\,l_{(i-m),(j-m)} \;\Big|\; m\lambda_{-m,m}\,l_{(i+m),(j-m)}} = \left(\frac{-i}{k+1}\right)V, \\
&\frac{m\lambda_{m,-m}l_{(i-m),(j+m)} \;\Big|\; m\lambda_{-m,-m}l_{(i+m),(j+m)}}{-m\lambda_{m,m}\,l_{(i-m),(j-m)} \;\Big|\; -m\lambda_{-m,m}\,l_{(i+m),(j-m)}} = \left(\frac{-j}{k+1}\right)V.
\end{aligned}\right\} \tag{23}$$

These two equations, like (16), express the general relation between the $(2m+1)^2$ coefficients in any polynomial of square form, and an equal number of coefficients forming a square group in any part of the expansion of the polynomial to the k power. It amounts to a "multinomial formula" for two variables. Its analogy to the formula for one variable, as given by me in ANALYST, Sept., 1879, p. 142, is apparent. The position of the sq're group of coefficients l in the expansion is fixed by the ranks i and j of its middle term reckoned from the middle coefficient of the whole expansion, so that, referring the polynomial and its expansion to the same axes, the coordinates of the middle coefficient of the group will be

$$x = i\Delta x, \qquad\qquad y = j\Delta y. \tag{24}$$

It will be seen that the terms in (23) are made up of products of λ into l, and the particular ones to be multiplied together are readily found by rotating the λ table in (22) through 180°, in its own plane, and then applying it to the l table, and multiplying together the terms which coincide. The sum of all the products thus formed is V. And it will be seen that if these products are regarded as parallel forces acting perpendicularly to the plane of XY, and the intervals between them are taken as unity of distance, their moments about vertical and horizontal axes through the middle of the group, constitute the first members of the two final equations in (23). The relation here proved is a general one, holding true for any given polynomial, whether its coefficients λ are positive or negative.

[To be continued.]

REPLY TO "CRITICISMS".

BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

It is fortunate that writers differ in their manner of presenting a subject. Some amplify to diffuseness, others condense to abruptness, while a few hit the happy mean. I am pleased to see Sec. 9 of pp. 36 and 127, of the preceding volume of the Analyst, in Mr. Christie's own clear and adequately emphatic style; and although that topic might be further amplified, it is deemed unnecessary to do so for present purposes.

In regard to the definition given to $U\beta$, we might have taken refuge under the plea that it was a new meaning applied to U; but as I considered it in conflict with that given by Hamilton,* I at once abandoned it. Were it simply *worthless*, it would be *harmless*. The critic's remark in regard to the term *implies*, is attacking a mode of expression rather than the subject matter under consideration.

On p. 67 we wrote

$$k = \frac{j}{i}, \qquad -k = \frac{i}{j};$$

from which it was inferred that "the reciprocal of the fraction *changes the sign* of the vector axis instead of producing its reciprocal." Although this was criticised, we observe that *no other conclusion can be drawn from the principles stated up to that point.* Positive rotation only had been considered. Hamilton says (*Elements*, p. 121) "It is evident that the angle of the reciprocal remains *unchanged*, but that the axis is *reversed* in direction", which agrees exactly with our statement. But afterwards—pp. 68 and 69 of our article—the notation for and effect of negative rotation was considered, and it was shown that $k^{-1} = -k$ and, following Hamilton's notation, we write $k^{-1} = \frac{1}{k}$, and *call* the expression 'the reciprocal of a vector'. It was shown that k^{-1} implies (or *expresses*) a negative quadrantal rotation about a positive axis; so that we have

$$k = \frac{j}{i}, \qquad -k = k^{-1} = \frac{i}{j};$$

hence, as stated at p. 128, "When the direction of rotation of the fraction and its reciprocal are in opposite senses, the reciprocal of the axis is the axis of the reciprocal." *This is correct.* The remark of the critic that "It is a matter of easy and direct perception" makes it none the less correct.

*Prior to writing my articles, my efforts at getting even a sight of Hamilton's *Elements* had been unsuccessful, but later I had access to a copy. It is clear, concise, comprehensive.

Having failed to draw this inference after negative rotation had been considered—which fact was called to mind by reading Mr. Christie's communication—the "emendation" referred to was made expressly for the purpose of calling the attention of the reader, by the contrast, to the fact that negative rotation had not previously been considered. Tait says "The versor of the reciprocal differs from the versor of the quaternion merely by the reversal of its representative angle" (Tait's *Quaternions*, p. 28), which agrees with the conclusion above. Hamilton calls these *Conjugate versors* (*Lect's*, p. 87). As a *result*, it is true that the axis of the reciprocal is the reciprocal of the axis, but in the operation of producing the reciprocal *one* or *the other* of the above processes will necessarily be used. (Lectures, p. 123.)

In regard to the expression

$$kk = \frac{-j}{i} \cdot \frac{i}{j} = \frac{-i}{i} = -1,$$

the critic states that it should be written

$$kk = \frac{-i}{j} \cdot \frac{j}{i} = \frac{-i}{i} = -1.$$

Both are correct *in this case*, and the matter might be passed with the single remark that the quaternions are complanar and hence commutative. This is a case, strictly speaking, of the multiplication of quaternions, but it should be observed that no rules have thus far been given for this operation; *that* was reserved for a future article. The expression was not designed to teach multiplication, but was simply an expedient for accomplis'ng a certain end, and it only remains to be shown that the manner in which it was used is admissible.

While the non-commutative principle is a characteristic of this science, yet there are exceptions to it. Thus, if a and β be two parallel vectors, we have $a\beta = +\beta a$, or they are commutative (ANALYST, Vol. VII, p. 125); but in no other case (p. 124). Now, considering division in the sense given by the critic (p. 187), where we have $k = i \div j$, $k = -j \div i$, it is evident that in the expression kk it is immaterial which k is written first in order. The same result becomes apparent in regard to the second member of the equation, for by beginning with the second quadrant, instead of the first we have

$$kk = \frac{-i}{j} \cdot \frac{-j}{-i} = \frac{-i}{i} = -1,$$

where the order of the factors in the second member have become practic'ly reversed, and stand in the proper order for cancellation in the *general case* of multiplication. But these are right quaternions, and although they include all that was involved in the case under consideration, yet we will

show that the principle is true for all complanar quaternions. We have for unit vectors (ANALYST, Vol. VII, p. 124),

$$q = \frac{\beta}{\alpha} = \cos\theta + i\sin\theta,$$

$$q' = \frac{\delta}{\gamma} = \cos\varphi + i\sin\theta;$$

where the axis i of these quaternions will be common, since they are complanar. An inspection of the second members of these equations shows that the first multiplied by the second will give the same result as the second multiplied by the first, and we have

$$qq' = q'q = \cos(\theta+\varphi) + i\sin(\theta+\varphi);$$

hence *they are commutative.* It is unnecessary, for present purposes, to consider the general case of the multiplication of diplanar quaternions.

We observe that $\beta\div\alpha$ is not, by *definition*, equal to $\beta\alpha^{-1}$; but that it is a *result* flowing from previous definitions. That it might have been so *defined* we admit, but for the same reason it might have been defined to equal $\alpha^{-1}\beta$, a form which would have changed in an important particular the existing relation between multiplication and division. Hamilton nor Kelland so treat it. (*Lectures*, p. 124; Kelland and Tait's *Introduction to Quaternions*, p. 45. See also ANALYST, Vol. VII, p. 124.)

We have now reached the point where I am charged with giving credit to Hamilton which does not belong to him.

We quote the following from the criticism (see p. 187): "Prof. W. says, 'In division, the versor operating on the divisor line is conceived to turn it, positively about the axis of the versor $(+i)$ through an angle equal to that of the versor (θ) to coincide in direction with the dividend line. In multiplication the versor operating on the multiplier line [as α, eq. (27)] is conceived to turn it in a positive direction about the axis of the versor $(+i)$ through an angle equal to the *supplement* of the angle of the versor $[\pi-(\pi-\theta)=\theta]$, making it coincide in direction with the multiplicand line (as β).'" Prof. W. cites p. 85 of the Lectures for the above, but I can assure the readers of the ANALYST that Hamilton does not deserve the credit of it. In the 'division'

$$\frac{\beta}{\alpha} = \cos\theta + i\sin\theta,$$

the versor $\cos\theta + i\sin\theta$, so far from 'operating upon the divisor line' (α), in reality *does not operate upon any line whatever, does not act at all,* but simply *constitutes the conception* $\beta\div\alpha$. So soon as the equivalency of the vector and the right part of the quaternion is established we may change the 'quotient' $\frac{\beta}{\alpha}$ into the 'product' $\beta.\frac{1}{\alpha} = -\beta\alpha$, and *then* we

have a *bona fide* 'operation', viz., β and the inversor (—) operate on α;" and on p. 188, "To recapitulate results: It is seen that in the *division* $\beta \div \alpha$ the versor $\cos\theta + i\sin\theta$, does not operate at all; in the equivalent *multiplication* $\beta \cdot \frac{1}{\alpha}$, β and the inversor (—) operate upon α to produce the versor $\cos\theta + i\sin\theta$. Let this be compared with the first part of prof. W.'s statement.

In the multiplication $\alpha\beta$, the versor $-\cos\theta + i\sin\theta$, so far from operating upon the multiplier line α, is the result of α operating upon β — the multiplier line upon the multiplicand line, as it should be.

The assertion of the critic that "in the division $\beta \div \alpha$, the versor *does not operate upon any line*" is general, applying to right as well as oblique vectors; and strikes directly — as it was doubtless intended to do — at the substance of what we have said upon this operation. We will see what the author of the system teaches. Of right quaternions, Hamilton in his *Lectures*, pp. 61, 62, says "In regard to the right quaternion $\beta \div \alpha$, if i (that author uses j in the same sense, while we retain i to correspond with our own notation) be joined (mechanically) perpendicularly to α, and one takes hold of the axis (i) and turns α positively through a quadrant to coincide with β, the required act of version will be performed. And since the (mechanical) *agent* in producing this (mechanical) rotation, has been an axis, I now propose to denote the *versor* itself, or the CONCEIVED AGENT of the *conceived* version, or the purely geometrical rotation from α to β by i..... the line turned being a subject of operation, and the turning line being a versor or operator". This is not a literal extract, some phrases not essential to the meaning being omitted. If space permitted, we would be pleased to make extended extracts from the context; but if we read Hamilton correctly he asserts that the versor (i) operates upon α, the divisor line, turning it mechanically through a quadrant.

Next, in regard to the versor in general, we have (*Lectures*, p. 83) "The symbol i^t denotes a versor which would cause any right line in a plane perpendicular to it, to revolve in that plane through t quadrants .... in producing which rotation the *versor* is *conceived* to be the agent". If there were any doubt whether this *versor* or agent were an *operator*, it would be removed by the preceding extract. On page 128 of *Lectures* we have $\beta \div \alpha = i^t$, where "rotation round the base-line i from the divisor-line α to the dividend-line β, is positive". We understand that these two extracts do imply (or *express*) that i^t as a versor operates on α turning it mechanically to coincide with β. This conception is fundamental and will not be called in question. (*Elements*, p. 364, Tait's *Quaternions*, p. 38, &c.)

The critic says "$\cos\theta + i\sin\theta$ simply *constitutes the conception* $\beta \div a$." Grant it; the *conception* is one of rotation through an angle θ about the axis i. Kelland says "$\cos\theta + i\sin\theta$ is an operator of the same character as i; with this difference only, that whereas i as an operator would turn a through a right angle, $\cos\theta + i\sin\theta$ turns it in the same direction only through the angle θ (Kelland and Taits *Quat.*, p. 45). The fact that $\beta \div a$ constitutes a vector-division does not prevent its being a versor, nor a quaternion; nor does the fact that i^t expresses a vector quotient prevent it from being a versor or a quaternion (Lectures, pp. 85, 124, 127, 128).

The critic, apparently willing to provide a way of escape, remarks, p. 188, that "It is possible that Prof. W. had in mind that

$$\frac{\beta}{a}\,.\,a = (\cos\theta + i\sin\theta)a = \beta."$$

But this is not the case we had in mind, and it differs essentially from the one criticised. Here the *multiplying factor* is a versor, and the operation is among the first described by Hamilton. It is essentially the same as $\beta \div a = i^t$. Hamilton says, in regard to the two forms $qa = \beta$ and $q = \beta \div a$, "they are indeed, *intrinsically*, the same, but present themselves under different forms (*Lectures*, pp. 90, 91).

Again, the critic says, p. 188, that in the multiplication $a\beta$, a operates upon β producing $-\cos\theta + i\sin\theta$. If this be true a must operate either on the length or direction of β (*Lectures*, p. 75). It does not operate on the length, for its length is unity; neither does it operate on the direction, for it would produce a quadrantal rotation, whereas the actual rotation is θ. "A unit vector as a factor may be considered as a quadrantal versor whose plane is perpendicular to the versor." (*Lectures*, p. 76, Tait's *Quat.* p. 37.) Also if $\theta = 0$, we find $aa = -1$, which would be explained according to the critic's principle, a operating on itself produces the symbol of reversion; a questionable explanation at least. But according to the paragraph under consideration, we would say that the inversor *minus*, operating upon the *multiplier* a turns it through the *supplement* of the angle of inversion (i. e. $180^\circ - 180^\circ$) 0° to coincide with the multiplicand a, which *result* is correct. We observe in passing that a^2 is itself an operator on a line to which it is perpendicular (*Elements*, p. 316). We conclude, therefore, almost in the words of the critic, that in the multiplication $a\beta$, a does not operate upon any line whatever, *does not act at all*, but is simply the first of two factors (so called) in the order of arrangement, for forming a vector product.

Passing over some minor points in the criticism, we proceed to the main point. We observe at the outset that this is a case of the *multiplication of*

two lines. Hamilton first discussed the case of a line multiplied by a versor producing another line, as $i^t\alpha = \beta$; after which he discussed the case $\alpha\beta = i^{2-t}$. The latter is referred to on p. 76 of *Lectures* as follows: "much less have we shown how to multiply generally *any one vector by another*". This case is first taken-up on p. 83, *after* the two cases $\beta \div \alpha = i^t$ and $i^t\alpha = \beta$ had been discussed, and the first inference is "The product $\alpha\beta$ as unit vectors is therefore a versor" (*Lectures*, p. 85). Note the difference between this case of multiplication and the preceding; here the multiplier is a line, there a versor; here the product is a versor, there a line. The next inference is "the angle of the versor is the supplement of the angle between the lines". He then sums up the matter as follows:—confining our extract to unit vectors—"The product $\kappa\lambda$ of any two unit vectors κ and λ is a versor, which versor is the power i^{2-t} of the vector-unit i, this unit i having the direction of the axis of right-hand rotation *from* the multiplier-line κ to the multiplicand-line λ; and the *supplement* t, of the exponent $2-t$ of the constant number 2, expressing the ratio of the angle of the last rotation to a right-angle (*Lectures*, p. 85). This is the paragraph to which we referred in support of our conclusion, but the critic assures us that Hamilton should not have the credit of sustaining us. The *statical relations* (so to speak) of all the elements are evidently the same in Hamilton's statement as in our conclusion; and hence the obj. must be to our *dynamic* statem't "The versor operating on the multiplier-line". But Hamilton, only two pages preceding, describes a versor as an agent—an operator—and here describes the versor and the amount of rotation to be performed. The reader can judge if too much credit has been given to Hamilton.

But, as if to show still further my error, the critic says, p, 188, "We do *not* have $\alpha\beta.\alpha = \beta$". This is merely setting up a man of straw and demolishing *him*, for we have not asserted that it was true. But we do assert that if $\alpha\beta$ as a versor turns α through an angle equal to $\pi - \angle\alpha\beta$, it will produce β. We did *not* refer to α being operated upon as a multiplicand-line, but if it is desired to place it in that position, let it be done legitimately; thus

$$i^{2-t} = \alpha\beta = -\alpha\beta^{-1} = -\frac{\alpha}{\beta};$$

and taking the reciprocal, we have

$$\frac{\beta}{\alpha} = -(i)^{t-2} = -(-i)^{2-t} = i^t \text{ [A}\textsc{nalyst}\text{, Vol. VII. p. 122, eq. (20);]}$$

$$\therefore \beta = i^t\alpha;$$

a form admitted by Mr. Christie as expressing a legitimate operation, and this we claim is a demonstration that if it can be asserted, in any sense, that the versor i^t as a multiplier operating upon α as a multiplicand produces β;

then it can with equal force and in the same sense be asserted that the versor i^{2-t} operating upon the multiplier-line α, will by turning it through the angle t (the supplement of $2-t$) produce the multiplicand line; for one follows the other by legitimate analysis. We do not therefore abandon our position, but affirm it; still, we will relieve Hamilton and all others of any responsibility for our conclusion by changing it to the following: —In the multiplication $\alpha\beta = i^{t-2}$, *we* conceive that the axis i is attached (mechanically) perpendicularly to the multiplier-line (α), and that the versor (i^{2-t}), turns the latter (mechanically) positively through an angle equal to the supplement of the exponent $2-t$, that is, through the angle t, making it coincide with the multiplicand-line (β).

In regard to the matter referred to by Prof. Johnson in the last March No. of the ANALYST, I should have said that a quaternion may be expressed in terms of the four elements *tensor*, *scalar*, *versor*, and *vector*, as was finally shown in my equation (54), p, 127 of the preceding Vol. of the ANALYST, which equation is

$$q = Tq(SUq + VUq).$$

RESPONSE TO A REQUEST, BY THE EDITOR.—We are requested by a subscriber to furnish a solution to the following problem:—

"A conical hay-stack, altitude a and radius of base r, is to be divided horizontally into three equal parts, by weight; if the density of horizontal strata is everywhere proportional to their distance from the vertex, what must be the vertical height of each part?"

Put x = the distance of any horizontal section from the vertex, then will its area be $\pi(rx \div a)^2$; and if we assume the density at the vertex to be unity and the thickness of the horizontal strata to be dx, we shall have for an element of the cone at the distance x from its vertex, $\pi(rx \div a)^2 x dx$; therefore, if x be the height of the upper part, we must have

$$\int_0^x \left(\frac{rx}{a}\right)^2 \pi x dx = \tfrac{1}{3}\int_0^a \left(\frac{rx}{a}\right)^2 \pi x dx;$$

whence we find

$$x = a(\tfrac{1}{3})^{\frac{1}{4}}.$$

And if x = the height of the upper two thirds of the stack we shall find

$$x = a(\tfrac{2}{3})^{\frac{1}{4}}.$$

Hence, the heights of the upper, middle and lower parts must be, respectively, $a(\frac{1}{3})^{\frac{1}{4}}$, $a[(\frac{2}{3})^{\frac{1}{4}}-(\frac{1}{3})^{\frac{1}{4}}]$, and $a[1-(\frac{2}{3})^{\frac{1}{4}}]$.

QUERY.—If in the foregoing prob. the density of the strata is supposed to vary as the superincumbent weight, how must the division be made?

MECHANICS BY QUATERNIONS.

BY PROF. E. W. HYDE, UNIVERSITY OF CINCINNATI.

(21). *Center of Gravity.*—WE have seen (Eq. 21) that the vector of the center of mass of a system of particles whose weights are w_1, w_2, w_3, etc., is given by the equation

$$_{\prime}\rho_0 = \frac{\Sigma(w_{\prime}\rho)}{\Sigma(w)}.$$

To apply this formula to finding the center of gravity of any line, surface, or volume we have only to substitute for w the element of the line, surface, or volume multiplied by a coefficient of heaviness, aud then to substitute integration for summation. Let h be the weight of a unit of volume of the substance. If it is the same at all points of the body it may be taken outside of the sign of integration, and will then cancel out; if it is *not* constant it must be expressed as a function of $_{\prime}\rho$ before the integration can be performed.

(22). *Center of Gravity of a Line.*—By this is meant the c. g. of such a body as a fine wire bent into some curve, the diameter being so small compared with the length that it may be regarded for practical purposes as a true mathematical line, i. e., as if the wire, still retaining its weight, should have its diameter reduced to nothing. Let the equation of the curve be

$$_{\prime}\rho = {}_{\prime}\varphi(t). \quad \therefore d_{\prime}\rho = {}_{\prime}\varphi'(t)dt,$$

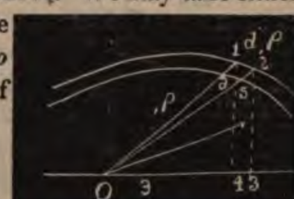
and $ds = Td_{\prime}\rho = T_{\prime}\varphi'(t).dt$ = element of curve. Therefore, for this case,

$$_{\prime}\rho_0 = \frac{\int_{\prime}\rho h T d_{\prime}\rho}{\int h T d_{\prime}\rho} = \frac{\int h_{\prime}\varphi(t) T_{\prime}\varphi'(t).dt}{\int h T_{\prime}\varphi'(t).dt}. \quad (56)$$

(23) *Center of Gravity of a Plane Area.*—Let the bounding curve be

$$_{\prime}\rho = {}_{\prime}\varphi(t),$$

and let ε be some unit vector in the plane of the curve. We may take either 1 2 3 4 as the element of area or 1 2 5 6. In the first case we have area 1 2 3 4 $= TV\varepsilon_{\prime}\rho S\varepsilon^{-1}d_{\prime}\rho$ (at the limit); so that the area between a pair of perpendiculars to ε will be



$$\int TV\varepsilon_{\prime}\rho S\varepsilon^{-1}d_{\prime}\rho = \int TV\varepsilon_{\prime}\varphi S\varepsilon^{-1}{}_{\prime}\varphi'.dt,$$

where for brevity $_{\prime}\varphi$ is written for $_{\prime}\varphi(t)$. Now the vector to the c. g. of the element is $_{\prime}\rho + \frac{1}{2}\varepsilon V\varepsilon_{\prime}\rho = {}_{\prime}\varphi + \frac{1}{2}\varepsilon V\varepsilon_{\prime}\varphi$; hence for the center of gravity of the area between ε and the curve

$$_{\prime}\rho_0 = \int h({}_{\prime}\varphi + \tfrac{1}{2}\varepsilon V\varepsilon_{\prime}\varphi) TV\varepsilon_{\prime}\varphi S\varepsilon^{-1}{}_{\prime}\varphi'.dt \div \int hTV\varepsilon_{\prime}\varphi S\varepsilon^{-1}{}_{\prime}\varphi'.dt. \quad (57)$$

In order to get the element 1 2 5 6 we must evidently change the tensor of $_{\prime}\rho$ independently of the versor. Therefore write the equation $_{\prime}\rho = u_{\prime}\varphi(t)$. By giving u a succession of values we shall obtain a series of curves such as 5 6. If we differentiate with respect to u only $T_{\prime}\rho$ varies, so that the end of $_{\prime}\rho$ moves say from 1 to 6, while if we differentiate with respect to t the end of $_{\prime}\rho$ moves along such a curve as 1 2 or 6 5. Let

$$\frac{d_{\prime}\rho}{dt} = D_t \text{ and } \frac{d_{\prime}\rho}{du} = D_u,$$

then the area of element 1 2 5 6 will be $TVD_uD_t.du\,dt$. But $D_u = {}_{\prime}\varphi$, and $D_t = u_{\prime}\varphi'$, by the equation assumed above. Therefore

$$_{\prime}\rho_0 = \int\int h_{\prime}\varphi TV_{\prime}\varphi_{\prime}\varphi'.u^2dt\,du \div \int\int hTV_{\prime}\varphi_{\prime}\varphi'.u\,dt\,du; \tag{58}$$

or integrating for u from 0 to 1, to cover the space from the origin to the bounding curve,

$$_{\prime}\rho_0 = \tfrac{1}{3}\int h_{\prime}\varphi TV_{\prime}\varphi_{\prime}\varphi'.dt \div \tfrac{1}{2}\int hTV_{\prime}\varphi_{\prime}\varphi'.dt. \tag{59}$$

This integration however can only be performed when h is *not a function* of u.

Equations (58) and (59) hold equally well when $_{\prime}\rho = {}_{\prime}\varphi(t)$ does not represent a plane curve. They give in any case the c. g. of the surface swept over by the radius vector.

(24). *Any Surface.*—Let the the equation of the surface be $_{\prime}\rho = {}_{\prime}\varphi(x, y)$. Then $D_x = \frac{d_{\prime}\rho}{dx}$, and $D_y = \frac{d_{\prime}\rho}{dy}$ are vectors $\parallel$ to tangents to the surface at the end of $_{\prime}\rho$, for by supposing x to be constant we have one curve on the surface, and by supposing y to be constant we have another. By giving successive constant values to x and y the surface will be divided up into four-sided figurés which when small enough may be regarded as parallelograms. The area of such an elem. parallelogram will be $TVD_xD_y.dx\,dy$. $\therefore$

$$_{\prime}\rho_0 = \int\int h_{\prime}\rho TVD_xD_y.dx\,dy \div \int\int hTVD_xD_y.dx\,dy. \tag{60}$$

(25) *Any Solid.*—If the equation of the bounding surface is $_{\prime}\rho = {}_{\prime}\varphi(x, y)$, then to obtain a parallelopipedical element we must vary the tensor of $_{\prime}\rho$. Therefore write, as in Art. 23, $_{\prime}\rho = u_{\prime}\varphi(x, y)$. By giving u successive values differing by du, and extending from 0 to 1 we divide up the solid into a series of shells. The solid element will then be $SD_uD_xD_y.du\,dx\,dy$. But

$$D_u = {}_{\prime}\varphi,\ D_x = u\frac{d_{\prime}\varphi}{dx},\ D_y = u\frac{d_{\prime}\varphi}{dy};\ \therefore$$

$$_{\prime}\rho_0 = \int\int\int hu^3{}_{\prime}\varphi S_{\prime}\varphi\frac{d_{\prime}\varphi}{dx}\frac{d_{\prime}\psi}{dy}.dx\,dy\,du \div \int\int\int hu^2S_{\prime}\varphi\frac{d_{\prime}\varphi}{dx}\frac{d_{\prime}\varphi}{dy}.dx\,dy\,du. \tag{61}$$

If h is not a function of u we can integrate for u between 0 and 1, thus obtaining

$$,\psi_0 = \tfrac{1}{4}\int\int h,\varphi S,\varphi \frac{d,\varphi}{dx}\frac{d,\varphi}{dy}.dx\,dy \div \tfrac{1}{3}\int\int hS,\varphi \frac{d,\varphi}{dx}\frac{d,\varphi}{dy}.dx\,dy. \qquad (62)$$

(26). *Surface of Revolution.*—We will now apply eq. (60) to the general equation of a surface of revolution. Let $,\rho = ,\varphi(t)$ be the equation of any curve, plane or tortuous. If this curve be revolved about an axis along which ε is a unit vector, and the origin be taken at a point of this axis, a surface of revolution will be generated whose equation will be

$$,\rho = \varepsilon^{\frac{\theta}{\pi}},\varphi(t)\,\varepsilon^{-\frac{\theta}{\pi}}.\text{*} \qquad (63)$$

This will appear from the fact that if θ be constant in eq. (63), the eq'n represents the generating curve turned through the angle θ about ε, while if t be constant the equation is that of a circle generated by the extremity of $,\varphi(t)$ revolving about ε.

To apply (60) we have to evaluate TVD_xD_y.

$$D_x = D\theta = \varepsilon^{\frac{2\theta}{\pi}} V\varepsilon,\varphi \text{ and } D_y = D_t = \varepsilon^{\frac{\theta}{\pi}},\varphi'\varepsilon^{-\frac{\theta}{\pi}},$$

in which the t is omitted for brevity.

$$\begin{aligned}\therefore TVD_xD_y &= TVD\theta\, Dt = TV.\varepsilon^{2\theta\div\pi} V\varepsilon,\varphi.\varepsilon^{\theta\div\pi},\varphi'\varepsilon^{-\theta\div\pi} \\ &= TV[\varepsilon V\varepsilon,\varphi S\varepsilon,\varphi'\cos\theta - V\varepsilon,\varphi S\varepsilon,\varphi'\sin\theta + \varepsilon S.\varepsilon,\varphi' V\varepsilon,\varphi] \\ &= \sqrt{(S^2.\varepsilon,\varphi' V\varepsilon,\varphi - V^2\varepsilon,\varphi S^2\varepsilon,\varphi')} = \sqrt{(,\varphi'^2 V^2\varepsilon,\varphi - S^2\varepsilon,\varphi,\varphi')} \\ &= TV.,\varphi' V\varepsilon,\varphi. \quad \text{Therefore (60) becomes}\end{aligned}$$

$$,\rho_0 = \int\int h\varepsilon^{\theta\div\pi},\varphi\varepsilon^{-\theta\div\pi} TV.,\varphi' V\varepsilon,\varphi\, dt\, d\theta \div \int\int hTV.,\varphi' V\varepsilon,\varphi\, dt\, d\theta. \quad (64)$$

If h be not a function of θ we may integrate for θ from 0 to θ_1, obtaining

$$,\rho_0 = \int h[\theta_1\varepsilon^{-1}S\varepsilon,\varphi - \left(\varepsilon^{2\theta_1\div\pi} - 1\right) V\varepsilon,\varphi] TV.,\varphi' V\varepsilon,\varphi dt \div \theta_1 \int hTV.,\varphi' V\varepsilon,\varphi dt \qquad (65)$$

If the integration be for a complete revolution so that $\theta_1 = 2\pi$, then the equation reduces to

$$,\rho_0 = \int h\varepsilon^{-1}S\varepsilon,\varphi TV.,\varphi' V\varepsilon,\varphi\, dt \div \int hTV.,\varphi' V\varepsilon,\varphi\, dt. \qquad (66)$$

If the generating curve lie in a plane passing through the axis we shall have $S\varepsilon,\varphi,\varphi' = 0$, so that in this case $TV.,\varphi' V\varepsilon,\varphi = T,\varphi' V\varepsilon,\varphi$.

(27). *Solid of Revolution.*—For the solid generated by the radius vector, when its extremity generates the surface of eq. (63), we have as in Art. 25,

$$,\rho = u\varepsilon^{\theta\div\pi},\varphi(t)\varepsilon^{-\theta\div\pi},$$

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*See Tait's Quaternions, Art. 351

We need these to evaluate

$$SD_uD_\theta D_t = SD_u VD_\theta D_t = u^2 S\varepsilon^{\frac{\theta}{\pi}}{}_{,}\varphi\varepsilon^{-\frac{\theta}{\pi}} V.\varepsilon^{\frac{2\theta}{\pi}} V\varepsilon{}_{,}\varphi'.\varepsilon^{\frac{\theta}{\pi}}{}_{,}\varphi\varepsilon^{-\frac{\theta}{\pi}},$$

or by the value of $VD_\theta D_t$ of the last Art.

$$SD_uD_\theta D_t = u^2 S(\varepsilon^{-1}S\varepsilon{}_{,}\varphi + \varepsilon^{-1}V\varepsilon{}_{,}\varphi\cos\theta + V\varepsilon{}_{,}\varphi\sin\theta)$$
$$\times(\varepsilon V\varepsilon{}_{,}\varphi S\varepsilon{}_{,}\varphi'\cos\theta - V\varepsilon{}_{,}\varphi S\varepsilon{}_{,}\varphi'\sin\theta + \varepsilon S.\varepsilon{}_{,}\varphi' V\varepsilon{}_{,}\phi)$$
$$= u^2[S.\varepsilon{}_{,}\varphi' V\varepsilon{}_{,}\varphi S\varepsilon{}_{,}\varphi - V^2\varepsilon{}_{,}\varphi S\varepsilon{}_{,}\varphi'] = -S.{}_{,}\varphi{}_{,}\varphi' V\varepsilon{}_{,}\varphi.u^2.$$

Hence substituting this value and that of ${}_{,}\rho$ above in eq. (61) we have

$${}_{,}\rho_0 = \iiint hu^3\varepsilon^{\frac{th}{\pi}}{}_{,}\psi\varepsilon^{-\frac{th}{\pi}} S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi.dt\,d\theta\,du \div \iiint hu^2 S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi.dt\,d\theta\,du. \quad (67)$$

[Here, and through the remainder of this paper, for want of sorts, ψ is substituted for φ and for subscripts and indices *th* is written for θ.]

If h be not a function of θ we can integrate as in last article from 0 to θ_1, obtaining

$${}_{,}\rho_0 = \iint hu^3\left[\theta_1\varepsilon^{-1}S\varepsilon{}_{,}\psi - \left(\varepsilon^{\frac{2th}{\pi}} - 1\right)V\varepsilon{}_{,}\psi\right]S.{}_{,}\psi{}_{,}\varphi' V\varepsilon{}_{,}\psi.dt\,du$$
$$\div\theta_1\iint hu^2 S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi.dt\,du. \quad (68)$$

If $\theta_1 = 2\pi$, for a complete revolution,

$${}_{,}\rho_0 = \varepsilon^{-1}\iint hu^3 S\varepsilon{}_{,}\psi S.{}_{,}\varphi{}_{,}\psi' V\varepsilon{}_{,}\psi.dt\,du \div \iint hu^2 S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi.dt\,du. \quad (69)$$

If the generating curve be in a *plane through the axis*, so that $V.V{}_{,}\psi{}_{,}\psi' - V\varepsilon{}_{,}\psi = 0$, we may replace the quantity $S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi$ in eqs. (67), (68) and (69) by the equivalent expression $TV{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi$. For with the above condition we have

$$V^2{}_{,}\psi{}_{,}\psi' V^2\varepsilon{}_{,}\psi = T^2 V{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi = S^2.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi; \therefore TV{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi$$
$$= \pm S.{}_{,}\psi{}_{,}\psi' V\varepsilon{}_{,}\psi.$$

(28). *Guldinus' Properties.*—We have by Art. 26, the area of a surface of revolution formed by revolving a plane curve about an axis in its plane through an angle $2\pi \div n$,

$$S = \frac{2\pi}{n}\int T{}_{,}\psi' V\varepsilon{}_{,}\psi.dt:$$

also by eq. (56) omitting the h, i. e., supposing the density uniform,

$${}_{,}\rho_0 = \int {}_{,}\psi T{}_{,}\psi'.dt \div \int T{}_{,}\psi'.dt.$$

Multiply both sides of this eqn. by $(2\pi \div n)\int T{}_{,}\psi'.dt$ and operate by $TV\varepsilon$;

$$\therefore \frac{2\pi}{n}TV\varepsilon{}_{,}\rho_0.\int T{}_{,}\psi' dt = \frac{2\pi}{n}\int T{}_{,}\psi' V\varepsilon{}_{,}\psi.dt = S. \quad (70)$$

The factor outside the sign of integration in the first member is the nth part of the circumf. of the circle in which the c. g. of the curve ${}_{,}\rho = {}_{,}\psi(t)$

moves, while the integral in that member is the length of the same curve; hence the area of the surface generated by a plane curve revolving about an axis in its own plane is equal to the length of the curve multiplied by the length of the path of the center of gravity of the curve.

Again, the volume generated by revolving the area bounded by a plane curve about an axis in its own plane through an angle $2\pi \div n$ is, by Art. 27,

$$V = \frac{2\pi}{n}\int\int TV_{,}\phi_{,}\phi' \, V\varepsilon_{,}\phi . u^2 du\, dt,$$

and by eq. (58), omitting h as before, the c. g. of a plane area is given by

$$_{,}\rho_0 = \frac{2\pi}{n}\int\int {}_{,}\phi TV_{,}\phi_{,}\phi' . u^2 du\, dt \div \frac{2\pi}{n}\int\int TV_{,}\phi_{,}\phi' . u\, du\, dt.$$

or clearing and operating by $TV.\varepsilon$

$$\frac{2\pi}{n} TV\varepsilon_{,}\rho_0 . \int\int TV_{,}\phi_{,}\phi' . u\, du\, dt = \frac{2\pi}{n}\int\int TV_{,}\phi_{,}\phi' \, V\varepsilon_{,}\phi . u^2 du\, dt \quad (71)$$
$$= V.$$

Hence the volume of the solid formed by revolving a plane area about an axis in its own plane is equal to the plane area multiplied by the length of the path of the center of gravity.

(29) These properties may be extended to the cases of a plane curve or a plane area moving so that the center of gravity of the curve or area generates some curve $_{,}\rho = {}_{,}\phi(t)$ to which the plane of the curve or area is always normal. Let l be the length of the moving curve and A its area, S and V respectively the surface and volume generated and $_{,}\rho = {}_{,}\phi(t)$ the path of the center of gravity. Then for the length of an elementary arc $d_{,}\rho$ the curve will coincide with the arc of its osculating circle for that point, and therefore Guldinus' properties will hold; hence $dS = lTd_{,}\rho = lT_{,}\phi'.dt$, and $dV = AT_{,}\phi'.dt$; and by integration

$$S = l\int T_{,}\phi'.dt; \qquad V = A\int T_{,}\phi'.dt. \qquad (72)$$

(30). A few examples to illustrate the application of these formulæ will now be given.

Center of Gravity of the arc of a Cycloid.—If θ be the angle turned thro' by the rolling circle, and a the radius of this circle; then the equation of the cycloid with the origin at a cusp may be written

$$_{,}\rho = {}_{,}\phi(\theta) = ai(\theta - \sin\theta) + aj(1 - \cos\theta),$$

in which i is a unit vector along the base, and j a unit vector along the tangent at the origin. Therefore $d_{,}\rho \div d\theta = {}_{,}\phi'(\theta) = ai(1 - \cos\theta) + aj\sin\theta$, and $T_{,}\phi'(\theta) = a\sqrt{[2(1-\cos\theta)]} = 2a\sin\frac{1}{2}\theta$. Therefore, by eq. (56), putting θ for t, and supposing h constant,

$$_{,}\rho_0 = a\int [i(\theta - \sin\theta) + j(1 - \cos\theta)]\sin\tfrac{1}{2}\theta . d\theta \div \int \sin\tfrac{1}{2}\theta . d\theta.$$

By integrating this equation from 0 to π, i. e. from the cusp to the vertex of the curve, we have ${}_{\prime}\rho_0 = \frac{4}{3}a(i+j)$. If we go from the origin to the next cusp, the limits will be 0 and 2π, and ${}_{\prime}\rho_0 = a(\pi i+\frac{4}{3}j)$.

As another example take the tortuous curve whose equation is

$$ {}_{\prime}\rho = a(it+\tfrac{1}{2}\sqrt{2}.j\,t^2+\tfrac{1}{3}kt^3), $$

and which is projected on the three reference planes into a common, cubic, and semi-cubic parabola respectively. $d{}_{\prime}\rho \div dt = {}_{\prime}\psi'(t) = a(i+jt\sqrt{2}+kt^2)$;

$$ \therefore\ T{}_{\prime}\psi'(t) = a(1+t^2). $$

$$ \therefore\ {}_{\prime}\rho_0 = a\int(it+\tfrac{1}{2}jt^2\sqrt{2}+\tfrac{1}{3}kt^3)(1+t^2)dt \div \int(1+t^2)dt. $$

If the integration be from 0 to 1 we shall have

$$ {}_{\prime}\rho_0 = \tfrac{9}{16}i+\tfrac{1}{5}\sqrt{2}.j+\tfrac{5}{48}k. $$

For an example of the application of eq. (57) take the parabola whose equation is ${}_{\prime}\rho = {}_{\prime}at+\varepsilon\frac{1}{2}t^2$, in which ε is a unit vector along the axis, and ${}_{\prime}a$ is a vector along the tangent at the vertex, so that $S\varepsilon{}_{\prime}a = 0$.

$\therefore\ {}_{\prime}\psi'(t) = {}_{\prime}a + \varepsilon t,\ V\varepsilon{}_{\prime}\psi = tV\varepsilon{}_{\prime}a = t\varepsilon{}_{\prime}a,\ \varepsilon V\varepsilon{}_{\prime}\psi = -t{}_{\prime}a,\ TV\varepsilon{}_{\prime}\psi = at$ and $S\varepsilon^{-1}{}_{\prime}\psi' = t$. Suppose h to be constant, then

$$ {}_{\prime}\rho_0 = \int_0^t \tfrac{1}{2}({}_{\prime}at+\varepsilon t^2)at^2.dt \div \int_0^t at^2dt = \frac{\frac{1}{4}t^4{}_{\prime}a+\frac{1}{5}\varepsilon t^5}{\frac{2}{3}t^3} $$

$$ = \tfrac{3}{8}t{}_{\prime}a+\tfrac{3}{10}t^2\varepsilon = \tfrac{3}{8}{}_{\prime}at+\tfrac{3}{5}\varepsilon\tfrac{1}{2}t^2. $$

Let us apply eq. (59) to the equation of the parabola just used, now however regarding ${}_{\prime}a$ as a vector along *any* tangent and ε as a unit vector along the diameter through the point of contact of the same. Then

$$ T\dot{V}{}_{\prime}\psi{}_{\prime}\psi' = tTV({}_{\prime}a + \tfrac{1}{2}\varepsilon t)({}_{\prime}a + \varepsilon t) = t^2TV({}_{\prime}a\varepsilon - \tfrac{1}{2}{}_{\prime}a\varepsilon) = \tfrac{1}{2}t^2TV{}_{\prime}a\varepsilon. $$

$$ \therefore\ {}_{\prime}\rho_0 = \tfrac{2}{3}\int_0^t({}_{\prime}a+\tfrac{1}{2}\varepsilon t)t^2dt \div \int_0^t t^2dt = \tfrac{1}{2}{}_{\prime}at+\tfrac{2}{5}\varepsilon\tfrac{1}{2}t^2. $$

This gives the c. g. of any segment of a parabola.

The equation of a circle may be written ${}_{\prime}\rho = {}_{\prime}\psi(\theta) = a(i\cos\theta+j\sin\theta)$; $\therefore\ {}_{\prime}\psi' = a(-i\sin\theta+j\cos\theta)$, and $TV{}_{\prime}\psi{}_{\prime}\psi' = a^2$. Therefore, by eq. (58), if h = constant, and $\theta = t$,

$$ {}_{\prime}\rho_0 = a\int_{-th}^{+th}\int_u^1(i\cos\theta+j\sin\theta)u^2d\theta\,du \div \int_{-th}^{+th}\int_u^1 ud\theta du $$

$$ = \tfrac{1}{3}a(1-u^3)[i\sin\theta-j\cos\theta]_{-th}^{+th} \div \tfrac{1}{2}(1-u^2)[\theta]_{-th}^{+th} $$

$$ = \frac{2}{3}\cdot\frac{1-u^3}{1-u^2}\cdot\frac{i\sin\theta}{\theta}.a. $$

This gives the c. g. of a portion of a ring contained bet. two concentric cir.

$$ \frac{1-u^3}{1-u^2} = \frac{(1-u)(1+u+n^2)}{(1-u)(1+u)} = \frac{1+u+u^2}{1+u};\ \therefore\ {}_{\prime}\rho_0 = \frac{2}{3}\cdot\frac{1+u+u^2}{1+u}\ \frac{\sin\theta}{\theta}.ia. $$

If $u = 1$, we have for an arc of a circle, ${}_{\prime}\rho_0 = (\sin\theta \div \theta).ia.$

Ellipsoid.—The equation of this surface may be written

$$,\rho = ,a\cos x + ,b\sin x\cos y + c\sin x\sin y,$$

in which a, b. c are the semi-axes. The c. g. of the solid ellipsoidal shell will be found first by eq. (61), and then that of the surface, from this by making the thickness of the shell infinitesimal, for the reason that the scalar in eq. (61) is much simpler than the tensor in eq. (60) in the case of the ellipsoid. We have then

$$\frac{d,\phi}{dx} = -,a\sin x + ,b\cos x\cos y + ,c\cos x\sin y,$$

$$\frac{d,\phi}{dy} = -,b\sin x\sin y + ,c\sin x\cos y. \quad \text{Whence}$$

$$S,\phi\frac{d,\phi}{dx}\frac{d,\phi}{dy} = S,a,b,c\sin x = -abc\sin x.$$

Thus eq. (61) becomes, if h is constant as usual,

$$,\rho_0 = \int_0^x\int_0^y\int_u^1 (,a\cos x + ,b\sin x\cos y + ,c\sin x\sin y)u^3\sin x.dx\,dy\,du$$
$$\div \int_0^x\int_0^y\int_u^1 u^2\sin x.dx\,dy\,du,$$

in which equation the limits are so taken as to include a shell bounded by two similar ellipsoids, the plane $,a,b$, a plane through $,a$ inclined to the plane of $,a$ and $,b$ at an angle $\tan^{-1}[(c\div b)\tan y]$, and a cone with its vertex at the origin and a section of the ellipsoid by a plane $\|$ to the plane of $,b$ and $,c$ for a directrix, the distance of this last plane from the origin being $,a\cos x$. By integration we find

$$,\rho_0 = \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\cdot\frac{,a\,y\sin^2 x + ,b\sin y\,(x-\sin x\cos x) + ,c(1-\cos y)(x-\sin x\cos x)}{2y(1-\cos x)}$$

For one-eighth part of the ellipsoidal shell $x = y = \frac{1}{2}\pi$, and

$$,\rho_0 = \frac{3}{8}\left(\frac{1-u^4}{1-u^2}\right)(,a + ,b + ,v).$$

For one-fourth $x = \pi$, $y = \frac{1}{2}\pi$,

$$\therefore ,\rho_0 = \frac{3}{8}\left(\frac{1-u^4}{1-u^3}\right)(,b + ,c).$$

If we had integrated from $-y$ to $+y$ instead of from 0 to y the result above would have been unchanged except that the term containing $,c$ would not appear; if then x be taken from 0 to π we shall have a lune lying between two planes through $,a$ making equal angles with the plane of $,a$ and $,b$, of which the c. g. will be given by

$$,\rho_0 = \frac{3\pi ,b}{16}\cdot\frac{\sin y}{y}\cdot\frac{1-u^4}{1-u^3}.$$

To obtain the solids reaching to the center we have only to make $u = 0$ in each of the above equations. For the surface of the ellipsoid we have

$$\frac{1-u^4}{1-u^3} = \frac{1+u+u^2+u^3}{1+u+u^2} = \frac{4}{3},$$

where $u = 1$; and this value substituted in each of the above expressions will give the desired results. Of course we have only to make $a = b = c$ to get the corresponding results for the sphere.

As an example of the use of (64) let us apply it to the torus generated by the revolution of the circle ${}_{,}\rho = bj + a(i\cos t + j\sin t)$ about the axis of i.

We shall have then $\varepsilon = i$ and for the equation of the torus

$${}_{,}\rho = i^{t\theta\div\pi}[ai\cos t + (b + a\sin t)j]i^{-t\theta\div\pi}.$$

$V\varepsilon{}_{,}\psi = Vi[ai\cos t + (b + a\sin t)j] = (b+a\sin t)k$, ${}_{,}\psi'(t) = a(-i\sin t + j\cos t)$, $T{}_{,}\psi' V\varepsilon{}_{,}\psi = a(b+a\sin t)$, $S\varepsilon{}_{,}\varphi = Si{}_{,}\psi = -a\cos t$. Therefore

$${}_{,}\rho_0 = \int_{+t}^{\pi-t}\left[ia\theta\cos t - i^{\frac{2t\theta}{\pi}}k(b+a\sin t)\right]_{-t\theta}^{+t\theta}(b+a\sin t)\,dt$$

$$\div 2\theta\int_{+t}^{\pi-t}(b+a\sin t)\,dt,$$

The limits are taken from $-\theta$ to $+\theta$ and from t to $\pi - t$ so as to get a portion of the surface symmetrical about j. From this we find on integr'n,

$${}_{,}\rho_0 = \frac{[b^2(\pi-2t)+4ab\cos t+\frac{1}{2}a^2(\pi-2t-2\sin t\cos t)]j\sin\theta}{\theta(b\pi-2bt+2a\cos t)},$$

from which by giving proper values to θ and t we may find the c. g. of any portion of the surface symmetrical about the planes of i and j, and j and k.

We will next apply eq. (67) to the case of a hyperboloid of revolution of one sheet formed by revolving the straight line ${}_{,}\rho = {}_{,}\psi(t) = bj + t\varepsilon$ about the axis of i. For convenience we will suppose $S\varepsilon j = 0$, so that the generatrix in its initial position meets j at a distance b from the origin and is $\perp$ to j.

For the equation of the surface we have then ${}_{,}\rho = i^{t\theta\div\pi}(bj+t\varepsilon)i^{-t\theta\div\pi}$, i replacing the ε of eq. (67) $\therefore$ ${}_{,}\psi'(t) = \varepsilon$, $Vi{}_{,}\psi = Vi(bj + t\varepsilon) = bk + tVi\varepsilon$, $V{}_{,}\varphi{}_{,}\psi' = V(bj+t\varepsilon)\varepsilon = bj\varepsilon$, $S{}_{,}\psi{}_{,}\psi' Vi{}_{,}\psi = bSj\varepsilon(bk+tVi\varepsilon) = -b^2Si\varepsilon$, $Si{}_{,}\psi = tSi\varepsilon$.

$$\therefore {}_{,}\rho_0 = \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\cdot\int_{t_1}^{t_2}\left[\theta ti^{-1}Si\varepsilon - i^{\frac{2t\theta}{\pi}}(bk+tVi\varepsilon)\right]_0^{t\theta_1}dt \div \theta_1\int_{t_1}^{t_2}dt$$

$$= \frac{3}{4\theta_1(t_2-t_1)}\cdot\frac{1-u^4}{1-u^3}\left[\tfrac{1}{2}\theta_1(t_2^2-t_1^2)i^{-1}Si\varepsilon - \left(i^{\frac{2t\theta_1}{\pi}}-1\right)[bk(t_2-t_1)+\tfrac{1}{2}(t_2-t_1)Vi\varepsilon]\right]$$

$$= \frac{3}{4}\cdot\frac{1-u^4}{1-u^3}\left[\tfrac{1}{2}(t_2+t_1)i^{-1}Si\varepsilon - \theta_1^{-1}\left(i^{\frac{2t\theta_1}{\pi}}-1\right)[bk+\tfrac{1}{2}(i_2+t_1)Vi\varepsilon]\right].$$

Suppose $t_1 = 0$, $u = 0$ and $\theta_1 = \pi$, for a half revolution,

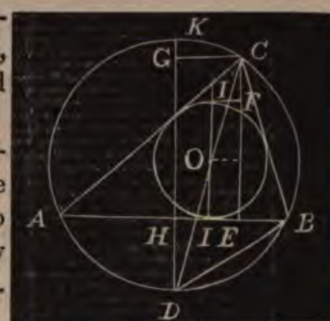
$$\therefore {}_{,}\rho_0 = \tfrac{3}{8}i^{-1}Si\varepsilon t_2 + \frac{3}{4\pi}Vi\varepsilon t_2 + \frac{3b}{2\pi}k.$$

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. VII.

SOLUTIONS of problems in No. 6, Vol. VII, have been rec'd as follows: From Prof. W. P. Casey, 325, 326, 327, 329; Geo. M. Day, 325, 326, 328, 329; Prof. A. B. Evans, 327; Alexander Evans, 328; George Eastwood, 325, 326, 329; Wm. Hoover, 325, 326, 329; Prof. J. H. Kershner, 325, 326, 328; Prof. F. P. Matz, 325; Octavian L. Mathiot, 325; Prof. D. J. Mc Adam, 328; Prof. P. H. Philbrick, 325, 326, 328; Prof. E. B. Seitz, 325, 326, 328, 329, 330; Prof. J. Scheffer, 325, 326, 327, 328, 329.

325. "Given the altitude and radius of the circumscribed and inscribed circles of a plane scalene triangle; to find the three sides."

SOLUTION BY PROF. F. P. MATZ, KING'S MOUNTAIN, N. C.



Let ABC represent the triangle, CE, the given altitude, O the center of the inscribed circle, R and r radii of the circumscribed and insc'bd circles, respectively, and put $CE=a$.

Through O, draw CD, and draw DK perpendicular to AB; then is DK a diam. of the circumscribed circle parallel to CE. Also through O draw II' parallel to CE, and draw CG and $I'F$ each parallel to AB. Then (Eucl. Book VI., Prop. XXIV) we have

$$DH : IO(=r) :: OI'(=r) : FC(=a-2r);$$

$$\therefore DH=\frac{r^2}{a-2r}.$$

By property of the circle

$$AH=\surd(DH\times HK)=\frac{r\surd[2R(a-2r)-r^2]}{a-2r};$$

$$\therefore AB=\frac{2r\surd[2R(a-2r)-r^2]}{a-2r}.$$

Similarly we find

$$GC=HE=\frac{(a-r)\surd[2R(a-2r)-(a-r)^2]}{a-2r}.$$

Therefore we have

$$AC=[(AH+HE)^2+a^2]^{\frac{1}{2}}=\frac{(a-r)[2R(a-2r)-r^2]^{\frac{1}{2}}+r[2R(a-2r)-(a-r)^2]^{\frac{1}{2}}}{(a-2r)},$$

$$BC=[(AH-HE)^2+a^2]^{\frac{1}{2}}=\frac{(a-r)[2R(a-2r)-r^2]^{\frac{1}{2}}-r[2R(a-2r)-(a-r)^2]^{\frac{1}{2}}}{(a-2r)}.$$

326. "Draw a line bisecting a given triangle so that the part lying within shall be, 1st, a minimum, 2nd, a maximum."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.

Let a denote the area of the triangle, and x, y and z the sides of the triangle formed by the bisecting line; and put $\phi =$ the angle opposite z.

We have

$$xy \sin\phi = a; \qquad (1)$$

$$z^2 = x^2+y^2-2xy\cos\phi. \qquad (2)$$

Substituting in (2) the value of y from (1), and then differentiating and placing the result equal to zero, we get

$$x = \sqrt{\frac{a}{\sin\phi}} \text{ and from (1), } y = \sqrt{\frac{a}{\sin\phi}}.$$

Substituting these values for x and y in (2), and reducing we find

$$z^2 = 2a\tan\tfrac{1}{2}\phi.$$

Therefore for a maximum the bisecting line must be taken opposite the greatest angle, and for a minimum, opposite the least angle.

SOLUTION BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY.

Let ABC represent the triangle, A the least and C the greatest angle. Take $Am = x$, $An = y$; the sides as usual a, b, c; $mn = l$. Then is

$$l^2 = x^2+y^2-2xy\cos A. \qquad (1)$$

This expression for given values of x and y is a minimum when A is least and a maximum when A is greatest or equal C.

Let mn represent the minimum line required and we also have

$$2xy = bc. \qquad (2)$$

Substituting for x in (1), differentiating and equating to zero, we get

$$2y^2 = bc = 2xy, \therefore x = y = \sqrt{(\tfrac{1}{2}bc)}.$$

2. Let l_1 be the length of the maximum line, and v and w distances of its extremities from C, on CA and CB. Then

$$l_1^2 = v^2+w^2-2vw\cos C, \qquad (4)$$

$$2vw = ab. \qquad (5)$$

By putting $v = w + z$ and substituting we get $z^2+ab(1-\cos C)$ a max. And this occurs when $v = b$ and $w = \frac{1}{2}a$. A line from A to the middle of BC is therefore the line required.

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327. "The poles of the radical axis of two circles taken with respect to each circle, and the two centers of similitude of the circles, are four harmonic points. (Ex. 8, p. 367, Chauvanet's Modern Geom.)"

SOLUTION BY PROF A. B. EVANS, LOCKPORT, N. Y.

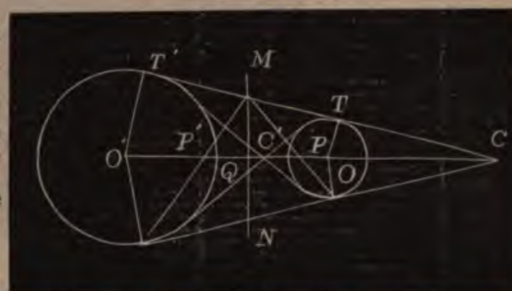
Let P and P' be the poles of the radical axis MN with respect to the circles O and O' whose centres of similitude are C and C'.

Put $OO' = a$, $O'T' = R$, $OT = r$; then

$$OC = \frac{ar}{R-r},\ OC' = \frac{ar}{R+r},\ O'C = \frac{aR}{R-r},\ O'C' = \frac{aR}{R+r};$$

and since $O'Q^2 - OQ^2 = R^2 - r^2$ and $O'Q + OQ = a$,

$$OQ = \frac{a^2-R^2+r^2}{2a},\qquad O'Q = \frac{a^2+R^2-r^2}{2a}.$$



The radius being a mean proportional between the distances of the polar and its pole from the centre of the circle, $r^2 = OP \times OQ$, and $R^2 = O'P' \times O'Q$, and therefore $OP = 2ar^2 \div (a^2 - R^2 + r^2)$, $O'P' = 2aR^2 \div (a^2 + R^2 - r^2)$.

Now $PC' = OC' - OP$ and $P'C' = O'C' - O'P'$; therefore

$$\frac{P'C'}{PC'} = \frac{R}{r}\left[\frac{a^2-R^2+r^2}{a^2+R^2-r^2}\right]. \tag{1}$$

Again $PC = OC + OP$ and $P'C = O'C - O'P'$; therefore

$$\frac{P'C}{PC} = \frac{R}{r}\left[\frac{a^2-R^2+r^2}{a^2+R^2-r^2}\right]. \tag{2}$$

From (1) and (2)

$$\frac{P'C'}{PC'} = \frac{P'C}{PC},$$

and therefore P, P' and C, C' are four harmonic points.

SOLUTION BY PROF. JOSEPH H. KERSHNER.

Let C and C be the centres of similitude, P and P' the poles, and R any point on the radical axis. Draw RP, RP' cutting the circles in A and A'. Connect AP', $A'P$ and RC', which will meet in a common point: See any good work on Mod. Geom. By the transversal CA' we have

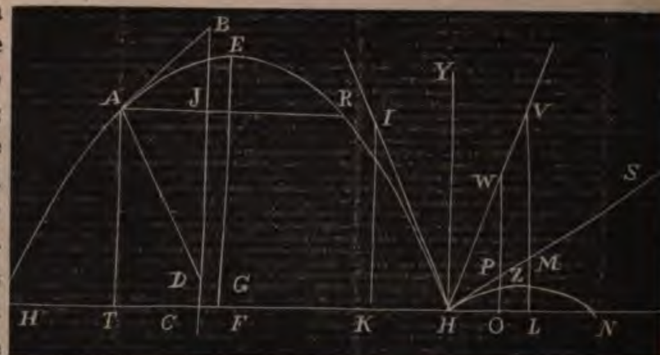
$$CP' . AP . A'R = CP . AR . A'P';$$

also $$C'P'. AP . A'R = C'P. AR . A'P'.$$

Dividing, $$CP' : C'P' :: CP : C'P.$$

328. "A body is projected from the top of a tower 100 ft high at an angle of elevation of 45°, with a velocity of 60 ft per second. Find the distance from the point at which it first strikes the horizontal plane to the second point at which it strikes the plane. The modulus of elasticity being $\frac{1}{3}$, and the resistance of the atmosphere neglected."

SOLUTION BY ALEXANDER EVANS, ESQ., ELKTON, MD.



Draw AR, a horizontal line through the top of the tower AT; make the angle $RAB = 45°$, and take $AB = 60$ equal the velocity at A. Cons't the parabola $AERH$, draw the principal axis EF, and through B, the line BC parallel to EF. BC will be the Hodograph to the parabola. [See *Elements of Dynamics*, by W. K. Clifford, F. R. S., p. 67.]

Draw the horizontal line THN through the bottom of the tower: and through H, the tangent HI; and draw AD parallel to HI to intersect the hodograph in D; AD will represent in magnitude and direction the velocity of the projectile at H.

Draw HY parallel to EF and make the angle YHW equal to YHI. From any point W draw WO parallel to EF, and take PO equal one-third of WO; draw from H the line $HPMS$, this will be the direction of the projectile after reflection.

Take HI on the tangent equal to AD, and draw IK parallel to EF. KH is the horizontal velocity at H; and since the horizontal velocity is not changed after the impact, take $HL = HK = AJ$; through L draw LV parallel to EF; VL will be the hodograph to HZN, and HM will represent in magnitude and direction the velocity of the projectile at H after rebounding.

With HM the velocity, and the angle MHN, construct the parabola HZN; HN will be the range on the rebound $= 79.7$ nearly.

As the velocity at the point H' on the other side of the principal axis, is the same as at H, if we suppose this velocity to be V, the time of flight in the first parabola will be $(2V \div g) \sin \angle IHK = (2V \div g) \sin \theta$; and in

the second parabola, $(2V\div g)e\sin\theta$, and as the velocities horizontally are uniform, the range in the second parabola will be to that in the first as e to 1, that is one-third of the first; and as the range in the first, $2GH=239.2$, that in the second $=79.7$ nearly.

[Prof. Philbrick, and Prof. Scheffer, each gave a very elegant solution of this problem, and each shows that the range in the second parabola is to that in the first as e to 1; and Prof Scheffer draws the inferrence, that the body would rest, after an infinite number of rebounds, at a distance from $H'=[1\div(1-e)]\times H'H$.]

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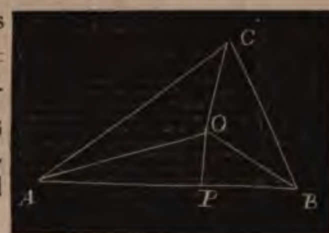
329. "Three points, A, B, C, being given, to find a point M, whose distance from A, B and C, shall be a minimum."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.



Let ABC be the given triangle, a, b, c its sides, and O the required point. Put $AO=u$, $BO=v$, $CO=w$. Draw OP perpendicular to AB and let $AP=x$ and $PO=y$; also let $\angle AOP=\theta$, $BOP=\varphi$, $COP=\beta$.

Then $u^2=x^2+y^2$, $v^2=(c-x)^2+y^2$, and $w^2=(b\cos A-x)^2+(b\sin A-y)^2$.

And in order that $u+v+w$ may be a maximum we must have

$$\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}=0,\quad \frac{du}{dy}+\frac{dv}{dy}+\frac{dw}{dy}=0.$$

Now $\frac{du}{dx}=\frac{x}{u}=\sin\theta$, $\frac{dv}{dx}=-\frac{c-x}{v}=-\sin\varphi$, $\frac{dw}{dx}=-\frac{b\cos A-x}{w}=-\sin\beta$; $\frac{du}{dy}=\frac{y}{u}=\cos\theta$, $\frac{dv}{dy}=\frac{y}{v}=\cos\varphi$ and $\frac{dw}{dy}=-\frac{b\cos A-y}{w}=\cos\beta$. Hence we have the conditions,

$$\sin\theta=\sin\varphi+\sin\beta,$$
$$\cos\theta=-\cos\varphi-\cos\beta.$$

Squaring and adding we find $\cos(\beta-\varphi)=-\frac{1}{2}$, or $\beta-\varphi=120°$; $\therefore$ the angle $BOC=120°$. And in a similar way it may be shown that $\angle AOC=120°=\angle AOB$. Therefore if on any two sides of the triangle segments of a circle be described containing angles of 120° their intersect'n will determine the point O.

This problem is the same as Problem 257, ANALYST, Vol. VI, p. 93, if we take m, n, r in that problem each $=1$. This problem possesses considerable interest in the history of mathematics.

330. "Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken at random in the circumference; find the average area of the triangle formed by joining the points."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $ACBD$ be the given circle, AB the given diameter, M, N two random points on opposite sides of AB, CD the chord through them, P a random point in the circumference, and O the center of the circle. Draw the diameter EF perpendicular to CD, and PK perpendicu'r to EF.

Let $OA=r$, $RM=x$, $RN=y$, $RC=u$, $RD=v$, $\angle COH=\theta$, $BOH=\varphi$, and $POE=\psi$. Then we have

$$u=r\sin\theta+r\cos\theta\tan\varphi, \qquad v=r\sin\theta-r\cos\theta\tan\varphi,$$

$$\text{area } MNP=\tfrac{1}{2}r(x+y)(\cos\psi-\cos\theta)=u_1, \text{ when } \psi<\theta,$$

and $$\text{area } MNP=\tfrac{1}{2}r(x+y)(\cos\theta-\cos\psi)=u_2, \text{ when } \varphi>\theta,$$

An element of surface at M is $r\sin\theta\,d\theta dx$, and at N it is $(x+y)d\varphi dy$, and an element of the circumference at P is $rd\psi$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $-\theta$ and θ, and doubled; of x, 0 and u; of y, 0 and v; and of ψ, 0 and θ, and θ and π.

By limiting P to the semi-circumference ECF, the whole number of ways the three points can be taken is $\frac{1}{4}\pi^2r^4.\pi r$; hence the required average is

$$\frac{8}{\pi^3r^5}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v\left\{\int_0^\theta u_1rd\psi+\int_\theta^\pi u_2rd\psi\right\}r\sin\theta\,d\theta d\varphi dx(x+y)dy$$

$$=\frac{4}{\pi^3r^2}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}\int_0^u\int_0^v(\pi-2\theta+2\tan\theta)\sin\theta\cos\theta d\theta d\varphi dx(x+y)^2dy$$

$$=\frac{2r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{+\theta}(\pi-2\theta+2\tan\theta)(7\tan^2\theta+\tan^2\varphi)(1-\cos^2\theta\sec^2\varphi)\sin\theta\cos^3\theta d\theta d\varphi$$

$$=\frac{4r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}[(\pi-2\theta)\cos\theta+2\sin\theta](24\theta\sin^2\theta-3\theta+3\sin\theta\cos\theta-22\sin^3\theta\cos\theta)\sin\theta d\theta$$

$$=\frac{r^2}{\pi}\left[\frac{13}{18}+\frac{352}{81\pi^2}\right].$$

SOLUTION OF MISCEL. PROB. (2), P. 149, VOL. VII, BY PROF. SCHEFFER.

If in the series $S=ax+bx^2+cx^3+dx^4+ex^5+\dots$ we put $x=y\div(1+vy)$, we get $S=ay(1+vy)^{-1}+by^2(1+vy)^{-2}+cy^3(1+vy^{-3}+dy^4(1+vy)^{-4}+\dots$

Expanding by the Binomial Theorem, we obtain

$$ay-(av-b)y^2+(av^2-2bv+c)y^3-(av^3-3bv^2+3cv-d)y^4+\dots$$

Substituting for y its equiv't $x\div(1-vx)$, we have the req'd transformation.

[This poblem was solved in a similar manner by Prof. Kershner.]

PROBLEMS.

332. *By Prof. M. L. Comstock.*—Find all the values of x and y in the following equations:

$$x^2 + xy^3 = 18, \qquad (1)$$

$$xy + xy^2 = 12. \qquad (2)$$

333. *By William Hoover.*—Find the value to x terms of the con. fract.

$$\cfrac{2}{1+\cfrac{2}{1+\&c.}}$$

334. *By Prof. J. H. Kershner.*—Pairs of tangents which meet always at the same angle are drawn to a given ellipse. Find the envelope of the chords of contact.

335. *Communicated by Pof Root. (from Exam. Prob's, H. Col.)*—The curve whose rectangular equation is $x^{2/3} + y^{2/3} = r^{2/3}$ revolves around the axis of x. Determine the volume of the solid thus described between the limits $x = 0$ and $x = r$.

336. *By George Eastwood.*—In a locomotive engine there are given: The impressed force of the steam on the piston, the radius of the crank, and the length of the connecting rod: To find the uniform force which, if applied at right angles to the end of the crank, would do the same work as the impressed force.

337. *By Prof. Elias Schneider.*—Required the constant quantity into which if we divide the periodic time of any planet, multiplied by its third root, the quotient will be the distance such planet falls from a tangent to its orbit in one second of time: i. e., solve the equation,

$$\frac{\text{Constant Quantity}}{(\text{Periodic Time})^{4/3}} = \text{Fall from tang.}$$

338. *By Prof. Asaph Hall.*—A comet moves around the sun in a given parabolic orbit; find the right ascension and declination of the point on the heavens towards which the comet approaches as it recedes from the sun and earth.

QUERY BY PROF. J. SCHEFFER.—If of any curve we find the evolute, and of the latter the evolute, and so on ad infin., the ultimate evolute is a cycloid. How is this proved?

NOTE OF THANKS. — We tender our thanks to our subscribers for the many kind words of greeting we have received, and for their patronage and contributions for the ANALYST. We are under special obligations to Prof. Casey for his efforts in behalf of the ANALYST on the "Pacific Slope".

We embrace this opportunity to call the attention of our readers to the *map* on p. 4 of cover. Many of them, no doubt, are personally acquainted with the "*Rock Island Road*" and its officers and do not need our endorsement to select that road as the best, and most direct, line of transit from Chicago to the Pacific coast; but to those who are not, we can say, from an experience of twenty years with that road, that travelers will find, on the *Rock Island Road*, all the comforts and facilities for travel to be found on first class roads anywhere; and officers who are accommodating and attentive to their passengers, and who spare no pains nor expense to keep their road in good condition and their trains on time.

J. E. HENDRICKS.

PUBLICATIONS RECEIVED.

Bulletin of the Philosophical Society of Washington. Volumes I, II and III. 1871 to 1880.

The *Bulletin* contains notes of the meetings of the Society (semi-monthly), with abstracts of the more important papers read.

Discussion of a Geometrical Problem, with Biographical Notes. By MARCUS BAKER, U. S. Coast Survey. [Read before the Phil. Soc. of Wash. and published in the *Bulletin.*]

Journal de Mathematiques Elementaires et Speciales. Quatrieme Annee. No. 10.—Octobre, 1880. Paris.

The Observatory, a Monthly Review of Astronomy. Edited by W. H. M. CHRISTIE, M. A. Dec. 1880. London.

Ciel et Terre. Decembre 1880. Bruxelles.

Science, a Weekly Review of Scientific Progress. JOHN MICHELS, Editor. 229 Broadway, New York.

An Elementary Treatise on Plane Trigonometry. By J. MORRISON, M. D., M. A. 332 pp. 12mo. Toronto: Canada Publishing Co. 1880.

In discussing the principles of the science, the author has introduced in this book a variety of examples that will not only interest the student but will serve to impress him with the utility of the science.

The Teacher's Hand-Book of Algebra. By J. A. MCLELLAN, M. A., LL. D. 229 pp. 12mo. Toronto: W. J. Gage and Co. 1879.

Our space will not permit us to notice the various claims this book presents to the teacher. We quote a single paragraph from the preface:

"It gives complete explanations and illustrations of important topics which are strangely omitted or barely touched upon in the ordinary books, such as the Principle of Symmetry, Theory of Divisors, Factoring, Applications of Horner's Division, &c."

ERRATA.

On page 9, formula (21), for $B_{(r-n)}$ read $B_{(r-n),s}$.
" " 12, line 4, for $i \sin\theta$ read $i \sin\phi$.
" " 16, " 7, for i^{t-2} read i^{2-t}.

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VOL. VIII. MARCH, 1881. No. 2.

NOTES ON THE THEORIES OF JUPITER AND SATURN.

BY G. W. HILL, NAUTICAL ALMANAC OFFICE, WASHINGTON, D. C.

ON account of their large masses and the near approach to commensurability of their mean motions, Jupiter and Saturn offer the most interesting, as well as the most difficult, field for research, in the planetary perturbations of the solar system. In the following remarks, without treating the subject in a complete manner, which would be impossible here, I intend only to point out a method of procedure and give a few illustrations of its use.

At present we shall notice only the introaction of the sun, Jupiter and Saturn. It will facilitate matters much if we employ differential equations in which the potential function is the same for both planets. This is accomplished by an orthogonal transformation of variables. Let us suppose that the coordinates of the sun in space are denoted by

$$X,\ Y \text{ and } Z,$$

those of Jupiter by

$$X + x + \varkappa x',\ Y + y + \varkappa y' \text{ and } Z + z + \varkappa z',$$

and those of Saturn by

$$X + x' + \varkappa x\,,\ Y + y' + \varkappa y \text{ and } Z + z' + \varkappa z\,,$$

where $\varkappa$ is a small constant to be so determined that the variables x, x', &c., may be orthogonal.

M, m and m' denoting severally the masses of the sun, Jupiter and Saturn, the *vis viva* T of the system is represented by the equation

$$2Tdt = MdX^2 + m(dX + dx + \varkappa dx')^2 + m'(dX + dx' + \varkappa dx)^2$$
$$+ \text{ similar terms in } Y, y, y' \text{ and } Z, z, z'$$

$$\frac{dL}{dt}=\frac{dR}{dl},\quad \frac{dl}{dt}=-\frac{dR}{dL},\quad \frac{dL'}{dt}=\frac{dR}{dl'},\quad \frac{dl'}{dt}=-\frac{dR}{dL'},$$
$$\frac{dG}{dt}=\frac{dR}{dg},\quad \frac{dg}{dt}=-\frac{dR}{dG},\quad \frac{dG'}{dt}=\frac{dR}{dg'},\quad \frac{dg'}{dt}=-\frac{dR}{dG'},$$
$$\frac{dH}{dt}=\frac{dR}{dh},\quad \frac{dh}{dt}=-\frac{dR}{dH},\quad \frac{dH'}{dt}=\frac{dR}{dh'},\quad \frac{dh'}{dt}=-\frac{dR}{dH'},$$

in which it is understood that R is expressed in terms of the elements L, G, H, &c.

As r is a function of the three elements L, G, l only, and r' of L', G', l' only, it follows that the six elements H, g, h, H', g' and h' enter in R only through s; hence we have the equations

$$\frac{dR}{dg}=\frac{dR}{ds}\frac{ds}{dg},\qquad \frac{dR}{dH}=\frac{dR}{ds}\frac{ds}{dH},\qquad \frac{dR}{dh}=\frac{dR}{ds}\frac{ds}{dh},$$
$$\frac{dR}{dg'}=\frac{dR}{ds}\frac{ds}{dg'},\qquad \frac{dR}{dH'}=\frac{dR}{ds}\frac{ds}{dH'},\qquad \frac{dR}{dh'}=\frac{dR}{ds}\frac{ds}{dh'}.$$

The expression for s being given by

$$rr's = xx' + yy' + zz',$$

and v and v' denoting the true anomalies, the rectangular coordinates have the equivalents

$$\begin{aligned}
x &= r[\cos h\cos(v+g)-\cos i\sin h\sin(v+g)],\\
y &= r[\sin h\cos(v+g)+\cos i\cos h\sin(v+g)],\\
z &= r\sin i\ \sin(v+g),\\
x' &= r'[\cos h'\cos(v'+g')-\cos i'\sin h'\sin(v'+g')],\\
y' &= r'[\sin h'\cos(v'+g')+\cos i'\cos h'\sin(v'+g')],\\
z' &= r'\sin i'\ \sin(v'+g').
\end{aligned}$$

Whence the following expression for s:

$$\begin{aligned}
s = {} & \cos(h-h')\cos(v+g)\cos(v'+g')+\cos i\cos i'\cos(h-h')\sin(v+g)\sin(v'+g')\\
&+\cos i\sin(h-h')\cos(v+g)\sin(v'+g')\\
&-\cos i\sin(h-h')\sin(v+g)\cos(v'+g')\\
&+\sin i\sin i'\sin(v+g)\sin(v'+g').
\end{aligned}$$

Remembering that v and v' contain only the same elements as r and r', and that

$$\cos i=\frac{H}{G},\quad \sin i=\frac{\sqrt{(G^2-H^2)}}{G},\quad \cos i'=\frac{H'}{G'},\quad \sin i'=\frac{\sqrt{(G'^2-H'^2)}}{G'},$$

it will be found that

$$\frac{d}{dt}\Big[\sqrt{(G^2-H^2)}\cos h+\sqrt{(G'^2-H'^2)}\cos h'\Big]=0,$$

$$\frac{d}{dt}\Big[\sqrt{(G^2-H^2)}\sin h+\sqrt{(C'^2-H'^2)}\sin h'\Big]=0,$$

$$\frac{d}{dt}\Big[H+H'\Big]\qquad\qquad\qquad\qquad=0.$$

Hence we have the following integrals of the differential equations,

$$\sqrt{(G^2-H^2)}\cos h+\sqrt{(G'^2-H'^2)}\cos h=\text{a constant},$$

$$\sqrt{(G^2-H^2)}\sin h+\sqrt{(G'^2-H'^2)}\sin h=\text{a constant},$$

$$H+H'=\text{a constant}.$$

These integrals may be employed to diminish the number of differential equations. Thus far the system of planes, to which x, y, z, &c. are referred, has been left indefinite: let us now assume that the plane of maximum areas, called by Laplace the invariable plane, is chosen for the plane of xy. In this case it is well known that the constants of the first two of the integrals, given above, become zero. Then we shall have

$$\sqrt{(G^2-H^2)}\cos h+\sqrt{(G'^2-H'^2)}\cos h'=0,$$

$$\sqrt{(G^2-H^2)}\sin h+\sqrt{(G'^2-H'^2)}\sin h'=0,$$

$$H+H'=c,$$

c being an arbitrary constant. But since i and i' are supposed contained between 0° and 180°, the radicals in these expressions must be taken positively. Consequently the equations are equivalent to

$$h'=h+180^\circ,\qquad H+H'=c,\qquad H-H'=\frac{G^2-G'^2}{c}.$$

These equations determine the values of the elements H, H' and h' in terms of the rest, and they may be used to eliminate them from R. Then it is plain, from the expression of s, given above, that h will also disappear from R, and we shall have

$$R=\text{function }(L,\ G,\ L',\ G',\ l,\ g,\ l',\ g'),$$

and s takes the much simpler form

$$s=-\cos(v-v'+g-g')+\frac{(G+G')^2-c^2}{2GG'}\sin(v+g)\sin(v'+g').$$

As to the partial derivatives of R with respect to L, L', l, l', g, g', they are evidently unchanged by this elimination of the elements H, H', h, h'. But $\left(\frac{dR}{dG}\right)$ and $\left(\frac{dR}{dG'}\right)$ denoting the derivatives of R on the supposition of its containing the elements H, H', h, h', we have

$$\left(\frac{dR}{dG}\right)=\frac{dR}{dG}-\frac{dR}{dH}\frac{dH}{dG}-\frac{dR}{dH'}\frac{dH'}{dG},$$

$$\left(\frac{dR}{dG'}\right)=\frac{dR}{dG'}-\frac{dR}{dH}\frac{dH}{dG'}-\frac{dR}{dH'}\frac{dH'}{dG'}.$$

But we also have

$$\frac{dR}{dH}-\frac{dR}{dH'}=\frac{d(h'-h)}{dt}=0,$$

hence

$$\left(\frac{dR}{dG}\right)=\frac{dR}{dG}-\frac{dR}{dH}\frac{d(H+H')}{dG}=\frac{dR}{dG},$$

$$\left(\frac{dR}{dG'}\right)=\frac{dR}{dG'}-\frac{dR}{dH}\frac{d(H+H')}{dG'}=\frac{dR}{dG'}.$$

Moreover

$$\frac{dR}{dc}=\frac{dR}{dH}\frac{dH}{dc}+\frac{dR}{dH'}\frac{dH'}{dc}=\frac{dR}{dH}\frac{d(H+H')}{dc}=\frac{dR}{dH}.$$

Thus the system of differential equations still retains its canonical form, and is

$$\frac{dL}{dt}=\frac{dR}{dl},\quad \frac{dL'}{dt}=\frac{dR}{dl'},\quad \frac{dG}{dt}=\frac{dR}{dg},\quad \frac{dG'}{dt}=\frac{dR}{dg'}.$$

$$\frac{dl}{dt}=-\frac{dR}{dL},\quad \frac{dl'}{dt}=-\frac{dR}{dL'},\quad \frac{dg}{dt}=-\frac{dR}{dG},\quad \frac{dg'}{dt}=-\frac{dR}{dG'}.$$

After this system of eight differential equations is integrated, the value of h is found by a quadrature from the equation

$$\frac{dh}{dt}=-\frac{dR}{dc}.$$

These integrations introduce nine arbitrary constants which, together with c, make ten. The reference of the coordinates to any arbitrary planes introduces three more, but one of these coalesces with the constant which completes the value of h.

The time t does not explicitly enter R, hence the complete derivative of it with respect to t is

$$\frac{dR}{dt}=\frac{dR}{dL}\frac{dL}{dt}+\frac{dR}{dl}\frac{dl}{dt}+\ \&c..$$

If, in this, are substituted the values of $\frac{dL}{dt}, \frac{dl}{dt}$, &c.., from the equations just given, we shall find that it vanishes; hence

$$R = \text{a constant}$$

is an integral of the system of differential equations. This integral may be employed to eliminate one of the elements, as L, from the equations. We can also take one of the elements, as l, for the independent variable in place of t. The system of equations, to be integrated, is then reduced to the six

$$\frac{dL'}{dl}=-\frac{\frac{dR}{dl'}}{\frac{dR}{dL}},\qquad \frac{dG}{dl}=-\frac{\frac{dR}{dg}}{\frac{dR}{dL}},\qquad \frac{dG'}{dl}=-\frac{\frac{dR}{dg'}}{\frac{dR}{dL}},$$

$$\frac{dl'}{dl}=\frac{\frac{dR}{dL'}}{\frac{dR}{dL}},\qquad \frac{dg}{dl}=\frac{\frac{dR}{dG}}{\frac{dR}{dL}},\qquad \frac{dg'}{dl}=\frac{\frac{dR}{dG'}}{\frac{dR}{dL}}.$$

A simpler form can be given to them. If the solution of $R=$ a constant gives

$$L = \text{function } (L', G, G', l', g, g', l),$$

and L is supposed to stand for the right member of this, the foregoing equations can be written

$$\frac{dL'}{dl}=\frac{dL}{dl'},\qquad \frac{dG}{dl}=\frac{dL}{dg},\qquad \frac{dG'}{dl}=\frac{dL}{dg'},$$

$$\frac{dl'}{dl}=-\frac{dL}{dL'},\qquad \frac{dg}{dl}=-\frac{dL}{dG},\qquad \frac{dg'}{dl}=-\frac{dL}{dG'}.$$

When the values of L', G, G', l', g and g' in terms of l have been derived from the integrals of these, they can be substituted in the equation $\frac{dl}{dt}=-\frac{dR}{dt}$, which will then give t in terms of l, by a quadrature. By inverting this we shall have l in terms of t; and by substituting this in equations previously obtained we shall have the values of all the other elements in terms of t.

It will be noticed that R is a homogeneous function of L, L', G, G' and c of the degree -2; hence we shall have

$$L\frac{dR}{dL}+L'\frac{dR}{dL'}+G\frac{dR}{dG}+G'\frac{dR}{dG'}+c\frac{dR}{dc}=-2R=\text{a const.}$$

and, as a consequence of this,

$$L\frac{dl}{dt}+L'\frac{dl'}{dt}+G\frac{dg}{dt}+G'\frac{dg'}{dt}+c\frac{dh}{dt}=\quad 2R=\text{a const.}$$

Thus, if the rate of motion of each elementary argument l, l' &c., be multiplied by the element which is conjugate to it, the sum of the products is invariable.

The sines of half the inclinations of the orbits on the plane of maximum areas are

$$\sin\frac{i}{2} = \sqrt{\left[\frac{(G+G'-c)(G'-G+c)}{4cG}\right]},$$

$$\sin\frac{i'}{2} = \sqrt{\left[\frac{(G'+G-c)(G-G'+c)}{4cG'}\right]}.$$

Thus, in the special case where the two planets move in the same plane, we have

$$G + G' = c.$$

This equation may be employed to eliminate one of the elements G or G' from R. In the same case, the expression for s is reduced to

$$s = -\cos(v - v' + g - g').$$

Then, if we put

$$G - G' = \Gamma, \qquad g - g' = \gamma,$$

R will be a function of L, L', Γ, l, l', γ, and we shall have, for determining these variables, the system of differential equations

$$\frac{dL}{dt} = \frac{dR}{dl}, \qquad \frac{dL'}{dt} = \frac{dR}{dl'}, \qquad \frac{d\Gamma}{dt} = \frac{dR}{d\gamma},$$

$$\frac{dl}{dt} = -\frac{dR}{dL}, \qquad \frac{dl'}{dt} = -\frac{dR}{dL'}, \qquad \frac{d\gamma}{dt} = -\frac{dR}{d\Gamma}.$$

After these are integrated, the value of $g+g'$ will be got by a quadrature from the equation

$$\frac{d(g+g')}{dt} = -\frac{dR}{dc}.$$

If the value of L is obtained from the solution of $R =$ a constant, and we have

$$L = \text{function}\ (L', \Gamma, l', \gamma, l),$$

and l is adopted as the independent variable in place of t, the solution of this special case is reduced to the integration of the four equations

$$\frac{dL'}{dl} = \frac{dL}{dl'}, \quad \frac{dl'}{dl} = -\frac{dL}{dL'}, \quad \frac{d\Gamma}{dl} = \frac{dL}{d\gamma}, \quad \frac{d\gamma}{dl} = -\frac{dL}{d\Gamma}.$$

The angle between the planes of the orbits of Jupiter and Saturn is about $1\frac{1}{4}°$. This is small enough to make the terms, which are multiplied by the square of the sine of half it, and which are besides of two or more dimensions with respect to disturbing forces, practically insignificant. Thus, while we are engaged in developing those terms of the coordinates which demand the highest degree of approximation relatively to disturbing forces, we shall assume that the planes coincide; the determination of the effect of non-coincidence of these planes being reserved to the end, when it will be always sufficient to limit ourselves to the first power of the disturbing force.

[To be continued.]

LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST.

[Continued from page 9.]

In demonstrating the important formulas (23) in the first part of this paper, we followed, as stated at p. 6, the analogy of a process employed by previous writers in finding a "multinomial formula" for the case of one variable. After he work was printed I perceived that for our present purpose it can be considerably shortened, without making it less intelligible.

Let us dispense with the notation of a, B and n as in formulas (4) and (5), and designate the coefficients of the first power of the polynomial by λ as in (2), while those of the k power are represented by l. Then the given polynomial is

$$u=\frac{\lambda_{-m,m}\,\xi^{-m}\eta^{m}\;\Big|\;\lambda_{m,m}\,\xi^{m}\eta^{m}}{\lambda_{-m,-m}\xi^{-m}\eta^{-m}\;\Big|\;\lambda_{m,-m}\xi^{m}\eta^{-m}},$$

and its expansion to the k power is denoted by

$$U=\frac{l_{-km,km}\,\xi^{-km}\eta^{km}\;\Big|\;l_{km,km}\,\xi^{km}\eta^{km}}{l_{-km,-km}\xi^{-km}\eta^{-km}\;\Big|\;l_{km,-km}\xi^{km}\eta^{-km}}.$$

Differentiate these with respect to ξ, and in the first of the two equations (8),

$$kU\left(\frac{du}{d\xi}\right)=u\left(\frac{dU}{d\xi}\right),$$

substitute the expressions for u, U, $\frac{du}{d\xi}$ and $\frac{dU}{d\xi}$. This gives an equation similar to (10), each member of which contains the product of two rectangular polynomials. Forming the coefficient of $\xi^{i-1}\eta^{j}$ in each member, and equating the two to each other by the principle of indeterminate coefficients, we get

$$k\left(\frac{-m\lambda_{-m,m}\,l_{(i+m),(j-m)}\;\Big|\;m\lambda_{m,m}\,l_{(i-m),(j-m)}}{-m\lambda_{-m,-m}l_{(i+m),(j+m)}\;\Big|\;m\lambda_{m,-m}l_{(i-m),(j+m)}}\right)$$
$$=\frac{(i+m)\lambda_{-m,m}\,l_{(i+m),(j-m)}\;\Big|\;(i-m)\lambda_{m,m}\,l_{(i-m),(j-m)}}{(i+m)\lambda_{-m,-m}l_{(i+m),(j+m)}\;\Big|\;(i-m)\lambda_{m,-m}l_{(i-m),(j+m)}}. \qquad (24a)$$

Likewise, differentiating u and U with respect to η, and substituting in the second of the equations (8),

$$kU\left(\frac{du}{d\eta}\right)=u\left(\frac{dU}{d\eta}\right),$$

we get a second equation similar to (10), and equating to each other the coefficients of $\xi^{i}\eta^{j-1}$ in the product in each member, we have

$$k\left(\frac{m\lambda_{-m,m}\ l_{(i+m),(j-m)}}{-m\lambda_{-m,-m}l_{(i+m),(j+m)}}\left|\frac{m\lambda_{m,m}\ l_{(i-m),(j-m)}}{-m\lambda_{m,-m}l_{(i-m),(j+m)}}\right.\right)$$

$$=\frac{(j-m)\lambda_{-m,m}\ l_{(i+m),(j-m)}}{(j+m)\lambda_{-m,-m}l_{(i+m),(j+m)}}\left|\frac{(j-m)\lambda_{m,m}\ l_{(i-m),(j-m)}}{(j+m)\lambda_{m,-m}l_{(i-m),(j+m)}}\right.. \qquad (24b)$$

The relation existing between the coefficients λ and l which form the square groups (22), is expressed by these equations (24*a*) and (24*b*). Transposing in each of them, from the second member to the first, the portion which does not have the coefficient i or j, and rotating each table thro' 180° in its own plane, and changing the signs of both members, we get the desired formulas (23).

Now when the exponent k is made very large, or infinite, the coefficients l of the expansion become ordinates z to the limiting surface, and supposing them to be set close together so as to be consecutive, Δx and Δy are represented by dx and dy, and (24) reduces to

$$x = idx, \qquad y = jdy. \qquad (25)$$

The square group of coefficients l in (22) contains $(2m + 1)^2$ terms, and the whole expansion contains $(2km+1)^2$ terms. Let k be made an infinity of the second order, as in the analogous case already cited (ANALYST, Sept. 1879, p. 147). The limiting surface will then extend over the whole of the infinite plane XY, and $(2km+1)^2$ will represent the whole number of points in that plane, or the whole number of consecutive ordinates z, while $(2m+1)^2$ is the number of such ordinates included in the square group under consideration; so that the portion of the plane XY occupied by this group is infinitesimal, both in its length and breadth. The corresponding portion of the surface may therefore be regarded as an element of a plane surface, that is, an element of the plane which is tangent to the limiting surface at the middle point of the group. Let z be restricted to mean the middle one of this square group of ordinates, answering to $l_{i,j}$ in (22), and let d_xz and d_yz denote the differentials of z taken in the x and y directions. Then the values of all the coefficients l in (22), at the limit, may be found from z by successive additions and subtractions of d_xz and d_yz, and will stand in the rectangular order

$$\frac{z-md_xz+md_yz}{z-md_xz-md_yz}\left|\frac{z+md_xz+md_yz}{z+md_xz-md_yz}\right.. \qquad (26)$$

Substituting them for the corresponding values of l in (23), collecting separately the coefficients of z, d_xz and d_yz in the result, remembering that

$$\frac{\lambda_{-m,m}}{\lambda_{-m,-m}}\left|\frac{\lambda_{m,m}}{\lambda_{m,-m}}\right. = 1 \qquad (27)$$

since the sum of all the given probabilities is certainty, also observing that we have identically

$$\left.\begin{aligned}\frac{m(-m)\lambda_{m,-m} \mid m(m)\lambda_{-m,-m}}{-m(-m)\lambda_{m,m} \mid -m(m)\lambda_{-m,m}} &= \frac{-m(\ \ m)\lambda_{-m,m} \mid m(\ \ m)\lambda_{m,m}}{-m(-m)\lambda_{-m,-m} \mid m(-m)\lambda_{m,-m}} \\ \frac{-m(\ \ m)\lambda_{m,-m} \mid m(\ \ m)\lambda_{-m,-m}}{-m(-m)\lambda_{m,m} \mid m(-m)\lambda_{-m,m}} &= \qquad \textit{ditto},\end{aligned}\right\} \tag{28}$$

and using α_1, α_2, β_1, β_2 and γ as auxiliary letters, thus,

$$\left.\begin{aligned}\alpha_1 &= \frac{-m\lambda_{-m,m} \mid m\lambda_{m,m}}{-m\lambda_{-m,-m} \mid m\lambda_{m,-m}}, & \alpha_2 &= \frac{m\lambda_{-m,m} \mid m\lambda_{m,m}}{-m\lambda_{-m,-m} \mid -m\lambda_{m,-m}}, \\ \beta_1 &= \frac{(-m)^2\lambda_{-m,m} \mid m^2\lambda_{m,m}}{(-m)^2\lambda_{-m,-m} \mid m^2\lambda_{m,-m}}, & \beta_2 &= \frac{m^2\lambda_{-m,m} \mid m^2\lambda_{m,m}}{(-m)^2\lambda_{-m,-m} \mid (-m)^2\lambda_{m,-m}}, \\ \gamma &= \frac{(-m)m\lambda_{-m,m} \mid mm\lambda_{m,m}}{(-m)(-m)\lambda_{-m,-m} \mid m(-m)\lambda_{m,-m}}, & &\end{aligned}\right\} \tag{29}$$

and writing k instead of $k+1$, which is permissible because k is infinite, we shall find that (23) reduces to the form

$$\left.\begin{aligned}k(-\alpha_1 z+\beta_1 d_x z+\gamma\ d_y z) &= -i(z-\alpha_1 d_x z-\alpha_2 d_y z), \\ k(-\alpha_2 z+\gamma\ d_x z+\beta_2 d_y z) &= -j(z-\alpha_1 d_x z-\alpha_2 d_y z).\end{aligned}\right\} \tag{30}$$

These are differential equations of the limiting surface sought, and α, β, &c. are constant parameters whose significance we now inquire into. In all of them, the given probabilities λ occur in the same order as in (2) and (22), and regarding them as the masses of material points in a system, and taking the intervals between them, Δx and Δy, as units of distance, we see that here as in (3), α_1 and α_2 are the statical moments of the system about the Y and X axes through the middle of the group. If therefore the given elementary errors are so distributed about the point of no error, which is at the middle of the group, as to make this point the centre of gravity of the system of masses λ, the statical moments about any axes through this centre will be null, and we shall have $\alpha_1=0$ and $\alpha_2=0$. As for β_1 and β_2, the masses λ in each are multiplied by the squares of their distances from the Y and X axes respectively, so that β_1 and β_2 are the moments of inertia of the system about those axes, and they also represent the squares of the radii of gyration, since the sum of all the masses λ is unity. Lastly, γ is the sum of all the products formed by multiplying each λ into its distances from the Y and X axes.

It is proved in mechanics that if we have a system of masses M of material points in a plane, and draw any rectangular axes through their centre of gravity, and denote by u and v the coordinates of each point M, the "free axes" of the system in that plane, or the principal axes through the centre of gravity, will be determined in position by the equation

$$\tan 2\varphi = \frac{2\Sigma(Muv)}{\Sigma(Mu^2)-\Sigma(Mv^2)}, \tag{31}$$

where φ is the angle which a free axis makes with the assumed axis of the abscissas u. (See for instance the demonstration in Weisbach's *Mechanics*.) If the free axis coincides with this axis of abscissas, so that $\varphi=0$, we shall have $\Sigma(Muv) = 0$. Now γ in (29) is formed in the same way as $\Sigma(Muv)$, whence it appears that if the given elementary errors are so distributed about the assumed X and Y axes through the point of no error, as to make these the free axes of the system of masses λ, we shall have not only $a_1=0$ and $a_2=0$, but $\gamma=0$.

Of course, if the distribution of the elementary errors about the point of no error is entirely arbitrary, that point will not in general be the centre of gravity of their probabilities λ, and the assumed X and Y axes will not be free axes. If, however, we suppose these X and Y axes to be rotated about the origin, so as to become for instance parallel to the free axes, this will make no difference in the expansion. When the positions of the coefficients λ in a rectangular polynomial such as (2) or (22), and those of the coefficients l in its expansion, are referred to the same rectangular axes, these positions are independent of the directions of the axes so long as the origin remains the same. For example, let x_1y_1 and x_2y_2 be co-ordinates of two given points of error, referred to horizontal and vertical axes, and let $x'_1y'_1$ and $x'_2y'_2$ be co-ordinates of the same points referred to rectangular axes having the same origin but making an angle θ with the former. Multiplying together the two corresponding terms of the polynomial, omitting their coefficients λ, we have in the two systems

$$\left.\begin{aligned} \xi^{x_1}\eta^{y_1}\times\xi^{x_2}\eta^{y_2} &= \xi^{x_1+x_2}\eta^{y_1+y_2}, \\ \xi^{x'_1}\eta^{y'_1}\times\xi^{x'_2}\eta^{y'_2} &= \xi^{x'_1+x'_2}\eta^{y'_1+y'_2}. \end{aligned}\right\} \tag{32}$$

The exponents in the second members are the coordinates of the place of the coefficient of the product, in the two systems. The known relations between such systems of coordinates as are here supposed, are

$$\left.\begin{aligned} x_1 &= x'_1\cos\theta - y'_1\sin\theta, & x_2 &= x'_2\cos\theta - y'_2\sin\theta, \\ y_1 &= x'_1\sin\theta + y'_1\cos\theta, & y_2 &= x'_2\sin\theta + y'_2\cos\theta, \end{aligned}\right\} \tag{33}$$

and addition gives

$$\left.\begin{aligned} x_1+x_2 &= (x'_1+x'_2)\cos\theta - (y'_1+y'_2)\sin\theta, \\ y_1+y_2 &= (x'_1+x'_2)\sin\theta + (y'_1+y'_2)\cos\theta. \end{aligned}\right\} \tag{34}$$

These equations, being of the same form as (33), show that just as x_1y_1 and $x'_1y'_1$ are coordinates of the same point in the two systems, so the point whose coordinates are x_1+x_2 and y_1+y_2 in the first system, is the same point whose coordinates in the other system are $x'_1+x'_2$ and $y'_1+y'_2$. In other words, the product occupies the same position under either system. And this being true for all the partial products which make up the total, it appears that whatsoever the directions of the assumed rectangular axes may

be, the origin being unchanged, the places of the coefficients in the expanded polynomial will be the same, and consequently the limiting surface will be unchanged.

Supposing now that the assumed X and Y axes have been rotated so as to be parallel to the free axes, we may suppose further, that while the axes retain this new direction, the origin is transferred to some new point, for instance, to the centre of gravity of the masses λ. This also will make no difference in the positions relatively to each other, of the coefficients in the expansion to the k power. The only difference will be, that relatively to the origin, the whole system of coefficients in this expansion will be transferred through distances, in the x and y directions, k times as great as the distances through which the origin was transferred.

The change of origin amounts simply to increasing or diminishing all the exponents of ξ in (2) by one constant number, and the exponents of η by another, and this, as we have already noticed, does not alter the values of the coefficients in the expansion. If the expansions in the two systems are afterwards brought together so as to coincide, then their origins, or the places of exponent zero in each, will be separated by intervals, in the x and y directions, k times as great as the intervals between the origins in the first power. The case is similar to that of a polynomial of one variable, as treated in my former article (ANALYST, Sept. 1879, p. 146). It appears therefore, that no matter what point is taken as an origin, or what direction is given to the rectangular axes, the coefficients in the expansion of the given polynomial are always the same and in the same relative positions, and consequently their limiting surfaces are the same, differing only in position relatively to the axes. In any given case, we can choose the axes which are most convenient, and we shall choose the free axes of the system of masses λ, the origin being at the centre of gravity.

To show that our previous demonstration is applicable here, we observe that the units of measure Δx and Δy, which form the intervals between the positions of successive coefficients λ in the given polynomial (2), and correspond to differences of unity in the exponents of ξ and η, may be made as small as we please, or even infinitesimals, without affecting the positions of the coefficients, provided all the exponents up to m are made large in proportion. Fractional exponents can be converted into whole numbers by multiplying all the exponents of ξ by one sufficiently large number, infinite if necessary, and all those of η by another, and this will not alter the values of the coefficients in the expansion.

Our system of notation, by the way, represents the coefficient of any term in the first power of the polynomial by λ with sub-indices equal to the ex-

ponents which ξ and η may receive in that term. If now the origin, or place of $\xi^{\alpha}\eta^{\beta}$, is transferred from one term to another, by adding or subtracting a constant number in all the exponents of ξ and another in those of η, this new origin can be made the middle of a square polynomial, by adding to the given terms a sufficient number of terms with zero coefficients, to complete the square. These terms are imagined merely to show that the conditions of the demonstration are satisfied, and their coefficients $\lambda=0$ disappear in the result (23).

Suppose for instance that the polynomial is such that its coefficients occupy the nine angular points of the four small squares into which the square $ABCD$ is divided in the marginal figure, and that we wish to change the positions of the axes so that instead of being parallel to the sides of this square, with their origin at its centre, they shall have new directions and a new origin. Describe the square $EFGH$ with its centre at the desired origin and its sides parallel to the new axes, and large enough to include within it the square $ABCD$. We may conceive that every point of the plane within this new square is occupied by a coefficient λ, only all of them are equal to zero except those at the nine angular points in the original sq. $ABCD$. All that our demonstration requires is that the coefficients λ in the given polynomial should form a square group with one term in the centre, the sides of the square being parallel to the axes of X and Y; and these conditions are fulfilled in the square $EFGH$. The number of terms which the square contains is $(2m+1)^2$, and m here is of course infinite. But the relation (23) between these terms λ and the terms l in any square group of equal size in the expansion will still hold true, when the products λl are formed in the manner there stated. The first members of the two last eq's in (23) will represent the moments of the system of products λl, about axes passing through the centre of the square group and parallel to the new axes of X and Y. The numbers i and j will represent the coordinates of the centre of the square group of expanded terms l in the same X and Y system, the centre of the whole expansion being at the origin, so that i and j may be any whole numbers, up to infinity, and idx and jdy are the coordinates themselves. But a distance, or a lever arm, is not altered by changing the unit of measure. It makes no difference, for example, whether we

say 3 feet, or 36 inches, or $(36\times\infty)$ times (1 inch $\div\infty$). We may restore then, if we please, in (22) and (23), the finite units of measure Δx and Δy, regarding m, i and j as finite numbers, only they will now, in general, be fractional. The same is true of the m in (26), for $d_x z$ and $d_y z$ are the increments of z corresponding to those increments in x and y respectively which are equal to the units of measure Δx and Δy, and which in the limiting surface are reprseented by dx and dy. Thus the validity of the differential eq's (30) of the limiting surface remains unimpaired by the change of axes, provided it is understood that the moments α, β, γ of the given system of masses λ, as expressed in (29), are now to be taken with reference to the new axes, that is, the free axes of the system, while the coordinates x and y, or idx and jdy, refer also to those axes.

Adopting therefore the centre of gravity of the coefficients λ as the origin, and their free axes as the axes of X and Y, we have in (29), as before stated, α_1, α_2 and γ equal to zero, and (30) takes the simple form

$$k\beta_1 d_x z = -iz, \qquad k\beta_2 d_y z = -jz. \tag{35}$$

Giving to i and j their values from (25) we can obtain

$$\frac{d_x z}{z} = \frac{-xdx}{k\beta_1(dx)^2}, \qquad \frac{d_y z}{z} = \frac{-ydy}{k\beta_2(dy)^2}. \tag{36}$$

According to what was said in connection with (29), $\beta_1(dx)^2$ and $\beta_2(dy)^2$ represent the squares of the radii of gyration of the masses λ about the X and Y axes, when Δx and Δy are represented by dx and dy at the limit. Let us write

$$r_1^2 = \beta_1(dx)^2, \qquad r_2^2 = \beta_2(dy)^2. \tag{37}$$

By the theorem which I gave in ANALYST, May 1880, p. 78, kr_1^2 and kr_2^2 represent the squares of the radii of gyration of the coefficients l in the whole expansion to the k power, so that $r_1\sqrt{k}$ and $r_2\sqrt{k}$ are the "quadratic mean errors" in the x and y directions. It is immaterial to the validity of that theorem, whether the radii of gyration are taken with reference to axes passing through the centre of gravity and parallel to the X and Y axes originally assumed, or are taken with reference to the free axes which stand at an angle φ with the others, for if the positions of the coefficients in the first power of the polynomial and its expansion are referred to any rectangular axes with origin at the centre of gravity, it will always hold true, as in my article cited, that the squares of the radii of gyration in the k power are k times the the squares of the corresponding radii in the first power. In that demonstration the exponents m, n, p and q may be as large as we please, even to infinity, and the units of measure Δx and Δy as small as we please, or infinitesimals. Whatever the direction of the axes may be, the polynomial can always be put in the form of a rectangle with sides parallel

to the axes, by adding a sufficient number of terms having zero coefficients, as we have already noticed.

It is a known mechanical property of the free axes, that the radius of gyration is a maximum about one free axis and a minimum about the other. If then a given polynomial is raised to successive powers, making $k = 2$, $k = 3$, $k = 4$, &c., and the coefficients in these expansions are all referred to the same axes passing through the centre of gravity of the coefficients in the first power, the radii of gyration in the k power are $\sqrt{k}$ times what they are in the first power, for any given position of the axes, so that they are still a maximum when taken about one of the free axes of the first power, and a minimum abeut the other. Hence, the free axes retain a constant position, in all the successive powers.

If we now write

$$h_1^2 = \frac{1}{2kr_1^2}, \qquad h_2^2 = \frac{1}{2kr_2^2}, \tag{38}$$

the equations (36) become

$$\frac{d_x z}{z} = -2h_1^2 x\,dx, \qquad \frac{d_y z}{z} = -2h_2^2 y\,dy. \tag{39}$$

Their integral is

$$z = ce^{-(h_1^2x^2+h_2^2y^2)}. \tag{40}$$

They are readily derived from it by differentiating it with respect to x and y and dividing by z. To find the value of c, we consider that as the sum of all the coefficients λ in the first power of the polynomial is unity, the sum of all the coefficients z in its expansion to the k power must be unity. Hence arises the condition

$$\frac{1}{dx\,dy}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\,dx\,dy = 1, \tag{41}$$

which gives

$$\frac{c}{dx\,dy}\int_{-\infty}^{+\infty} e^{-h_1^2x^2}\,dx \int_{-\infty}^{+\infty} e^{-h_2^2y^2}\,dy = 1.$$

The known values of the two definite integrals are $\sqrt{\pi} \div h_1$ and $\sqrt{\pi} \div h_2$, so that we get

$$\frac{c}{dx\,dy}\left(\frac{\pi}{h_1 h_2}\right) = 1, \qquad \therefore c = \frac{h_1 h_2\,dx\,dy}{\pi}, \tag{42}$$

and the complete equation of the limiting surface stands

$$z = \frac{h_1 h_2\,dx\,dy}{\pi}\,e^{-(h_1^2x^2+h_2^2y^2)}. \tag{43}$$

[To be continued.]

MECHANICS BY QUATERNIONS.

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[Continued from page 24.]

THUS the distance of the center of gravity along i is equal to $\frac{3}{8}$ of the projection of εt_2 on i, along j to $3 \div 4\pi$ times the projection of εt_2 on the plane of j and k, while the distance along k is independent of t_2.

For the c. g. of the *surface* we have as before

$$\frac{1-u^4}{1-u^3} = \frac{1-u+u^2+u^3}{1+u+u^2} = \frac{4}{3},$$

where $u = 1$; which value is to be substituted in the above equations.

The volume whose c. g. is given by the general equation above is bounded by two similar hyperboloids, two circular cones having their vertices at the origin and for directrices the two circles

$$_{\prime}\rho = i^{\theta \div \pi}(bj + t_1\varepsilon)i^{-\theta \div \pi} \text{ and } {}_{\prime}\rho = i^{\theta \div \pi}(bj + t_2\varepsilon)i^{-\theta \div \pi},$$

and two planes passing respectively through the origin and the generatrix in its initial position, and in its position after turning thro' the angle θ_1.

(31) *Friction.*—If two bodies, being in contact, have motion relatively to each other a resistance to this relative motion is developed which is called *friction*. If one body rolls on the other the resistance is called *rolling* friction, if it slides, the resistance is called *sliding* friction. The latter only will be here considered.

Sliding friction is generally assumed to be,

(*a*). Directly proportional to the normal pressure between the bodies:

(*b*). Independent of the area of the surface of contact:

(*c*). Independent of the relative velocity of the two bodies.

These laws were originally stated as the results of experiment, but later experiment has shown that the last, at least, is not strictly true. Within reasonable limits however they are probably near enough to the truth for practical purposes. The friction we have to consider here is that called into play when a body is just at the point of beginning to move upon another; this may be slightly greater than the friction after motion has been established.

First consider a particle resting against a rough surface in such a way that the forces acting on it press it against the surface.

Let ${}_{\prime}\nu$ be a vector along the normal to the surface, ${}_{\prime}R$ the resultant of all the forces acting on the particle except friction and the normal pressure of the surface against it, ${}_{\prime}N$ this normal pressure, and ${}_{\prime}F$ the friction called

into play. Then, as in equation (5), we shall have for equilibrium

$$V_{,}\nu(_{,}R+_{,}F+_{,}N) = V_{,}\nu(_{,}R+_{,}F) = 0, \quad (73)$$

the term $V_{,}\nu_{,}N$ being dropped because equal to zero. Also since $_{,}F$ acts tangentially to the surface, we must hve $S_{,}\nu_{,}F = 0$, therefore by (73)

$$_{,}\nu_{,}F = S_{,}\nu_{,}F + V_{,}\nu_{,}F = -V_{,}\nu_{,}R, \text{ or } _{,}F = -_{,}\nu^{-1} V_{,}\nu_{,}R. \quad (74)$$

Equation (74) expresses the friction actually called into play so long as the particle does not move. By the first law given above the greatest am't of friction that can be called into play is a multiple of the normal pressure, i. e., $F = \mu N$, in which μ is a numerical multiplier determined experimentally and called the coefficient of friction.

The normal pressure must be equal and opposite to the component of $_{,}R$ along $_{,}\nu$, i. e., $_{,}N = -_{,}\nu^{-1}S_{,}\nu_{,}R$; therefore

$$F = \mu N = \mu T_{,}N = \mp \mu S_{,}\nu_{,}R.T_{,}\nu^{-1},$$

in which the upper or lower sign is taken according as the angle between $_{,}\nu$ and $_{,}R$ is acute or obtuse. Comparing this value of F with (74) we have

$$TV_{,}\nu_{,}R \pm \mu S_{,}\nu_{,}R = 0, \quad (75)$$

as the limiting condition of equilibrium when the possible friction is all called into play, and the particle is at the point of starting.

The particle will remain at rest so long as $TV_{,}\nu_{,}R$ is not numerically greater than $\mu S_{,}\nu_{,}R$. If $_{,}R$ be regarded as variable and $_{,}\nu$ constant eq'n (75) becomes the equation of a cone of revolution about $_{,}\nu$ as axis, within, or on the surface of which $_{,}R$ must remain if the particle is not to move.

If the surface be a plane inclined at the angle a at which the particle is at the point of starting; if $\varepsilon = U_{,}\nu =$ vector perpendicular to the plane, drawn so as to make an acute angle with $_{,}W$, the weight of the particle, and $_{,}R = {}_{,}W$; then by (75) $TV\varepsilon_{,}W + \mu S\varepsilon_{,}W = 0$, or

$$\mu = -\frac{TV\varepsilon_{,}W}{S\varepsilon_{,}W} = \tan a. \quad (76)$$

a is called the angle of friction.

Particle Constrained to Remain on a Rough Curve.—We have by Equation (4), since $S_{,}N_{,}\varphi'(t) = 0$,

$$S(_{,}R+_{,}F)_{,}\psi'(t) = 0. \quad (77)$$

Hence, as $V_{,}F_{,}\varphi'(t) = 0$, we have $_{,}F_{,}\varphi'(t) = -S_{,}R_{,}\varphi'(t)$; therefore

$$_{,}F = -[_{,}\varphi'(t)]^{-1}S_{,}R_{,}\varphi'(t): \text{ also } _{,}N = -[_{,}\varphi'(t)]^{-1} V_{,}\varphi'(t)_{,}R.$$

$$\therefore \; F = \mu N = \mu TV_{,}\varphi'(t)_{,}R.T[_{,}\varphi'(t)]^{-1} = \mp S_{,}R_{,}\varphi'(t).T[_{,}\varphi'(t)]^{-1},$$

in the limiting case, when F has its greatest value. The signs are to be taken, as before, so as to make the right hand member positive. We have thus for equilibrium.

$$\mu TV_{,}R_{,}\varphi'(t) \pm S_{,}R_{,}\varphi'(t) > \text{ or } = 0. \quad (78)$$

This shows that, if the particle is to remain at rest, $_{,}R$ must be outside the circular cone whose equation is

$$\mu TV_{,}\rho_{,}\varphi'(t) \pm S_{,}\rho_{,}\varphi'(t) = 0,$$

in which t is taken constant, and $_{,}\varphi'(t)$ is a vector along the axis, or else on the surface of this cone.

(32). *Equilibrium of Inextensible Flexible Strings.*—By a flexible string will be understood one which offers no resistance whatever to bending.

The cross-section of the strings treated will be regarded as very small in comparison with their length. Let $_{,}P$ be the pull or tension at any point of the string. It will evidently act tangentially to the string, so that

$$U_{,}P = Ud_{,}\rho = \frac{d_{,}\rho}{Td_{,}\rho} = \frac{d_{,}\rho}{ds} = {}_{,}\rho'.$$

In treating this subject we shall use *primes* to indicate differentiation with respect to s, the length of any portion of the string, as above; thus $_{,}P' = d_{,}P \div ds$, etc. $_{,}P'$ is then the rate of variation of $_{,}P$ as we go along the string, and, if this be multiplied by ∂s, we have the total variation of $_{,}P$ in the infinitesimal distance ∂s.

In order that the element ∂s may be in equilibrium the sum of all the forces acting on it must be zero. Let $_{,}R\partial s$ be the resultant of all the exterior forces acting on the element: then the other forces are $_{,}P$ at one end, and $-{}_{,}P + {}_{,}P'\partial s$ at the other; therefore

$$_{,}P + {}_{,}R\partial s - {}_{,}P + {}_{,}P'\partial s = 0, \text{ or } {}_{,}P' + {}_{,}R = 0. \tag{79}$$

Whence
$$_{,}P = -\int_{,}Rds. \tag{80}$$

As we have seen above that $U_{,}P = {}_{,}\rho'$, we have

$$_{,}P' = \frac{d}{ds}\Big(P_{,}\rho'\Big) = P'_{,}\rho' + P_{,}\rho'' = -{}_{,}R. \tag{81}$$

If $_{,}r$ is the vector radius of curvature of the string at the end of $_{,}\rho$, we have (Tait's Quat., Art. 283)

$$_{,}\rho'' = -{}_{,}r^{-1}; \therefore P'_{,}\rho' - P_{,}r^{-1} = -{}_{,}R. \tag{82}$$

If P be constant, so that $P' = 0$, this equation shows that $_{,}R$ acts along the principal normal to the curve, and that R varies as $1 \div r$. Operate on (81) by $S.V_{,}R_{,}\rho'$;

$$\therefore S_{,}R_{,}\rho'_{,}\rho'' = 0, \tag{83}$$

which shows that $_{,}R$ always lies on the osculating plane of the curve.

Operate on (81) by $S._{,}\rho'$; then, since $T_{,}\rho' = 1$ and $S_{,}\rho'_{,}\rho'' = 0$ (Tait's Quat., Art. 282), we have

$$P' = S_{,}R_{,}\rho', \text{ or } P = \int S_{,}Rd_{,}\rho. \tag{84}$$

Next operate on (81) by $S._{,}\rho''$; $\therefore P_{,}\rho''^{2} = -S_{,}R_{,}\rho''$, or

$$P = -S_{,}R_{,}\rho''^{-1} = + S_{,}R_{,}r. \tag{85}$$

Comparing (84) with (85) we have

$$S_{,}R_{,}\rho''^{-1} = -\int S_{,}Rd_{,}\rho. \tag{86}$$

This being a relation between $_{,}\rho$ and $_{,}R$ independent of $_{,}P$ is the equation of the curve of the string.

Suppose as a particular case that the string lies in the plane of i and j, and that $_{,}R$ is parallel to j; so that $Sk_{,}\rho = Sk_{,}\rho' = Sk_{,}R = 0$, and $Si_{,}R = 0$, or $_{,}R = \pm Rj$. Then by (79)

$$Si_{,}P' = 0;\ \therefore Si_{,}P = PSi_{,}\rho' = \text{constant} = -H, \text{ say},$$

$$\therefore P = -H \div Si_{,}\rho'. \tag{87}$$

It appears therefore that when $_{,}R$ is parallel to j the horizontal component of the tension is always constant. Again by (79)

$$Sj_{,}P' = Sj\frac{d}{ds}(P_{,}\rho') = \frac{d}{ds}(PSj_{,}\rho') = -H\frac{d}{ds}\left(\frac{Sj_{,}\rho'}{Si_{,}\rho'}\right) = -Sj_{,}R = \pm R. \tag{88}$$

The upper or lower sign of the last member is to be taken according as $_{,}R$ acts upward or downward.

As another special case, let the force act from or towards a fixed point. If the origin be taken at this point we shall have $_{,}R = \pm RU_{,}\rho$, according as the force acts from or towards the origin.

By (83) $S_{,}\rho_{,}\rho'_{,}\rho'' = 0$, which shows that the string lies in a plane thro' the origin. By (84) and (85)

$$P = \pm\int RSU_{,}\rho.d_{,}\rho = \mp\int RSd_{,}\rho.U_{,}\rho^{-1} = \mp\int RdT_{,}\rho = \mp\int Rd\rho$$

$$= \mp RSU_{,}\rho._{,}\rho''^{-1}. \tag{89}$$

This is the equation of the curve of the string. We may however derive the equation in a form not involving $_{,}\rho''$. We have by (79)

$$_{,}P' = -_{,}R = \mp RU_{,}\rho;\ \therefore V_{,}\rho_{,}P' = 0.$$

Now $V_{,}\rho'_{,}P = PV_{,}\rho'_{,}\rho' = 0$; $\therefore$ adding and integrating $V_{,}\rho_{,}P = _{,}c$;

$$\therefore TV_{,}\rho_{,}P = PTV_{,}\rho_{,}\rho' = c,$$

whence by comparison with (89)

$$P = c(TV_{,}\rho_{,}\rho')^{-1} = \mp\int Rd\rho. \tag{90}$$

This equation gives by integration the curve of the string, and will give the same curve whether the upper or lower sign of the right hand member be taken. The sign ought to be so taken however as to make the integral $\int Rd\rho$ positive.

(33). Let us now apply our general equation to the case of a string stretched over a curve. We will first suppose the curve smooth and the string without weight. On account of the smoothness of the curve the only

action possible between it and the string will be in the direction of the principal normal to the curve at the point considered. Call this action—NU,r, minus because $,r$ is directed inwards. Then

$$,R = -NU,r = -NU,\rho'',$$

and by (81)

$$P',\rho' + P,\rho'' = NU,\rho'', \text{ or } P',\rho' = (N - Pr^{-1})U,\rho''.$$

This equation can only be satisfied by equating to zero separately the coefficients of $,\rho'$ and U,ρ''. Thus we have

$$P = \text{constant, and } N = Pr^{-1}. \tag{91}$$

The total pressure on the curve will be

$$\int N ds = \int Pr^{-1} ds. \tag{92}$$

If we consider the weight of the string, we shall have $,R = -NU,\rho'' - Wj$, if j be a vertical unit vector and W be the weight of a unit of length of the string. By (84) $P' = S,R,\rho' = -WSj,\rho'$; and by (85) $P = -S,R,\rho''^{-1} = S(NU,\rho'' + Wj),\rho''^{-1} = rN - WSj,r$: whence

$$P = rN - WSj,r = -W\int Sjd,\rho. \tag{93}$$

As the equation of the curve is known $,r$ and d,ρ are known, and therefore P and N may be found by (93).

Finally, suppose the curve to be rough, then on each element δ,ρ of the string there will act a force of friction equal to $-\mu NUd,\rho = -\mu N,\rho'$; so that $,R = -NU,\rho'' - Wj - \mu N,\rho'$. Hence by (84)

$$P' = S,R,\rho' = -WSj,\rho' + \mu N. \tag{94}$$

By (85) $$P = S,R,r = -WSj,r + rN. \tag{95}$$

Eliminating N between (94) and (95) we have

$$P' - \frac{\mu}{r}P = WSj(\mu U,\rho'' - ,\rho'), \tag{96}$$

a differential equation for determining P. These equations hold for tortuous as well as plane curves.

We will next consider the case of a string stretched over a smooth surface. The action between the string and surface must be normal to the surface; therefore let $-NU,\nu$ be this normal pressure, and let $,Q$ be the resultant of all the other forces acting on the string per unit of length.

Then $,R = ,Q - NU,\nu$, which in (86) gives

$$S(,Q - NU,\nu),\rho''^{-1} = -\int S,Qd,\rho.$$

But by (83) $$S,\rho',\rho'',R = 0 = S,\rho',\rho''(,Q - NU,\nu);$$

$$\therefore N = \frac{S,\rho',\rho'',Q}{S,\rho',\rho''U,\nu}.$$

Substituting this value of N in the previous equation and reducing we have

$$S\left({}_{,}Q+{}_{,}\rho''\int S_{,}Qd_{,}\rho\right){}_{,}\rho'{}_{,}\upsilon=0, \tag{97}$$

a differential equation for determining the curve. If ${}_{,}Q$ is normal to the surface and $= C_{,}\upsilon$, say, then $S_{,}Q_{,}\rho'{}_{,}\upsilon = 0$, and therefore $S_{,}\rho'{}_{,}\rho''{}_{,}\upsilon = 0$, so that the normal to the surface lies in the osculatory plane of the curve, a property of the shortest line that can be drawn on the surface between two points of it.

(34). *Extensible Strings.*—Under the action of a force, small compared with that which will break it, a string is found to be stretched to an amount very approximately proprtional to the applied force. Thus if s is the original length of a straight string and the forces P_1 and P_2, applied at one end in the direction of its length, stretch it to the lengths s_1 and s_2 respectively, the above law is expressed by the equation

$$\frac{s_1-s}{s_2-s}=\frac{P_1}{P_2}.$$

If this law be supposed to hold for forces of *any* magnitude, and P_2 be so taken as to stretch s to double its original length, so that $s_2 = 2s$, then P_2 is called the *modulus of elasticity*, and is usually represented by E. If we call the stretched length of the string σ, and drop the suffix of the P_1, our formula then becomes

$$\sigma-s=sPE^{-1}. \tag{98}$$

All the equations previously found apply equally well to extensible st'gs if we consider them after the extension has occurred. Thus (79) becomes

$$\frac{d_{,}P}{d\sigma}=-{}_{,}R_1, \tag{99}$$

in which ${}_{,}R_1$ is written instead of ${}_{,}R$ because the mass of a unit of length of the string is changed by the stretching. To express ${}_{,}R_1$ in terms of the mass before stretching, let ${}_{,}R = m_{,}q$, and ${}_{,}R_1 = m_{1,}q$, and let δs and $\delta\sigma$ be the original and stretched lengths of an element of the string. Then $m\delta s = m_1\delta\sigma$, and by (98) $\delta\sigma = \delta s(1 + PE^{-1})$; $\therefore\ m = m_1(1+PE^{-1})$, and ${}_{,}R_1 = m_{,}q(1+PE^{-1})^{-1}$, whence

$$\frac{d_{,}P}{d\sigma}=-m_{,}qE(P+E)^{-1}. \tag{100}$$

(35). *Examples.*—1st. Common Catenary. This being the curve of a uniform string hanging freely and acted on by its own weight only, we shall have ${}_{,}R = -wj$, in which w is the weight per unit of length.

By (79) ${}_{,}P' = wj$; $\quad\therefore\ {}_{,}P = wjs + Hi$, (101)

in which Hi is the constant of integration, being the value of ${}_{\prime}P$ when $s=0$ if s be measured from the lowest point of the string. Taking tensors

$$P = \sqrt{(w^2s^2 + H^2)}.$$

By (84) $\qquad P = -w\int Sjd_{\prime}\rho = -wSj_{\prime}\rho + H,$

in which H is the value of P already found when $\rho = 0$. Put $H = hw$, so that H is measured in terms of the weight per unit of length of the string, and equate the values of P just found; hence

$$s^2 = S^2j\rho - 2hSj\rho. \qquad (102)$$

By (88) $R = w = hw\dfrac{d}{ds}\left(\dfrac{Sj\rho'}{Si\rho'}\right) = hw\dfrac{d}{ds}\left(\tan\varphi\right)$, if φ is the angle between the tangent to the curve and the vector i. Hence $\tan\varphi = s \div h$, no constant being required if $s = 0$ when $\varphi = 0$. This is the *intrinsic* eq. of the catenary. We have also

$$\frac{d(\tan\varphi)}{ds} = \frac{d(\tan\varphi)}{ds}\cdot\frac{Sid\rho}{Sid\rho} = \frac{d(\tan\varphi)}{Sid\rho}\cdot Si\rho' = -\frac{\sec^2\varphi d\varphi}{Sid\rho}\cos\varphi.$$

Hence $-h\sec\varphi d\varphi = Sid_{\prime}\rho$, and integrating,

$$-Si\rho = h\log(\sec\varphi + \tan\varphi) = h\log\left[(s\div h) + \sqrt{1+(s\div h)^2}\,\right]. \quad (103)$$

The constant will be zero if $Si\rho = 0$, when $\varphi = 0$.

If in (102) the origin be taken at a distance h below the lowest point of the curve instead of at that point, the equation becomes

$$s^2 = -h^2 + S^2j\rho. \qquad (104)$$

If s be eliminated between (103) and (104) we have

$$-Sj\rho = \tfrac{1}{2}h[e^{h^{-1}Si_{\prime}\rho} + e^{-h^{-1}Si_{\prime}\rho}]. \qquad (105)$$

2nd. Let the string be such that portions whose horizontal projections are equal have equal weights. Then ${}_{\prime}R = CjSi_{\prime}\rho'$, and by (88)

$$H\frac{d}{ds}\left(\frac{Sj\rho'}{Si\rho'}\right) = -CSi\rho',$$

whence

$$HSj\rho' = -CSi\rho Si\rho',$$

and

$$2HSj\rho = -CS^2i_{\prime}\rho. \qquad (106)$$

There will be no constant in either integration if the origin be taken at the lowest point of the string. The equation is that of a parabola, on the condition presupposed that $Sk_{\prime}\rho = 0$. Without this condition it would be that of a parabolic cylinder.

Also by (79) $\qquad {}_{\prime}P' = -{}_{\prime}R = -CjSi_{\prime}\rho';$

$$\therefore\ {}_{\prime}P = -CjSi_{\prime}\rho + Hi,$$

in which, as before, Hi is the value of ${}_{\prime}P$ at the lowest point.

THE MOONS OF MARS, AND THE NEBULAR HYPOTHESIS.

BY PLINY EARLE CHASE, LL. D.

IN the January number of the ANALYST, Professor Kirkwood says: "Since, therefore, the satellite could never have existed at its present distance in a nebular state, it must follow, if any form of the nebular hypothesis is to be accepted, that its original distance was much greater than the present."

Similar statements have often been made by others, and, as they indicate an apparent misconception of some of the activities of elastic media, it may be well to inquire whether they are strictly correct.

Sir John Herschel states his father's hypothesis as follows. "Admitting the existence of such a medium, dispersed in some cases irregularly through vast regions in space, in others confined to narrower and more definite limits, Sir William Herschel was led to speculate on its gradual subsidence and condensation by the effect of its own gravity, into more or less regular spherical or spheroidal forms, denser (as they must in that case be) towards the centre. Assuming that in the progress of this subsidence, local centres of condensation, subordinate to the general tendency, would not be wanting, he conceived that in this way solid nuclei might arise, whose local gravitation still further condensing, and so absorbing the nebulous matter, each in its immediate neighborhood, might ultimately become stars, and the whole nebulæ finally take on the state of a cluster of stars. . . . Even though we should feel ourselves compelled to reject the idea of a gaseous or vaporous 'nebulous matter', it looses little or none of its force. Subsidence, and the central aggregation consequent on subsidence, may go on quite as well am'g a multitude of discrete bodies under the influence of mutual attraction, and feeble or partially opposing projectile motions, as among the particles of a gaseous fluid." (Outlines of Astronomy, Sect. 871.)

Soon after his discovery, Prof. Asaph Hall sent me the query: "Will the inner moon of Mars fall into harmony or will it make a discord?"

I believe that the tendency of inertia, in all elastic media, to form harmonic nodes, is so great that the evidence of such nodes must be discoverable, wherever we look, both in cosmical and in molecular phenomena. As a necessary consequence of such subsidence as Herschel supposed, there must be an acceleration of velocity in all the nebular particles, the acceleration being more rapid in the nucleus than near the outer surface of the nebula.

Many indications point to the simultaneous, or nearly simultaneous initiation of numerous planetary centres, and it is very doubtful if either of

the two-planet belts, except, perhaps that of Neptune and Uranus, will be long regarded as having been "thrown off" by the mere increase of centrifugal velocity. But if the separation is owing to the velocity acquired by subsidence, a satellite may have any velocity which can be maintained by stable harmonic action.

One of the simplest sources of harmony is the division of a linear pendulum into three equal parts, and the synchronism of vibration, whether the suspension is made at the extremity, or at either of the dividing points. In the case of Mars, if we start from a point near the theoretical beginning of nebular condensation for the outer satellite (see Phil. Mag. for Oct. 1877, page 292), and take a series of harmonic divisors of the form $d_{n+1} = 3d_n - d_1$, we find the following accordances:

Divisors.	Quotients.	Observed.
13.7 $\quad d_1 = 1$	13.700	13.692 = Nebular radius.
$d_2 = 3d_1 - d_1 = 2$	6.850	6.846 = Deimus.
$d_3 = 3d_2 - d_1 = 5$	2.740	2.730 = Phobus.
$d_4 = 3d_3 - d_1 = 14$	.979	1.000 = Semidiameter.
$d_5 = 3d_4 - d_1 = 41$	.334	.333 = Centre of radial occil'n,
$d_6 = 3d_5 - d_1 = 122$		120.560 = Moon's major axis.
$d_7 = 3d_6 - d_1 = 365$		365.256 = Terrestrial year.

The reason for the accordance of the terrestrial and lunar harmonies with those of the Martial system, may be understood by considering that the Earth is midway between the secular perihelion of Mercury and the secular aphelion of Mars, and is, therefore, the centre of the belt of greatest condensation in the solar system.

Note on Prof. A. Hall's Query in Vol. VII, No. 4, by Prof. H. T. Eddy.—The eight values which $\nabla^2 v$ assumes at the surface of an attracting body, when v is the potential due to the attraction of the body are explained in sufficient detail in Mr. Todhunter's History of the Theory of Attraction and Figure of the Earth. In chapter XXXI, which he devotes entirely to the consideration of this function of v, he discusses the erroneous result $-2\pi\rho$ obtained by Poisson, as well as the erroneous results obtained by Ostragradsky at singular points of the surface.

I object to the statement made by Mr. Stone in answer to this query in Vol. VII, No. 5, where he states that these eight values are obtained on the supposition that the second differential coefficients of v with respect to x, y, z are independent. The word *independent* does not correctly convey

the idea. Each of the three differential coefficients of v has two ***distinct*** values, the one derived from v_1, the potential outside the body, the other from v_2, the potential inside the body. According as we select one or another set of values to make up $\nabla^2 v$ we arrive at one or other of eight distinct values, six of which have no physical significance, and the remaining two, 0 and $-4\pi\rho$ are said to hold at a point on the surface, in the sense that however near the point be to the surface on the outside the vulue is 0, while however near the point be to the surface on the inside the value is $-4\pi\rho$.

ANSWER TO QUERY AT P. 16 BY H. HEATON, PERRY, IOWA.—Put $v=$ the volum, and $w=$ weight of the portion of the stack above any stratum. Then the volume and weight of the stratum will be represented, respectively, by dv and dw. Let n be the ratio of weight to density, or density $= nw$. Then $dw = nwdv$, or $dw \div w = ndv$. Integrating, $\log w = nv + c$. When $v = 0$, $w = 0$ and $\log w = -\infty$; $\therefore$ $-\infty = c$. Whence it appears that unless we make an additional assumption the division cannot be made.

If we assume that a *weight*, b, however small, is placed on top of the stack, then when $v=0$ $w=b$, therefore $c = \log b$, and $\log(w \div b) = nv$, $w = be^{nv}$.

If W and V be the weight and volume of the stack, $W = b(e^{nv} - 1)$.

If v_1 and v_2 are respectively $\frac{1}{3}$ and $\frac{2}{3}$ the volume of the stack,

$$\tfrac{1}{3}b(e^{nv} - 1) = b(e^{nv_1} - 1), \text{ and } \tfrac{2}{3}b(e^{nv} - 1) = b(e^{nv_2} - 1);$$

$$\therefore\ v_1 = \log[\tfrac{1}{3}(2 + e^{nv})] \div n \text{ and } v_2 = \log[\tfrac{1}{3}(1 + 2e^{nv})] \div n.$$

[Professor De Volson Wood answered this query in a similar manner.]

NOTE ON "REPLY TO CRITICISMS." (SEE P. 10).—Mr. Christie dissents to the conclusions arrived at by Prof. Wood, and insists that his criticisms are in all respects valid and un-refuted by the Reply. We think it best, however, not to prolong the discussion in the ANALYST, as such of our readers as are interested in the subject can, doubtless, draw satisfactory conclusions from what has been said.—Editor.

SOLUTION OF PROB. 331 BY W. E. HEAL.—Denote the given curve by BCD. Form its reciprocal with the point O as origin, and denote this reciprocal by RST. Let s be a multiple point of the curve RST. Then the point p in which OS meets BCD will be a point of contact of a mult. tang. of BCD. Now, the Hessian of RST passes thro' all mult. points of RST.

Therefore the reciprocal, with respect to O, of this Hessian passes thro' all the points of contact of the multiple tangents of BCD. In like manner it is seen that it also passes thro' the p'ts of inflexion of BCD. *Q. E. D.*

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in number one have been received as follows:—

From Prof. L. G. Barbour, 335, 337; Prof. W. P. Casey, 332, 333; G. M. Day, 335; Geo. Eastwood, 336; Wm. Hoover, 332, 333, 335; A. Hall (son of Prof. A. Hall), 335; H. Heaton, 333; W. E. Heal, 332, 333, 334, 335; Prof. J. H. Kershner, 334; Prof. D. J. Mc Adam, 335, 336; Prof. J. Scheffer, 333, 335; Prof. E. B. Seitz, 333, 334, 335.

332. "Find all the values of x and y in the following equations:

$$x + xy^3 = 18, \qquad (1)$$
$$xy + xy^2 = 12." \qquad (2)$$

SOLUTION BY W. E. HEAL MARION, INDIANA.

Multiply the first equation by 2, the second by 3, subtract and divide by x and we have

$$\begin{aligned} 2(y^3+1)-3y(y+1) &= 2(y+1)(y^2-y+1)-3y(y+1) \\ &= (y+1)(2y^2-5y+1) = 0. \end{aligned}$$

Therefore $y+1=0$ and $2y^2-5y+1=0$; therefore

$$y = -1,\ 2,\ \tfrac{1}{2};$$
$$x = \infty,\ 2,\ 16.$$

333. "Find the value to x terms of the continued fraction

$$\cfrac{2}{1+\cfrac{2}{1+\&c.}}"$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $u_x \div v_x$, $u_{x+1} \div v_{x+1}$, $u_{x+2} \div v_{x+2}$ be the xth, $(x+1)$th, $(x+2)$th converging fractions. Then

$$\frac{u_{x+1}}{v_{x+1}} = 2 \div \left(1+\frac{u_x}{v_x}\right) = \frac{2v_x}{u_x+v_x},$$

whence $u_{x+1} = 2v_x$ (1), $v_{x+1} = u_x+v_x$; and simil'y $v_{x+2} = u_{x+1}+v_{x+1}$. (2)

From (1) and (2) we find $v_{x+2} = v_{x+1} + 2v_x$, an equation in finite differences whose solution gives $v_x = C_1(2)^x + C_2(-1)^x$. (3)

When $x=1$, $v_1 = 2C_1 - C_2 = 1$, and when $x=2$, $v_2 = 4C_1 + C_2 = 3$; whence $C_1 = \frac{2}{3}$, and $C_2 = \frac{1}{3}$; $\therefore\ v_x = \frac{1}{3}(2^{x+1} \pm 1)$, $u_x = 2v_{x-1} = \frac{1}{3}(2^{x+1} \mp 2)$,

and
$$\frac{u_x}{v_x} = \frac{2^{x+1} \mp 2}{2^{x+1} \pm 1},$$

the upper sign being taken when x is even, and the lower when x is odd.

334. **"Pairs of tangents which meet always at the same angle are drawn to a given ellipse. Find the envelope of the chords of contact."**

SOLUTION BY W. E. HEAL.

Let $(x^2 \div a^2)+(y^2 \div b^2) = 1$ be the equation of the given ellipse; $(a \cos \theta, b \sin \theta)$, $(a \cos \varphi, b \sin \varphi)$ the coordinates of the points of contact; α the constant angle at which the tangents intersect.

Then the equations of the tangents and chord of contact are

$$(x \div a) \cos \theta+(y \div b) \sin \theta = 1, \tag{1}$$

$$(x \div a) \cos \varphi+(y \div b) \sin \varphi = 1, \tag{2}$$

$$(x \div a) \cos \tfrac{1}{2}(\varphi+\theta)+(y \div b) \sin \tfrac{1}{2}(\varphi+\theta) = \cos \tfrac{1}{2}(\varphi-\theta). \tag{3}$$

By a well known formula the angle between the tangents is

$$\tan \alpha = \frac{\sin \theta \cos \varphi - \cos \theta \sin \varphi}{\sin \theta \sin \varphi + \cos \theta \cos \varphi} = \tan (\theta - \varphi). \tag{4}$$

$$\therefore \alpha = n\pi + (\theta - \varphi). \tag{5}$$

$$\therefore \varphi = (n\pi - \alpha) + \theta = 2[\tfrac{1}{2}(n\pi - \alpha)]+\theta = 2\beta+\theta. \tag{6}$$

Substituting in (3) we find for the equation of the chord of contact

$$(x \div a) \cos (\theta+\beta)+(y \div b) \sin (\theta+\beta) = \cos \beta. \tag{7}$$

Equating to zero the differential of (7) with respect to θ we have

$$(y \div b) \cos (\theta+\beta)-(x \div a) \sin (\theta+\beta) = 0. \tag{8}$$

The sum of the squres of (7) and (8) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \beta. \tag{9}$$

Equation (9) is the required envelope which is, therefore, an ellipse.

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335. "The curve whose rectangular equation is $x^{\frac{1}{2}} + y^{\frac{1}{2}} = r^{\frac{1}{2}}$ revolves around the axis of x. Determine the volume of the solid thus described between the limits $x = 0$ and $x = r$."

SOLNTION BY A. HALL, HARVARD COLLEGE.

The differential of the volume is

$$dV = \pi y^2 \, dx;$$

and, substituting the value of y^2 from the given equation and integrating, we have

$$V = \pi\left(r^2 x+\frac{x^3}{3}+3rx^2 -\frac{8}{3}r^{\frac{3}{2}}x^{\frac{3}{2}}-\frac{8}{5}r^{\frac{1}{2}}x^{\frac{5}{2}}\right).$$

Taking the limits and reducing,

$$V = \frac{\pi r^3}{15}.$$

336. "In a locomotive engine there are given: The impressed force of the steam on the piston, the radius of the crank, and the length of the connecting rod: To find the uniform force which, if applied at right angles to the end of the crank, would do the same work as the impressed force."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.

Let F = the area of the piston; S = distance traveled by piston before expansion begins; that is, before steam is cut off; r = length of crank; S_1 = entire length of stroke = $2r$. And let p = steam pressure per unit of area before expansion, expressed in atmospheres; p_1 = steam pressure per square unit of area after expansion; q = back pressure per unit, P = uniform pressure on end of crank which we are seeking.

Then, assuming expansion according to Mariotte's Law, the complete work of the steam per stroke (Weisbach, 2nd vol., Part second, p. 366) is

$$A = FS_1p_1\left[1+\log\left(\frac{S_1}{S}\right)-\frac{q}{p_1}\right] = S_1P_1, \text{ say,}$$

in which $p_1 = pS \div S_1$ and $q = 1$ for this non-condensing engine.

Now, by principle of virtual moments $P\pi r = S_1P_1 = 2rP_1$. Therefore $P = 2P_1 \div \pi$. That is, the uniform force is $2 \div \pi$ of the average pressure on the piston.

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337. "Required the constant quantity into which if we divide the periodic time of any planet, multiplied by its third root, the quotient will be the distance such planet falls from a tangent to its orbit in one second of time: i. e., solve the equation,

$$\frac{\text{Constant quantity}}{(\text{Periodic time})^{\frac{4}{3}}} = \text{Fall from tang.}$$

SOLUTION BY PROF. L. G. BARBOUR, RICHMOND, KY.

Let s = space any planet falls through in 1 second; $F = 2s$ = force of gravity, t being 1 second; T = number of seconds in periodic time of planet; R = radius vector; and let a = semi-major axis of elliptic orbit, then is mean value of $R = a$; and we have

$$s = \tfrac{1}{2}F = \frac{2\pi^2a^3}{T^2R^2} = \frac{2\pi^2a}{T^2} \text{ for mean value of } R.$$

Let $T' = 1$ second, then, by Kepler's third law, $T'^2 : T^2 :: a'^3 : a^3$, or

$$1 : T^2 :: a'^3 : a^3; \;\therefore\; T^2 = \frac{a^3}{a'^3};\; T^{\frac{4}{3}} = \frac{a^2}{a'^2}. \;\;\therefore\; sT^{\frac{4}{3}} = 2\pi^2a',$$

where a' = mean distance of a body revolving about the sun's center in one second, supposing the mass of the sun to be concentrated at its center.

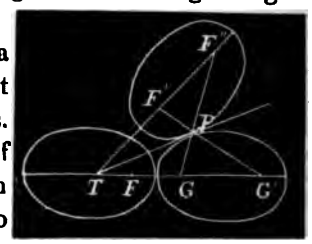
QUERY BY PROF. J. SCHEFFER.—"If of any curve we find the evolute, and of the latter the evolute, and so on ad infin., the ultimate evolute is a cycloid. How is this proved?"

ANSWER BY PROFESSOR KERSHNER.

The evolute of a cycloid is an equal cycloid (Todhunter's Integral Calc. 5th ed., art. 114) composed of two equal parts which are not concurrent (Tod. Diff. Calc., art. 359), but which radiate in curvilinear rays, from each extremity of the original cycloid as two centers, forever, the maximum dist. of two succeeding extremities of the rays being $2r$.

NOTE BY PROF. SCHEFFER.—*Mr. Editor:* The proposition on page 71 Vol. IV, demonstrated by you, suggests the following:—

Prop.—The locus of the focus of an ellipse or hyperbola which rolls on an equal ellipse or hyperbola is a circle, supposing that at the beginning of the motion the vertices coincide.



If F, F', F'', G, G' represent the foci, PT a tangent, FPG' and $F''PG$ are obviously strai't lines and of constant length, viz., = major axis. The locus, therefore, is a circle, the centre of which is the other focus and the radius of which is the major axis. The same reasoning applies to the hyperbola.

PROBLEMS.

739. *By Geo. H. Harvill. Colfax, La.*—Required the average distance from the center of a circle to all poiuts in the surface of a sector.

340. *By William Hoover, Wapakoneta, Ohio.*—Integrate $\frac{dx}{\sin x + \cos x}$.

141. *By William E. Heal, Marion, Ind.*—Show that "Every even number is the sum of two prime numbers, and every odd number is the sum of three prime numbers." Barlow's Theory of Numbers, page 259.

342. *By Prof. Kershner.*—Prove Schlömilch's Theorem: If D_a, D_b,... D_n are divisors of 10^k+1, so that $N_a=\frac{10^k+1}{D_a}$, $N_b=\frac{10^k+1}{D_b}$, $N_n=\frac{10^k+1}{D_n}$ the k digits or figures of the whole numbers D_a-1, D_b-1, D_n-1 are the k first figures of the circulator or period of $\frac{1}{N_a}$, $\frac{1}{N_b}$, $\frac{1}{N_n}$, respectively.

343. (*Selected*) *by Prof. H. T. Eddy.*—If E^2 be the sum of the squares of the edges of a tetrahedron, F^2 the sum of the squares of the areas of the faces and V the volume, show that the principal semi axes of the ellipsoid inscribed in the tetrahedron, touching each face in the center of gravity and having its center at the center of gravity of the tetrahedron, are the roots of

$$k^6 - \frac{E^2}{2^4.3}k^4 + \frac{F^2}{2^4.3^3}k^2 - \frac{V^2}{2^6.3} = 0.$$

344. *By Prof. E. B. Seitz.*—Through each of two points, taken at random within a circle, a random chord is drawn; find (1) the prob'y that the chords will intersect; and (2) if a third random chord be drawn thro' a 3d random p't, find the prob's that the 3 chords will intersect in 0, 1, 2, 3 p'ts.

345. *By Prof. W. W. Johnson.*—The great circle from $A\ (\varphi_1, \lambda_1)$ to $B\ (\varphi_2, \lambda_2)$ passes north of the parallel of latitude φ_0; what is the longitude, λ_0 of the point P on this parallel so that the course APB shall be the shortest course from A to B which does not pass north of this parallel?

QUERY BY H. HEATON.—Is there any known general method of elimination when we have two or more equations containing two or more unk'n quantities, the equations being of the third degree or higher?

DETERMINATION OF A MERIDIAN.

BY W. L. MARCY, U. S. DEP. MIN'L SURVEYOR, LEADVILLE, COL.

THE simplicity of any independent calculation for the azimuth of Polaris, involving nothing but a few tables of an Ephemeris or Almanac and a common case in Spherical trigonometry, should not certainly have made the operation a forbidden field to the average surveyor. It is not so from want of ability. By practice he retains his knowledge of plane trigonometry, forgetting his spherical trigonometry for the want of its application.

A surveyor of more than ordinary accuracy and skill, sent from the far East to do some special work for a mining company, surprised me by expressing the opinion that the elongation of Polaris was equal to the complement of its declination in all latitudes. Another surveyor sits up all night to watch the star to its maximam elongation—a very accurate method when the latitude is closely known, but at so inconvenient a season that few really make the trial.

The observation of the star should not take over 5 to 15 minutes, and it can be made so early in the evening or late in the morning as to dispense

with artificial light to see cross-hairs, read verniers &c. I have caught Polaris in the field of view before sunset, and have seen it after sunrise, when in the field of view. To make these observations, the position of the star with respect to the pole must be found. The following approximation to its position will answer for bringing the star into the field of view:—

Represent the hour angle by H, the azimuth by A, the vertical distance above or below the pole by d, and the complement of its declination by c; then, approximately, $A = (c . \sin H) \div \cos l$, $d = c . \cos H$. Now, having the magnetic variation, increase it by A if the star is on the opposite side of the meridian to the needle, or the reverse, and raise the telescope on the vertical arc to $l \pm d$, according as d is above or below the pole, first adjusting the focus to the most distant object in sight. If the star is not in the field of view, the magnetic meridian being variable, move the teles. slowly east and west until it is visible, then the rate and direction of motion will soon identify Polaris. The azimuth A, can be taken from the tables, by inspection, if the latitude is not too far removed. When the transit has no vertical arc a little more time and patience will generally succeed by slowly examining the sky in consecutive planes.

The principal errors in the meridian result from the inaccuracy of the time and latitude, but they would be much greater with other stars farther removed from the pole; the maximum error for one minute of time at the upper culmination of Polaris, in latitude 42° N., is 28″.7; and the max'm error for one degree of latitude is only 1′ 44″ or —1′ 38″.8, at the elongat'n.

To find the Hour Angle of Polaris.—If the observation is at evening, take the R. A. of mean sun, Greenwich mean noon for that day, but for the preceding day if the observation is in the morning; add to this the time of observation plus 12h, if in the morning, and also add the increase in R. A. corresponding to the time west from Greenwich and the time of observation, plus 12h, if in the morning. From these quantities deduct the apparent R. A. of Polaris for the day of observation if an Ephemeris is used, or the mean R. A. if an Almanac; the remainder, t^h, is the time the star is west of the upper culmination, but the hour angle will be estimated from the upper or lower culmination as follows:—

For t^h negative Polaris is East above pole, hour ang. = $t^h \times 15°$,
" $t^h < 6^h$ and $> 0^h$ Polaris is W. A. bole, $H = t^h \times 15°$,
" $t^h < 12^h$ " $> 6^h$ " " W. B. " , $H = (12^h - t^h) \times 15°$,
" $t^h < 18^h$ " $> 12^h$ " " E. B. " , $H = (t^h - 12^h) \times 15°$,
" $t^h < 24^h$ " $> 18^h$ " " E. A. " , $H = (24^h - t^h) \times 15°$,
" $t^h < 30^h$ " $> 24^h$ " " W. A. " , $H =$ &c.

Having obtained the hour angle (H) of Polaris, take from the Ephemeris, for the day of observation, the complement of the apparent declination

and we shall have, with the co-lat., two sides and the included angle of a spherical triangle to find the azimuth, or angle opposite the co-declination, or the azimuth can be taken from the table and corrected for time and latitude by columns D and V.

If mean right ascension is used, the greatest error occurs at culmination, and if mean declination is used the greatest error occurs at elongation.

These errors increase with the lat. but in lat. 42° the first will probably never exceed 45″, and the second, 60″. The following numbers are the approximate corrections to be applied to the mean co-dec. of Polaris to obtain the apparent co-declination for the year 1882.

Jan, 1st, — 25″, Feb. 1st, — 25″, March 1st, — 20″, April 1st, — 11″, May 1st — 1″, June 1st, + 5″, July 1st, + 7″, Aug. 1st, + 3″, Sept. 1st, —4″, Oct. 1st, —15″, Nov. 1st, —26″, Dec. 1st, —36″, Dec. 31st, —42″.

Compared with previous years, the plus quantities are increasing and the minus decreasing slowly, approximately at the rate of 1″ in a year.

Inaccuracy in time produces a max. error at culmination, and in latitude, at elongation, which can be interpolated from the tables, and a meridian obtained from an assumed time and latitude can be corrected. If the assumed latitude and time are revised by azimuth of sun or star from the trial meridian, a second or third correction may be necessary to converge to the required accuracy.

If from trial meridian, by azimuth or meridian passage of the sun, our time is found to be not more than 15^m fast or slow, a second trial meridian from corrected time will give the time within a few seconds and the meridian sufficiently accurate for ordinary purposes.

The formulas $N67''.8(100 \pm 2.3) \div 100$, and $N75''.5(100 \pm 2.8) \div 100$ give the azimuth correction for one deg. of lat. either north or south of latitude 39° or 42°, respectively, within one second of arc, and they can be extended to 5° of latitude with error of less than 1′. The diff. for 1° of lat., increased or diminished, as we go north or south, .05, will be the diff. for the next degree, and so on to 5° with no error exceeding 2″.

The expression $A = c\,(\sin H) \div \cos l$, is a very convenient approximation to the azimuth, the max'm error, when $H = 45°$, not exceeding, in lat. 42°, 70″. Let e be the difference between the upper and lower azimuth when $H = 45°$, then $A = c\,(\sin H) \div \cos l \pm \frac{1}{2} e \sin 2H$ will approximate closely to the true azimuth, the + sign corresponding to above, and the — sign to below pole. This expression is more simple than the trigonometrical formula when two sides and the included angle are given, and gives an error of less than 2″ at lat. 45°, and less than 1″ at lat. 36°.

Example.—Polari is observed on the 15th of March, 1881, $6^h 45^m$, P. M., Lat. 41° N., Long. 105° W. = 7^h west of Greenwich.

R. A. mean sun =	$23^h\ 32^m\ 54^s.3$
Time of observation	6 45
Increase in R. A.	2 15 .5
	30 20 09 .8
Apparent R. A. of Polaris	1 14 31 .4
	29 05 38 .4
	—24
$H = 76° 24' 36'' =$	$5^h\ 05^m\ 38^s.4$ above and west of pole.

The apparent decl'n of Polaris is 88° 40′ 45″, therefore $c = 1° 19' 15''$. Using the letters as above, we have by Chauvenet's Trig., page 180,

$$\tan\varphi = \tan c \cos H, \qquad \cot A = \frac{\sin(90°-l\pm\varphi)\cot H}{\sin\varphi},$$

the upper sign for φ must be used when the star is below, and the reverse when above the pole. The calculation gives

$$\begin{array}{rlrl} \tan c = & 8.3628023 & \cot H = & 9.3833498 \\ \cos H = & \underline{9.3710170} & \sin(90°-l-\varphi) = & \underline{9.8757224} \\ \tan\varphi = & 7.7338193, \quad \varphi = 18' 38''.5 & & 9.2590722 \\ & & \sin\varphi = & \underline{7.7338129} \\ A = & 1° 42' 32''.3. & \tan A = & 8.4747407 \end{array}$$

If we calculate A by the formula of approximation given above, we get

$$A = (c\sin H)\div\cos l + \tfrac{1}{2}c\sin 2H = 1° 42' 33''.2.$$

By the tables we have, interpolating 9″.8 for the 24′ 36″ in H,

for $H = 76°$, $\log N + \log c = 3.7951445 = \log$	6239″.4,
for 24′ 36″	9 .8
Azimuth for $H = 76° 24' 36''$ in Lat. 42° N. =	6249″.2
Correction for one degree of lat., from Col. V,	—96 .4
	6152 .8
Correction for 1881, 2nd÷100,	—.3
$A = 1° 42' 32''.5 =$	6152″.5.

The most convenient approximation to the meridian is by the use of a Solar Compass, but the result is often crude and uncertain. Defective and poorly adjusted solars (and the adjustment is not very simple) may be found varying from 10′ to 30′, and an error of 3′ to 5′ in a fairly adjusted instrument is perhaps not to be considered extraordinary.

There may be no better method of securing greater accuracy than to adjust the solar to a meridian and watch the position of the image when in strict conformity to the same at different hours of the day.

The tables that are here presented are often not as convenient as direct calculation, but they can serve as guides, and the determinations for the 1st and 2nd of each month can not materially change in several years.

TABLE I.

H.	Factors for determ'ng the Az. of Polaris below Pole: Lat. 39° N. Log N.	N.	M. Az.	Factors for determining the Az. of Polaris above Pole: Lat. 39° N. Log N.	N.	M. Az.	D.	V.
1°	8.343375	0.02205	0° 1′46″	8.359718	0.02289	0° 1′50″	0″.4	1″.5
2	8.644345	0.04409	3 31	8.660678	0.04578	3 39		
3	8.820332	0.06612	5 17	8.836653	0.068[illegible]5	5 29		
4	8.945130	0.08813	7 02	8.961430	0.09150	7 18		
5	9.041891	0.11013	8 47	9.058080	0.11431	9 08		
6	9.120808	0.13207	10 33	9.137046	0.13710	10 57		
7	9.187471	0.15398	12 18	9.203691	0.15984	12 46		
8	9.245151	0.17585	14 02	9.261333	0.18253	14 34		
9	9.295948	0.19767	15 47	9.312090	0.20516	16 23		
10	9.341309	0.21943	17 31	9.357405	0.22772	18 11	4″.3	15″.4
11	9.382262	0.24114	19 15	9.398301	0.25021	19 59	4″.2	14″.9
12	9.419569	0.26277	20 59	9.435554	0.27262	21 46		
13	9.453809	0.28432	22 42	9.469732	0.28494	23 33		
14	9.485428	0.30579	24 25	9.501287	0.31717	25 19		
15	9.514781	0.32718	26 07	9.530566	0.33929	27 05		
16	9.542160	0.34847	27 49	9.557871	0.36130	28 51		
17	9.567795	0.36965	29 31	9.583421	0.38320	30 36		
18	9.591878	0.39073	31 12	9.607432	0.40498	32 20		
19	9.614588	0.41171	32 52	9.630042	0.42662	34 04		
20	9.636046	0.43256	34 32	9.651395	0.44812	35 47	8″.5	30″.4
21	9.656368	0.45328	36 11	9.671627	0.46949	37 29	8″.2	29″.3
22	9.675667	0.47388	37 50	9.690811	0.49070	39 11		
23	9.694020	0.49433	39 28	9.709064	0.51176	40 52		
24	9.711503	0.51464	41 05	9.726442	0.53265	42 32		
25	9.728200	0.43481	42 42	9.743000	0.55335	44 11		
26	9.744153	0.55482	44 18	9.758836	0.57390	45 49		
27	9.759408	0.57466	45 53	9.773978	0.59426	47 27		
28	9.774045	0.59435	47 27	9.788475	0.61443	49 04		
29	9.788071	0.61386	49 01	9.802364	0.63440	50 39		
30	9.801536	0.63319	50 33	9.815689	0.65416	52 14	12″.4	44″.3
31	9.814474	0.65234	52 05	9.828480	0.67373	53 47	12″.0	42″.9
32	9.826909	0.67128	53 36	9.84075[illegible]	0.69304	55 20		
33	9.83[illegible]867	0.69003	55 06	9.852579	0.71216	56 52		
34	9.850400	0.70860	56 35	9.863927	0.73102	58 22		
35	9.861495	0.72693	58 03	9.874883	0.74969	59 51		
36	9.872200	0.74507	59 30	9.885425	0.76811	1 01 20		
37	9.882525	0.76300	1 00 55	9.895576	0.78628	1 02 47		
38	9.892489	0.78071	1 02 20	9.905361	0.80420	1 04 13		
39	9.[illegible]02108	0.79819	1 03 44	9.914795	0.82186	1 05 38		
40	9.911379	0.81542	1 05 06	9.923896	0.83926	1 07 01	15″.9	56″.9
							15″.5	55″.3

TABLE I—CONTINUED.

H.	Log N.	N.	M. Az.	Log N.	N.	M. Az.	D.	V.
41°	9.920340	0.83242	1°06′28″	9.932670	0.85639	1°08′23″		
42	9.929000	0.84918	1 07 48	9.941140	0.87325	1 09 44		
43	9.937358	0.86568	1 09 08	9.949307	0.88983	1 11 03		
44	9 945440	0.88194	1 10 26	9.957205	0.90616	1 12 21		
45	9.953243	0.89793	1 11 42	9.964810	0.92217	1 13 38		
46	9.960796	0.91368	1 12 57	9.972154	0.93790	1 14 54		
47	9.968088	0.92916	1 14 11	9.979241	0.95333	1 16 08		
48	9.975140	0.94437	1 15 24	9.986070	0.96844	1 17 20		
49	9.981946	0.95928	1 16 36	9.992672	0.98327	1 18 31		
50	9,988525	0.97392	1 17 46	9.999034	0.99778	1 19 40	19″.0	67″.7
51	9.994882	0.98828	1 18 55	0.005169	1.01197	1 20 48	18″.5	66″.0
52	0.001020	1.00231	1 20 02	0.011078	1.02584	1 21 55		
53	0.006936	1.01610	1 21 08	0.016773	1.03938	1 23 00		
54	0.012666	1.02959	1 22 13	0.022269	1.05261	1 24 03		
55	0.018185	1.04276	1 23 16	0.027558	1.06551	1 25 05		
56	0.023528	1.05567	1 24 17	0.032663	1.07811	1 26 05		
57	0.028660	1.06822	1 25 18	0.037562	1.09034	1 27 04		
58	0.033607	1.08046	1 26 16	0.042266	1·10221	1 28 01		
59	0.038369	1.09237	1 27 13	0.046790	1.11376	1 28 56		
60	0.042956	1.10396	1 28 09	0.051123	1.12492	1 29 49	21″.4	76″.3
61	0.047365	1.11523	1 29 03	0.055280	1.13574	1 30 41	21″.0	74″.8
62	0.051596	1.12615	1 29 55	0.059270	1.14622	1 31 31		
63	0.055665	1.13675	1 30 46	0.063083	1.15633	1 32 20		
64	0.059570	1.14701	1 31 35	0.066730	1.16608	1 33 07		
65	0.063311	1.15694	1 32 23	0.070220	1.17549	1 33 52		
66	0.066890	1.16651	1 33 09	0.073540	1.18451	1 34 35		
67	0.070316	1.17575	1 33 53	0.076699	1.19316	1 35 16		
68	0.073587	1·18464	1 34 36	0.079705	1.20144	1 35 56		
69	0.076701	1·19316	1 35 16	0.082567	1.20939	1 36 34		
70	0 079667	1·20134	1 35 56	0.085267	1.21693	1 37 10	23″.1	82″.5
71	0.082483	1·20915	1 36 33	0.087803	1.22405	1 37 44	22″.8	81″.4
72	0.085153	1·21661	1 37 09	0.090203	1.23084	1 38 17		
73	0.087677	1.22370	1 37 43	0.092455	1.23724	1 38 48		
74	0 090058	1.23043	1 38 15	0.094560	1.24325	1 39 16		
75	0.092296	1.23678	1 38 45	0.096524	1.24888	1 39 43		
76	0.094392	1.24277	. 39 14	0.098347	1.25413	1 40 08		
77	0.096349	1.24838	1 39 41	0.100023	1.25898	1 40 32		
78	0.098167	1.25362	1 40 06	0.101560	1.26346	1 40 53		
79	0.099848	1.25848	1 40 29	0.102971	1.26758	1 41 13		
80	0.101389	1.26295	1 40 51	0.104236	1.27126	1 41 30	24″.2	86″.1
81	0.102799	1.26706	1 41 10	0.105353	1.27453	1 41 46	24″.0	85″.6
82	0.104071	1.27077	1 41 28	0.106342	1.27744	1 42 00		
83	0.105203	1.27410	1 41 44	0.107193	1.27995	1 42 12		

TABLE I—CONTINUED.

H.	Log N.	N.	M. Az.	Log N.	N.	M. Az.	D.	V.
84°	0.106206	1.27704	1° 41′58″	0.107916	1.28208	1°42′23″	24″.4	
85	0.107077	1.27961	1 42 11	0.108501	1.28381	1 42 31	24 .3	
86	0.107827	1.28182	1 42 21	0.108956	1.28516	1 42 37		
87	0.108428	1.28360	1 42 30	0.109274	1.28610	1 42 42		
88	0.108893	1.28497	1 42 36	0.109466	1.28667	1 42 45		
88	55′ 18″			0.109523	1.28684	Maximum 1 42 45	24 .4	
89	0.199235	1.28598	1 42 41	0.109521	1 28683	1 42 45		
90	0.109443	1.28660	1 42 44	0.109443	1.28660	1 42 44	24 .4	1′29″ 1 25

TABLE II.

	Factors for determ'ng the Az. of Polaris below Pole: Lat. 42° N.			Factors for determining the Az. of Polaris above Pole: Lat. 42° N.				
H.	Log N.	N.	M. Az.	Log N.	N.	M. Az.	D.	V.
1°	8.361910	0.02301	0° 1′51″	8.380080	0.02399	0° 1′55″	0″.4	1″.7
2	8.662876	0.04601	3 40	8.681041	0.04798	3 50		
3	8.838863	0.06900	5 30	8.857016	0.07195	5 45		
4	8 963661	0.09197	7 20	8.981794	0.09589	7 39		
5	9.070428	0.11493	9 10	9.078441	0.11980	9 34		
6	9.139400	0.13785	11 00	9.157398	0.14368	11 28		
7	9.206013	0.16070	12 50	9 224043	0.16751	13 22		
8	9.263693	0.18352	14 39	9.281685	0.19129	15 16		
9	9.314490	0.20630	16 28	9.332432	0.21500	17 10	4 .6	18 .3
10	9.359851	0.22901	18 17	9.377747	0.23864	19 03	4 .4	17 ·3
11	9.400815	0.25166	20 06	9 418641	0.26221	20 56		
12	9.438122	9 27423	21 54	9.455894	0.28569	20 49		
13	9.472362	0.29673	23 42	9.490072	0.30908	24 41		
14	9.503981	0.31914	25 29	9.521627	0.33237	26 33		
15	9.533347	0.34147	27 16	9 550894	0.35554	28 24		
16	9.560725	0.36368	29 02	9.578190	0.37861	30 14		
17	9.586370	0.38581	30 48	9.603747	0.40156	32 04		
18	9.610453	0.40781	32 34	9.627748	0.42437	33 53		
19	9.633163	0.42970	34 19	9.650358	0.44705	35 42	8 .9	35 .5
20	9.654632	0.45147	36 03	9.671699	0.46956	37 30	8 .5	31 .3
21	9.674969	0.47311	37 46	9.691926	0.49196	39 17		
22	9.694264	0.49461	39 29	9.711132	0 51420	41 03		
23	9.712617	0.51596	41 11	9.729357	0.53624	42 49		
24	9.730111	0.53717	42 53	9.746723	0.55811	44 34		
25	9.746820	0.55824	44 34	9.763280	0.57980	46 18		
26	9.662772	0.57913	46 14	9.779110	0.60133	48 01		

TABLE II—CONTINUED.

H.	Log N.	N.	M. Az.	Log N.	N.	M. Az.	D.	V.
27°	9.778050	0.59986	0° 47′54″	9.794245	0.62265	0° 49′42″		
28	9.792680	0.62041	49 32	9.808720	0.64376	51 24		
29	9.806712	0.64079	51 10	9.822609	0.66467	53 04		
30	9.820188	0.66098	52 47	9.835922	0.68537	54 43	13″0	51″.7
31	9.833138	0.68099	54 23	9.848701	0.70583	56 21	12 .6	49 .9
32	9.845562	0.70075	55 57	9.860980	0.72607	57 59		
33	9.857542	0.72035	57 31	9.872788	0.74608	59 35		
34	9.869080	0.73974	59 04	9.884145	0.76585	1 01 09		
35	9.880188	0.75891	1 00 36	9.895069	0.78536	1 12 42		
36	9.890900	0.77786	1 02 06	9.905640	0.80471	1 14 15		
37	9.901232	0.79659	1 03 36	9.915782	0.82373	1 15 46		
38	9.911208	0.81509	1 05 05	9.925558	0.84248	1 17 16		
39	9.920828	0.83335	1 06 33	9.934980	0.86096	1 18 45		
40	9.930120	0.85136	1 07 59	9.944054	0.87914	1 10 12	16 .7	66 .4
41	9.939092	0.86915	1 09 24	9.952797	0.87901	1 11 38	16 .2	64 .3
42	9.947765	0.88668	1 10 48	9.961255	0.91465	1 13 02		
43	9.956135	0.90393	1 12 11	9.969415	0.93200	1 14 25		
44	9.964221	0.92092	1 13 32	9.977300	0.94907	1 15 47		
45	9.972040	0.93765	1 14 52	9.984890	0.96584	1 17 07		
46	9.979600	0.95411	1 16 11	9.992229	0.98227	1 18 26		
47	9.986900	0.97029	1 17 29	9.999307	0.99841	1 19 43		
48	9.993955	0.98618	1 18 45	0.006130	1.01422	1 20 59		
49	0.000787	0.00181	1 20 00	0.012707	1.02969	1 22 13		
50	0.007389	1.01714	1 21 13	0.019060	1.04487	1 23 26	19 .8	78 .9
51	0.013744	1.03215	1 22 25	0.025179	1.05969	1 24 37	19 .3	76 .8
52	0.019894	1.04687	1 23 36	0.031074	1.07417	1 25 46		
53	0.025838	1.06130	1 24 45	0.036759	1.08833	1 26 54		
54	0.031563	1.07538	1 25 52	0.042243	1.10215	1 28 01		
55	0.037096	1.08917	1 26 58	0.047520	1.11563	1 29 05		
56	0.042458	1.10268	1 28 03	0.052611	1.12878	1 30 08		
57	0.047596	1.11582	1 29 06	0.057495	1.14155	1 31 09		
58	0.052558	1.12865	1 30 07	0.062178	1.15392	1 32 08		
59	0.057339	1.14112	1 31 07	0.066692	1.16598	1 33 06		
60	0.061924	1.15325	1 32 05	0.071011	1.17763	1 34 02	22 .4	88 .9
61	0.066349	1.16506	1 33 02	0.075163	1.18894	1 34 56	21 .9	87
62	0.070600	1.17652	1 33 57	0.079129	1.19985	1 35 48		
63	0.074683	1.18763	1 34 50	0.082923	1.21037	1 36 39		
64	0.078600	1.19839	1 35 42	0.086560	1.22056	1 37 28		
65	0.082344	1.20876	1 36 31	0.090034	1.23036	1 38 15		
66	0.085940	1.21882	1 37 19	0.093340	1.23976	1 38 60		
67	0.089384	1.22852	1 38 06	0.096494	1.24880	1 39 43		
68	0.092670	1.23785	1 38 51	0.099473	1.15739	1 40 24		
69	0.095800	1.24680	1 39 35	0.102308	1.26561	1 41 04		
70	0.098783	1.25540	1 40 14	0.104994	1.27348	1 41 41	24 .2	96 .1
							23 .9	94 .8

TABLE II—CONTINUED.

H.	Log N.	N.	M. Az.	Log N.	N.	M. Az.	D.	V.
71°	0.101611	1.26359	1° 40′ 54″	0.107527	1.28093	1° 42′ 17″		
72	0.104296	1.27143	1 41 32	0.109911	1.28798	1 42 51		
73	0.106839	1.27890	1 42 07	0.112146	1.29462	1 43 23		
74	0.109232	1.28596	1 42 41	0.114237	1.30087	1 43 53		
75	0.111484	1.29265	1 43 13	0.116188	1.30673	1 44 20		
76	0.113598	1.29896	1 43 43	0.117994	1.31217	1 44 47		
77	0.115570	1.30487	1 44 12	0.119654	1.31720	1 45 11		
78	0.117404	1.31038	1 44 38	0.121179	1.32183	1 45 33		
79	0.119097	1.31551	1 45 03	0.122564	1.32605	1 45 53		
80	0.120655	1.32024	1 45 25	0.123810	1.32986	1 46 11	25″3	100″.4
81	0.122086	1.32456	1 45 46	0.124921	1.33327	1 46 28	25 .1	99 .7
82	0.123371	1.32852	1 46 05	0.125892	1.33625	1 46 42		
83	0.124523	1.33206	1 46 22	0.127431	1.33883	1 46 54		
84	0.125538	1.33518	1 46 38	0.127431	1.34101	1 47 5		
85	0.126419	1.33789	1 46 50	0.128001	1.34277	1 47 13	25 .5	
86	0.127180	1.34023	1 47 01	0.128445	1.34414	1 47 20	25 .4	
87	0.127796	1.34214	1 47 10	0.128741	1.34506	1 47 24		
88	0.128286	1.34365	1 47 17	0.128917	1.34560	1 47 27		
88	48′ 5″			0.128964	1.34575	Maximum. 1 47 27	25 .6	
89	0.128642	1.34475	1 47 23	0.129857	1.34573	1 47 27		
90	0.123864	1.34544	1 47 26	0.128864	1.34544	1 47 26	25 .6	1′.44″ 1′.39″

Explanation of Tables.—Tables I and II give the mean azimuth of polaris for the epoch 1880, in latitude 39° and latitude 42°, north, respectively, corresponding with the value of H (the hour angle) as indicated in 1st column. The mean azimuth, as given in the 4th and 7th col's, is the product of the factor N multiplied by the co-declination (c) of Polaris, the factor N being the equivalent of the formula $\sin H \div \sin(l \pm \cos H \times c)$. The azimuth thus determined is slightly in excess but less than 1″ at 45° and less than 2″ at 90°, and for values of H greater than 30° I have reduced N to correspond with more exact calculations. Column D contains a correction for annual variation in azimuth and is always *minus*, and column V, a correction for 1° of latitude, + when north and — when south. The double numbers in columns D and V correspond to the angle above and below pole, respectively; the upper in column V to be applied with the plus, and the lower, with the minus sign. If d be the difference between the upper and lower numbers, in column, D for any value of H in 1880, then, for any number n, of years after that epoch, the correction will be $2nd \div 100$, nearly.

When polaris is moving towards its culmination, positive increase in longitude and time diminishes its azimuth in table III, and vice versa.

Table III.

Time for observation on the first and second day of each month.	Lat. 39 deg. N., Long. 106 deg. W.: Azim'th for 1881.	Lat. 39 deg. N.: Azim'th for 1882.	Lat. 39 deg. N.: V'n of Azim. for 1 m, inc. is pos.	Lat. 39 deg. N.: V'n of Azim. for 15° long., w is pos.	Lat. 42 deg. N., Long. 106 deg. W.: Azim'th for 1881.	Lat. 42 deg. N.: Azim'th for 1882.	Lat. 42 deg. N.: V'n of Azim. for 1 m, inc. is pos.	Lat. 42 deg. N.: V'n of Azim. for 15° long., w is pos.	Position of Polaris with respect to pole, A = above; B = below; E = east; W = west of pole. When Polaris is A. E. or B. W. it is decreasing with positive time in azimuth.	Approx. am't to be added or subtracted from the Lat. to give the Alt. of Polaris.
	Special determinations of the azimuth of Polaris, the apparent Dec. and R. A. being used. Lat. 39 deg. N., Long. 106 deg. W.				Special determinations of the azimuth of Polaris, the apparent Dec. and R. A. being used, Lat. 42 deg. N., Long. 106 deg. W.					
	° ′ ″	° ′ ″	″	″	° ′ ″	° ′ ″	″	″		
Jan. 1, 5h 00m 00.0s p. m.	0 38 58	0 39 26	—25.0	—04.1	0 40 50	0 41 19	—26.0	—04.2	E. A. Pole, moving west	+01 13
Jan. 2, 7h 00m 00.0s a. m.	0 14 43	0 14 02	+26.0	+04.3	0 15 23	0 14 39	+27.0	+04.4	E. B. Pole, moving east	—01 18
Feb. 1, 5h 30m 00.0s p. m.	0 28 45	0 28 00	+26.0	+04.3	0 30 08	0 29 21	+27.5	+04.5	A. W. Pole, moving west	+01 16
Feb. 2, 6h 50m 00.0s a. m.	0 59 51	0 59 06	+21.0	+03.4	1 00 59	1 00 12	+22.5	+03.6	B. E. Pole, moving east	—01 04
Mar. 1, 6h 15m 00.0s p. m.	1 24 26	1 22 46	+15.0	+02.5	1 28 24	1 27 42	+16.5	+02.6	W. A. Pole, moving west	+00 46
Mar. 2, 6h 00m 00.0s a. m.	1 19 07	1 18 27	+16.7	+02.7	1 22 39	1 21 57	+17.7	+02.8	E. B. Pole, moving east	—00 49
April 1, 6h 45m 00.0s p. m*	1 41 53	1 41 28	—02.0	—00.3	1 46 32	1 46 06	—02.0	—00.3	W. B. Pole, mo'ing e. fr'm elong.	—00 04
April 2, 5h 20m 00.0s a. m.	1 36 27	1 36 09	+09.0	+01.5	1 40 48	1 40 29	+09.4	+01.5	E. B. Pole, moving east	—00 19
May 1, 7h 30m 00.0s a. m.	1 12 26	1 12 47	—19.0	—03.1	1 15 39	1 16 02	—19.7	—03.2	W. B. Pole, moving east	—00 55
May 2, 4h 30m 00.0s a. m.	1 42 14	1 41 50	+00.4	+00.0	1 46 55	1 46 30	+00.4	+00.0	E. B. Pole, mo'ing east to elong.	—00 01
			Near E	Elong.			Near E	Elong.		
June 1, 8h 00m 00.0s p. m.	0 14 00	0 14 35	—26.1	—04.3	0 14 37	0 15 13	—27.0	—04.4	W. B. Pole, moving east	—01 18
June 2, 4h 00m 00.0s a. m.	1 35 25	1 35 18	—10.5	—01.7	1 39 52	1 39 44	—10.5	—01.7	E. A. Pole, moving west	+00 30
July 1, 8h 00m 00.0s p. m.	0 36 48	0 36 05	+24.7	+04.0	0 38 24	0 37 39	+25.7	+04.2	E. B. Pole, moving east	—01 14
July 2, 4h 10m 00.0s a. m.	1 00 44	1 01 03	—22.2	—03.6	1 03 37	1 03 57	—23.0	—03.8	E. A. Pole, moving west	+01 04
Aug. 1, 7h 30m 00.0s p. m.	1 10 37	1 09 55	+19.0	+03.1	1 13 44	1 13 00	+20.0	+03.3	E. B. Pole, moving east	—00 57
Aug. 2, 4h 30m 00.0s a. m.	0 00 30	0 01 09	—27.5	—04.5	0 00 32	0 01 12	—28.7	—04.7	E. A Pole, moving west	+01 19
Aug. 2, 4h 31m 07.4s a. m.	†		+27.5	+04.5	†		+28.7	+04.7	A. Pole, moving west	+01 19
Aug. 2, 4h 32m 31.4s a. m.		†	+27.5	+04.5		†	+28.7	+04.7	A. Pole, moving west	+01 19
Sept. 1, 7h 00m 00.0s p. m.	1 33 36	1 33 00	+10.6	+01.7	1 37 48	1 37 11	+11.2	+01.8	E. B. Pole, moving east	—00 39
Sept. 2, 5h 00m 00.0s a. m.	1 01 35	1 00 53	+22.0	+03.6	1 06 27	1 05 44	+22.5	+03.7	W. A. Pole, moving west	+01 03
Oct. 1, 6h 15m 00.0s p. m.	1 41 30	1 41 05	+02.6	+00.4	1 46 08	1 45 41	+02.8	+00.4	E. B. Pole, moving east	—00 06
Oct. 2, 5h 15m 00.0s a. m.	1 37 00	1 36 27	+08.6	+01.4	1 41 30	1 40 56	+08.8	+01.4	W. A. Pole, moving west	+00 26
Nov. 1, 5h 30m 00.0s p. m.	1 38 49	1 38 38	—06.3	—01.0	1 43 24	1 43 12	—06.5	—01.1	E. A. Pole, moving west	+00 20
Nov. 2, 6h 00m 00.0s a. m.	1 33 37	1 33 03	—10.6	—01.7	1 37 16	1 37 13	—11.2	—01.8	W. B. Pole, moving east	—00 31
Dec. 1, 5h 00m 00.0s p. m.	1 22 08	1 22 12	—16.2	—02.7	1 27 59	1 25 04	—16.7	—02.7	E. A. Pole, moving west	+00 47
Dec. 2, 6h 40m 00.0s a. m.	0 00 11	0 46 35	—23.5	—03.8	0 48 13	0 48 [illegible]	—24.5	—04.0	W. B. Pole, moving east	—01 10

* Near Western Elongation. † Upper Culmination.

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LAW OF FACILITY OF ERRORS IN TWO DIMENSIONS.

BY E. L. DE FOREST.

[Continued from page 48.]

THIS surface (43) might be called the *probability surface*, just as (1) is the probability curve. Its form is such that if it is intersected by a plane parallel to the plane of XY, the lines of intersection will be ellipses whose centre and axes, projected on the plane of XY, coincide with the origin and axes of X and Y, and the axes of the ellipse in the x and y directions are inversely proportional to h_1 and h_2, and therefore directly proportional to r_1 and r_2. This may be shown by making z a constant in (43). The projected ellipses are the loci of points of equal probability.

From this symmetry of the surface with respect to the axes of X and Y, it follows that these are the free axes of the system of expanded coefficients z, regarded as masses in the plane of XY. Thus the given probabilities λ, and the resultant probabilities z, have both the same free axes. This also follows from the permanence of direction of the axes which give maximum and minimum values to the radii of gyration in successive powers, as already noticed. The value of z is a maximum at the origin, and this being the place of the centre of gravity of the whole system of masses z, it appears that the point of most probable error in the result of a very large number of observations, will be located at the centre of gravity of all the points which can possibly result under the given system of elementary errors, each possible point of error being taken with a weight proportional to the probability that such error will occur. This also follows from what we have already proved in connection with formula (3). In order that the most probable result should coincide with the true point or place of zero error, it would be necessary and sufficient that the probabilities or weights λ of the

elementary points of error should be such that their centre of gravity falls at the zero point.

Suppose the plane of XY to be divided up into elementary rectangles $dxdy$ by lines drawn parallel to the axes; then (43) gives the probability z that an error which occurs will fall on the rectangle whose centre is at the point $x = idx$, $y = jdy$. It gives an approximate result when instead of dx and dy we use small finite distances Δx and Δy. The values of h_1 and h_2 are computed by (37) and (38) if the elementary errors and their probabilities λ are known. But if these are unknown, as generally happens in practical applications, we must compute from a large number of observed errors, the quadratic mean errors in the x and y directions, answering to $r_1\sqrt{k}$ and $r_2\sqrt{k}$ in (38), and thus obtain h_1 and h_2. For example, if a large number of shots have been fired and have struck a target, and we wish to infer, from the distribution of the shot-marks, what is the law of probable distribution of other shots to be made under like conditions, our investigation leads to the following process. Regarding each given shot-mark as the mass of a material point, all such masses being equal, find the centre of gravity of the system, and then by (31) find its free axes. The distance of the centre of gravity from any assumed axis of reference is the arithmetical mean of the distances of all the shot-marks from that axis, distances on opposite sides of it having contrary signs. The free axes will be at right angles to each other at the centre of gravity, and we take either one of them as the X and the other as the Y axis. Find the quadratic mean errors in the x and y directions, that is, the square root of the mean of the squares of the distances of the shot-marks from the Y and X axis respectively. Denoting these by $e_1 dx$ and $e_2 dy$, we shall have as in (38),

$$(h_1 dx)^2 = 1 \div 2e_1^2, \qquad (h_2 dy)^2 = 1 \div 2e_2^2, \tag{44}$$

and (43) becomes

$$z = \frac{1}{2\pi e_1 e_2} e^{-(i^2 \div 2e_1^2) - (j^2 \div 2e_2^2)}. \tag{45}$$

For convenience, dx and dy may be taken equal to each other and to any small unit of measure, for instance one inch. Then (45) gives the probability z that any particular future shot will strike that square inch of the target whose coordinates are i and j inches.

All this is on the supposition that we know nothing about the probable distribution of the errors except what is to be inferred from the actual distribution of the shot-marks already made.

The principle that the axes of X and Y should be taken to coincide with the free axes of the given shot-marks, is one which has not, so far as I know, been brought out by any previous writer. The usual course has been, to

draw the axes horizontally and vertically through the centre of gravity of the marks. This, however, will in general be correct only when there is reason to believe that the elementary errors, whose probabilities are λ, are distributed so as to make the horizontal and vertical axes free axes, which will be the case for instance when these errors and their probabilities fall symmetrically on either side of the horizontal and vertical axes, making them axes of symmetry in the system of masses λ. Granting that the free axes are horizontal and vertical, formula (43) agrees with the one in general use. It can be written

$$z = \left(\frac{h_1 dx}{\sqrt{\pi}} e^{-h_1^2 x^2}\right)\left(\frac{h_2 dy}{\sqrt{\pi}} e^{-h_2^2 y^2}\right), \tag{46}$$

the second member being the product of two factors of the same form as in the probability curve (1). This property has enabled writers to give a demonstration of (43), or what has seemed to be one, in a very simple manner. (See for instance Didion's *Calcul des Probabilités appliqué au Tir des Projectiles*, Paris, 1858, p. 42.) If we suppose the plane of XY to be divided by lines drawn parallel to the Y axis, into bands of the width dx, and assume that the probability of deviation from the Y axis is the same at all parts of that axis, or in other words, is independent of y, then the probability that an error which occurs will fall within the band whose distance from the Y axis is x, is represented by the first of the two factors in (46). Likewise if lines are drawn parallel to the X axis, forming bands of the width dy, the probability that the error will fall within the band whose distance from the X axis is y, is represented by the second factor in (46). But the probability that two independent events will both occur, is the product of their separate probabilities, so that (46) gives the probability z that the error will fall within both bands, that is, at their intersection, and within the rectangle $dx\,dy$ whose coordinates are x and y. This demonstration, however, has been objected to, on the ground that the independence of the x and y deviations, if it exists, is not so evident as to be axiomatic. It really seems to be a thing to be proved. Various authors who have discus'd the point are named in Merriman's *List of Writings relating to the Method of Least Squares.* (Trans. Conn. Acad., Vol. IV, Part 1.) Our present investigation shows that the x and y deviations may be regarded as independent of each other, provided that the axes of X and Y are taken to coincide with the free axes of the probabilities λ of the elementary errors. When these errors are unknown, we approximate to the desired axes by taking the free axes of the observed points of error, or shot-marks. But if the axes in (43) are supposed to be rotated through the angle θ, and the coordinates to these new axes are x' and y', so that we have

$$x = x' \cos\theta - y' \sin\theta, \qquad y = x' \sin\theta + y' \cos\theta,$$

then substituting these for x and y in (43), the exponent of e will contain not only x'^2 and y'^2, but the product $x'y'$, so that it will not in general be possible to separate the second member into two factors, one containing the abscissa and the other the ordinate, as in (46). Thus the x and y deviations are not in general independent of each other unless the axes taken are free axes. But if we assume that the elementary errors are such as to make given deviations equally probable in all directions, every axis through the centre of gravity will be a free axis, and the deviations in horizontal and vertical directions will therefore be independent. That assumption is ordinarily made in practice, as it simplifies the calculations. The deviations of the shot-marks are then measured from the centre of gravity directly, instead of from the axes. Denoting the square root of the mean of the squares of these deviations by εdx, and taking $dx = dy$, we have

$$\varepsilon^2 = e_1^2 + e_2^2,$$

and since e_1 and e_2 are here equal, $\varepsilon^2 = 2e_1^2 = 2e_2^2$, and (44) gives

$$(h_1 dx)^2 = (h_2 dy)^2 = 1 \div \varepsilon^2, \quad \therefore h_1 = h_2 = 1 \div (\varepsilon dx).$$

(Compare Didion's work already cited, and Sonnet's *Dictionnaire des Mathématiques Appliquées*, article *Probabilité du Tir.*) The probability surface is what (43) becomes when $dx = dy$, $h_1 = h_2$ and $x^2 + y^2 = r^2$, the elliptical sections being reduced to circles. It has been found to agree sufficiently well with the results of observation in target shooting. Probably it will not be worth while to compute the place of the free axes by (31) unless the observed points of error or shot-marks form a decidedly elongated group. If the probability of deviation is really the same in all directions, still it is not likely that the actual distribution of the shots will be uniform, and a slight elongation of the group will not compel the inference that the elementary errors must be greater in that direction.

Returning now to the strict construction of the surface (43), let us refer the position of the rectangle $dx\,dy$ to polar coordinates by writing

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \tag{47}$$

so that we have

$$z = \frac{h_1 h_2 dx\,dy}{\pi} e^{-r^2(h_1^2\cos^2\phi + h_2^2\sin^2\phi)} \tag{48}$$

as the probability that an error which occurs will fall on the area $dxdy$ whose coordinates are r and φ. Suppose that the plane of XY is divided into rings by circles drawn from the origin as a centre, the breadth of each ring being $\surd(dxdy)$, and that it is also divided into sectors by radii drawn at angular intervals equal to $d\varphi$. The plane is thus partitioned into small elementary areas, each area at distance r from the origin being equal to $rd\varphi \times \surd(dx\,dy)$, and $dx\,dy$ is contained in it $rd\varphi \div \surd(dx\,dy)$ times, so that the

probability that an error which occurs will fall on this area, is

$$zrd\varphi \div \sqrt{(dxdy)}.$$

Hence the probability that it will fall on a given sector whose extreme radius is r will be

$$p = \frac{1}{dr}\int_0^r z\left(\frac{rd\varphi}{\sqrt{(dxdy)}}\right)dr,$$

where dr is to be taken equal to $\sqrt{(dxdy)}$; so that substituting the value of z from (48), we have

$$p = \frac{h_1 h_2 d\varphi}{\pi}\int_0^r e^{-r^2(h_1^2\cos^2\phi + h_2^2\sin^2\phi)}rdr,$$

and by integration

$$p = \frac{h_1 h_2 d\varphi}{2\pi(h_1^2\cos^2\varphi + h_2^2\sin^2\varphi)}\left\{1 - e^{-r^2(h_1^2\cos^2\phi + h_2^2\sin^2\phi)}\right\}. \qquad (49)$$

Making $r = \infty$ we get as the probability that the error will fall somewhere on this sector extended to infinity,

$$\rho = \frac{h_1 h_2 d\varphi}{2\pi(h_1^2\cos^2\varphi + h_2^2\sin^2\varphi)}. \qquad (50)$$

If now it is required that $p = \frac{1}{2}\rho$, we have the relation

$$e^{-r^2(h_1^2\cos^2\phi + h_2^2\sin^2\phi)} = \tfrac{1}{2},$$

and restoring x and y from (47),

$$e^{-(h_1^2x^2 + h_2^2y^2)} = \tfrac{1}{2}, \qquad \therefore\ h_1^2x^2 + h_2^2y^2 = (\log 2)\div(\log e).$$

Giving to h_1 and h_2 their values as in (44),

$$h_1^2 = 1\div 2(e_1 dx)^2, \qquad h_2^2 = 1\div 2(e_2 dy)^2,$$

we get the equation of an ellipse,

$$\frac{x^2}{[(\log 4)\div(\log e)](e_1 dx)^2} + \frac{y^2}{[(\log 4)\div(\log e)](e_2 dy)^2} = 1,$$

whose semi-axes are the square roots of the denominators in the first member. Their values are

$$a = 1.17741(e_1 dx), \qquad b = 1.17741(e_2 dy), \qquad (51)$$

so that they are proportional to $e_1 dx$ and $e_2 dy$, the quadratic mean errors in the x and y directions. Hence for example, in the group of shot-marks already considered, if we find the free axes and the quadratic mean deviations from them, and construct an ellipse with semi-axes a and b as in (51) coinciding with the free axes of X and Y respectively, this will be the *ellipse of probable error*, in the sense that it is an even chance whether a particular future shot will strike inside or outside the ellipse. And if two straight lines are drawn from its centre making any angle with each other, it is an even chance whether shots which fall between them will fall inside or outside of the ellipse.

There is a correspondence between some of our results and those which have been reached by Andrä and others in the theory of the error-ellipse (*die Fehlerellipse*), which is one of the more recent developm'ts of the method of least squares as applied to the location of a point by triangulations. The true position of the point should be at the common intersection of several straight lines, whose directions are given free from error, but whose positions laterally are subject to errors of observation, so that the lines do not in general have a common intersection. Each of them may be represented by an equation of the form

$$ax + by + l = 0,$$

where x and y are the coordinates of any point in the line, referred to rectangular axes taken in the vicinity of the desired point, a and $-b$ are the sine and cosine of the acute angle which the line makes with the X axis, on the $+X$ side, and l is the perpendicular from the origin upon the line. The angle and the perpendicular are essentially $+$ or $-$ according as thay lie above or below the X axis. The various intersecting lines furnish a number of observation equations of this form, which being solved by the method of least squares, give the desired coordinates x and y of the most probable point of common intersection. It is then shown that all other points whose probabilities are equal are ranged about the most probable point in ellipses which are similar and similarly situated and concentric. The directions of their major and minor axes are given by a formula which bears an external resemblance to (31). See for instance Jordan's *Handbuch der Vermessungskunde;* I., pp. 120 and 123.

The dimensions of the ellipse within which it is an even chance that the true point will fall, have been ascertained by Helmert. (*Ausgleichungsrechnung*, p. 236.) He makes its semi-axes bear to the corresponding quadratic mean errors of the point. the same ratio 1.17741 which I have obtained in a different way in (51). Hence they bear to the corresponding probable errors the ratio

$$1.17741 \div .6745 = 1.7456.$$

His demonstration will be found in Schlömilch's *Zeitschrift fur Math. und Phys.*, 1868, pp. 75 to 79. The probability of a given deviation of the p't from its most probable place, is expressed by the same form of function which I have found as the equation (43) of the probability surface.

It may be questioned, however, whether the theory of the error-ellipse affords proof of the universality of that law of facility, for all accidental errors in the situation of a point in a plane. It seems to me that when taken in this extended sense, the method is open to the same kind of theoretical objection which we noticed in connection with (46). An observed point of error, or intersection of two erroneous observed lines, is the result of the

concurrent happening of two independent errors, each of which, it is assumed by the theory, follows an exponential law of facility such as is expressed by the probability curve (1). The probability that any particular point in the plane is the true one, is regarded as proportional to the probabilty of the concurrent happening of all the independent errors represented by the perpendicular distances from that point to all of the observed lines. The most probable point is that which renders the sum of the squares of the distances, each square being multiplied by the weight due to its observed line, a minimum. (Helmert, in the *Zeitschrift*, pp. 76 and 89; also Jordan, II., p. 223.)

For the case of intersecting straight lines which lie in only two mutually perpendicular directions, the demonstration is evidently similar in principle to that which I remarked upon in connection with (46). My own proof of (43), however, is a direct proof, which does not take (1) into account at all, inasmuch as it does not assume that the x- and y- errors of a point happen independently of each other. But it includes the proof of (1) as a special case, for if the elementary points of error are all on one line which is made the axis of X, so that the given polynomial (2) is reduced to one dimension, all its coefficients being zero except those on the horizontal line through the middle of the square group, the last equation in (23) disappears, and of the two equations (30) only the first one remains, with $a_2 = 0$ and $\gamma = 0$. When the centre of gravity of the masses λ is made the origin and centre of the linear group, by imagining terms with zero coefficients to be added if necessary, we get $a_1 = 0$, and only the first of the two differential equations will exist in (35) and (39), the integral being

$$z = ce^{-h_1^2x^2}.$$

The condition (41) is reduced to

$$\frac{1}{dx}\int_{-\infty}^{+\infty} z dx = 1, \qquad \therefore c = \frac{h_1 dx}{\sqrt{\pi}},$$

and z becomes the ordinate to the probability curve, like y in (1).

The occurrence of errors in two dimensions can be further illustrated by the drawing of balls from an urn, each ball being marked with two numbers which may be positive or negative, the first number representing the x and the second the y coordinate of the point of error, reckoned from the true point, or place of error zero, as an origin. Each ball is replaced as soon as drawn, and they are all mixed so as to keep the chances the same.

Suppose for example that the urn contains 20 balls marked as follows; one of them is marked 0, 0, four are marked 1, 0, five 1, 1, four 0, 1, one —1, 1, three —1, —1, and two 0, —1. Then if k drawings are made, the probability that the algebraic sum of the first or x-numbers on the balls

drawn will be t, and that at the same time the sum of their sec'd or y-numbers will be v, is the coefficient of $\xi^t\eta^v$ in the expansion of the polynomial (52) to the k power, this polynomial being a case under the general form (2).

$$\frac{1}{20}\begin{pmatrix} \xi^{-1}\eta & +4\eta & +5\xi\eta \\ & +1 & +4\xi \\ +3\xi^{-1}\eta^{-1} & +2\eta^{-1} & \end{pmatrix} \tag{52}$$

To find the centre of gravity and the free axes of the coefficients λ, we may disregard for the present the divisor 20 common to them all, and write them in the rectangular form (53). Their centre of gravity is at the point whose coordinates reckoned from the middle term 1, are $u' = \frac{1}{4}\Delta u$, $v' = \frac{1}{4}\Delta v$, where Δu and Δv are the intervals between the terms in (53), horizontally and vertically. Drawing horizontal and vertical axes of U and V through this centre of gravity, the abscissas u of the terms in the three columns are

1	4	5	(53)
0	1	4	
3	2	0	

$$-\tfrac{5}{4}\Delta u, \qquad -\tfrac{1}{4}\Delta u, \qquad \tfrac{3}{4}\Delta u,$$

and the ordinates v of the terms in the three rows are

$$\tfrac{3}{4}\Delta v, \qquad -\tfrac{1}{4}\Delta v, \qquad -\tfrac{5}{4}\Delta v.$$

Denoting the terms in (53) by M, and applying (31), we find

$$\Sigma(Muv) = \tfrac{23}{4}\Delta u\Delta v, \qquad \Sigma(Mu^2) = \tfrac{47}{4}(\Delta u)^2, \qquad \Sigma(Mv^2) = \tfrac{55}{4}(\Delta v)^2.$$

$$\tan 2\varphi = 46\left(\frac{\Delta u}{\Delta v}\right) \div \left\{ 47\left(\frac{\Delta u}{\Delta v}\right)^2 - 55 \right\}.$$

Taking for convenience $\Delta u = \Delta v$, so that (53) is a square, we have $\tan 2\varphi = -\frac{23}{4}$, wherefore $\varphi = -40°4'$. This is the angle which one free axis makes with the horizontal, at the centre of gravity, and the second free axis is at right angles to the first. Designating these axes as X and Y, we get from (33) the relation between the new coordinates and the old,

$$x = u\cos\theta - v\sin\theta, \qquad y = u\sin\theta + v\cos\theta, \tag{54}$$

where $\theta = -\varphi = 40°4'$. Squaring x and y, multiplying them by M, and summing the results for all the terms M in the system, we have

$$\left.\begin{aligned} \Sigma(Mx^2) &= \Sigma(Mu^2)\cos^2\theta + \Sigma(Mv^2)\sin^2\theta - 2\Sigma(Muv)\sin\theta\cos\theta, \\ \Sigma(My^2) &= \Sigma(Mu^2)\sin^2\theta + \Sigma(Mv^2)\cos^2\theta + 2\Sigma(Muv)\sin\theta\cos\theta. \end{aligned}\right\} \tag{55}$$

We assign to the sums in the second members their values already found, and take for simplicity

$$\Delta x = \Delta y = \Delta u = \Delta v, \tag{56}$$

and so get

$$\Sigma(Mx^2) = 6.909(\Delta x)^2, \qquad \Sigma(My^2) = 18.588(\Delta y)^2.$$

Dividing these by 20, since $\lambda = M \div 20$, we obtain the squares of the radii of gyration of the masses λ about the Y and X axes,

$$r_1^2 = .3454(\Delta x)^2, \qquad r_2^2 = .9294(\Delta y)^2.$$

Suppose that the number of drawings from the urn is $k=4$. Then in the expanded polynomial the squares of the radii of gyration of the coeff's are

$$kr_1^2 = 1.382(\Delta x)^2, \qquad kr_2^2 = 3.718(\Delta y)^2,$$

so that in (44) we have

$$e_1^2 = 1.382, \qquad e_2^2 = 3.718,$$

and (45) becomes

$$\left.\begin{aligned} z &= .0702e^{-.362i^2-.134j^2}, \\ \therefore \log z &= \bar{2}.8464 - .15712i^2 - .05840j^2. \end{aligned}\right\} \qquad (58)$$

This is the equation of the limiting surface which represents approximately the contour of the coefficients in the expansion of (52) to the fourth power. Since the centre of gravity preserves in the expansion the same relative position which it had in the first power, its coordinates reckoned from the middle point will be four times as great, or

$$u' = 4u, \qquad v' = 4v.$$

The expansion contains 81 terms, wich occupy the centres of the small squares in the subjoined figure, and the centre of gravity falls at the next term diagonally from the middle one. The axes of U and V have their origin at this point, as do also those of X and Y. From the coordinates u and v for the several terms we compute the values of x and y by (54). A few nearest the origin being tabulated as in (59), the rest can be readily found from them by their differences. The values of z are now computed, and

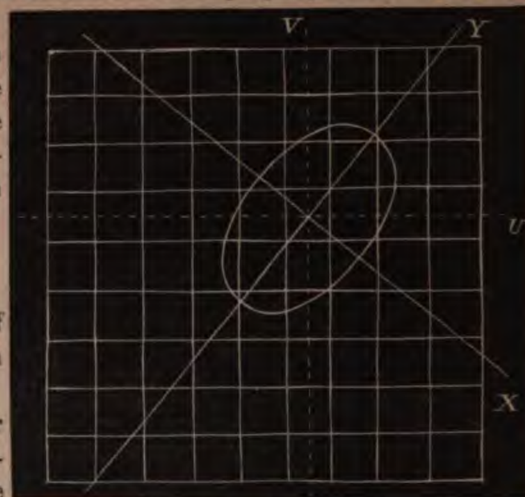
$$\begin{array}{|r|r|r|} \multicolumn{3}{|c|}{i = x \div \Delta x} \\ -1.409 & -.644 & .122 \\ -.765 & 0 & .765 \\ -.122 & .644 & 1.409 \end{array} \qquad \begin{array}{|r|r|r|} \multicolumn{3}{|c|}{j = y \div \Delta y} \\ .122 & .765 & 1.409 \\ -.644 & 0 & .644 \\ -1.409 & -.765 & -.122 \end{array} \qquad (59)$$

being multiplied by 1000, they are as shown in the first half of the accompanying table, omitting decimals. Also by actual expansion of (52) to the 4th power we get the true coefficients, which being multiplied by 1000 like the (1000z).

0	0	0	1	4	9	13	12	6	0	0	1	3	9	15	18	13	4
0	0	1	4	14	28	34	24	10	0	0	0	3	11	26	39	33	12
0	0	3	12	34	66	53	30	10	0	1	5	18	37	52	53	38	15
0	1	6	24	54	70	54	24	6	0	0	3	15	42	66	58	27	8
0	2	10	30	53	56	34	12	3	0	3	13	29	44	49	38	14	2
0	2	10	24	34	28	14	4	1	0	1	6	22	36	29	13	3	0
0	2	6	12	13	9	4	1	0	1	4	10	14	14	9	2	0	0
0	1	2	4	3	2	1	0	0	0	1	4	6	4	1	0	0	0
0	0	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0

others, are as shown in the second half of the table. The general agreement between the two sets of coefficients is evident, though it is not very close because $k = 4$ is not a large number. It will be seen that the largest tab-

ulated numbers form groups elongated in the direction of the free axis of Y. In accordance with (56), the numbers should properly be arranged in a square as in the figure.



The position of the axes of UV and of XY are there shown, and also the ellipse of probable error. The semi-axes of this ellipse as given by (51) and (57), are

$$a = 1.38dx,$$
$$b = 2.27dy,$$

where $dx = dy$ is the side of one of the small squares in the figure.

The ellipse encloses rather more than nine squares, allowing for fractions, but we can see that it is not far from an even chance that the algebraic sum of the x- and y-numbers on the balls drawn from the urn will be some one of the nine pairs.

0, 2	1, 2	2, 2
0, 1	1, 1	2, 1
0, 0	1, 0	2, 0

The most probable result is 1, 1. By following the analogy of this example we can construct an equation such as (58) to represent the expansion of any given polynomial, and to show the distribution of the errors resulting, in a given number of trials, from any set of elementary errors we please to assume.

It should be noticed, however, that the coefficients of a polynomial representing probabilities are all positive, whereas the general problem, to represent the limiting form of an expansion when some of the coefficients are negative, may give rise to special cases not included in our present investigation. The limiting surface, if there is one, will in gen'l be the probab'ty surface (43), but not always. In (29) it may happen that β_1 or β_2, or both of them, are zero, so that the differential equations (35) vanish, and our process fails. In such a case it is no longer sufficient to consider, as we did in connection with (26), that the element of the surface is a plane surface. Its curvature must be taken into account, and the limiting surface will most likely consist of an infinite series of undulations, analogous to those of the higher curves discussed in my former article. (See ANALYST, Sept. '79)

NOTES ON GAUSS' THEORIA MOTUS.

BY PROFESSOR ASAPH HALL.

WHILE at Harvard College Observatory in 1858, I began reading Davis' translation of the Theoria Motus. Since my mathematical knowledge had been got wholly from the small text books of our schools, the difficulty of reading such a work as this, and of obtaining a clear idea of the processes used by the author, was very great. I well remember that at first the style of the work fairly took me off my feet, and seemed to leave me dangling in the air for a month or two before I could get a firm foot-hold; and how at last the beauty and power of Gauss' methods were seen and felt. Having no teacher nor any one to assist me, I made it a rule to work out every equation and all the numerical examples before going on; and in this slow way I read to article 164. The rest of the book was passed over with less care. The whole reading occupied me nearly a year; and I consider myself very fortunate in having almost by chance hit upon one of the most perfect books ever written on theoretical astronomy. Admiral Davis' translation was sharply criticised in the American Journal of Science, Volume 26, p. 147, and some mistakes were pointed out. There are in fact a few errors of translation that confuse the meaning, but on the whole the translation seems to me well done, and the errors in the formulæ are very few.

I wish to notice here a few of the points and reductions that gave me the most trouble; and I have found since that these are also the ones over which students are apt to stumble.

(1). In article 3 Gauss uses a linear form for the equation of conic sections

$$r + \alpha x + \beta y = \gamma,$$

in which γ is always positive; and he calls this the characteristic equation of these curves. We see at once that this is an equation of the second degree, and hence the reader pays but little attention to it; but by referring to the Mécanique Celeste, Tome I, p. 165, the correctness and meaning of the statement will be seen.

(2). Among the formulæ of article 8, the following seem to me better for computing the true anomaly and the radius vector than those that Gauss recommends,

$$r \sin v = a \cos \varphi \sin E,$$

$$r \cos v = a (\cos E - e).$$

(3). The indirect method given in article 11 for the solution of Kepler's problem is probably the best way of solving this famous question that has ever been given in the case of orbits where the eccentricity is not very small. The opportunity that is left for the computer to use his judgement and make a close guess is a great advantage.

(4). Since the coefficient of $d\varphi$ in article 16 is generally found with difficulty I give my reductions. Substituting the values of $\frac{dp}{p}$ and $\frac{e \sin v}{1+e\cos v}$ $\times dv$, the value of dv being taken from the preceding article, we have

$$\frac{dr}{r}=\frac{da}{a}+\frac{a}{r}.\tan\varphi\sin v.dM+\left\{\frac{(2+e\cos v)e\sin v^2}{\cos\varphi(1+e\cos v)}-\frac{\cos\varphi\cos v}{1+e\cos v}-2\tan\varphi\right\}d\varphi$$

By successive reductions and noticing that $e=\sin\varphi$, the coefficient of $d\varphi$ takes the forms

$$+\frac{2\sin\varphi\sin v^2+\cos v\sin\varphi^2\sin v^2-2\cos v\sin\varphi^2-\cos v\cos\varphi^2-2\sin\varphi}{\cos\varphi(1+e\cos v)},$$

$$-\frac{2\sin\varphi\cos v^2+\sin\varphi^2\cos v^2+\cos v\sin\varphi^2+\cos v\cos\varphi^2}{\cos\varphi(1+e\cos v)};$$

$$-\frac{(1+e\cos v)^2.\cos v}{\cos\varphi(1+e\cos v)}.$$

Hence since

$$\frac{a}{r}=\frac{1+e\cos v}{\cos\varphi^2},$$

we have

$$dr=\frac{r}{a}.da+a\tan\varphi\sin v.dM-a\cos\varphi\cos v.d\varphi.$$

On sending the above reduction to Dr. Brünnow, then Prof'r of astronomy in the University of Michigan, in return I received from him the following solution.

All the equations between E, v and φ can be found from the formulæ of spherical trigonometry by means of a triangle given by B. Nicolai, a student with Gauss at Göttingen, and afterwards Director of the Observatory at Mannheim. From the equations given in article 8 we may assume the spherical triangle whose angles are $90°+E$; $90°-v$; and φ; the sides opposite being $90°-E$; $90°-v$, and φ. Thus we have

$$\sin E=\frac{\cos\varphi\sin v}{1+e\cos v}; \qquad \cos E=\frac{\cos v+e}{1+e\cos v};$$

$$\sin v=\frac{\cos\varphi\sin E}{1-e\cos E}; \qquad \cos v=\frac{\cos E-e}{1-e\cos E}.$$

From these equations we find

$$\cos E \cos v = \frac{\cos v^2 + e\cos v}{1 + e\cos v}; \quad \sin E \cos v = \frac{\cos\varphi \sin v \cos v}{1 + e\cos v};$$

$$\sin v \cos E = \frac{\cos\varphi \sin E \cos E}{1 - e\cos E}; \quad \cos v \sin E = \frac{\sin E \cos E - e\sin E}{1 - e\cos E}.$$

If we multiply the equation

$$\cos E \sin v = \frac{\cos v \sin v + e\sin v}{1 + e\cos v},$$

by $\cos\varphi$, and subtract the product from the value of $\sin E \cos v$ we have

$$\sin E \cos v - \cos E \sin v \cos\varphi = \frac{-e\cos\varphi \sin v}{1 + e\cos v} = -e\sin E.$$

Since $e = \sin\varphi$,

$$\cos E \cos v \cos\varphi = \frac{\cos\varphi - \cos\varphi \sin v^2 + \sin\varphi \cos\varphi \cos v}{1 + e\cos v}$$

$$= \cos\varphi - \sin E \sin v.$$

Hence

$$\cos\varphi = \sin v \sin E + \cos v \cos E \cos\varphi,$$

$$\sin\varphi \sin E = -\cos v \sin E + \sin v \cos E \cos\varphi,$$

$$\sin\varphi \cos E = \sin\varphi \cos E.$$

Similarly

$$\cos\varphi = \sin v \sin E + \cos v \cos E \cos\varphi,$$

$$\sin\varphi \sin v = \sin v \cos E - \cos v \sin E \cos\varphi,$$

$$\sin\varphi \cos v = \cos v \sin\varphi;$$

and these are the fundamental equations of spherical trigonometry.

If now we differentiate the equation

$$r = a(1 - e\cos E)$$

and then substitute the value of dE we find

$$dr = \frac{r}{a}.da + a\tan\varphi \sin v.dM + a.\left\{\sin\varphi \sin v \sin E - \cos\varphi \cos E\right\}.d\varphi.$$

Nicolai's triangle gives

$$\cos\varphi = \sin v \sin E + \cos v \cos E \cos\varphi,$$

$$\sin\varphi \sin v = \sin v \cos E - \cos v \sin E \cos\varphi.$$

Hence the coefficient of $ad\varphi$ in the value of dr is $-\cos\varphi \cos v$.

(5) In computing the maximum error for the hyperbola in section VII of article 32, I was obliged to solve a complete equation of the fourth degree. The numerical results agreed with those given by Gauss, but perhaps they may be got more easily, since in the other cases they are found by quadratic equations.

(6). In article 54 we have the elegant equations used by Gauss for the solution of a spherical triangle when from one side and the adjacent angles the other parts are to be found. These formulæ were published by Delambre in 1807 (Conn. des Tems, 1809, p. 445); and formulæ resembling them were given by Professor Mollweide in Germany; but it does not appear that Delambre or any one made use of these formulæ until they were applied by Gauss, and their convenience had been pointed out by him. In his review of the Theoria Mot. (Conn. des Tems 1812) Delambre says: "Quand j'eus trouvé ces formules j'en cherchai les applications qui pouvaient etre vraiment utile; n'en voyant aucune, je les donnai simplement comme curieuses." When we remember that Gauss had used these formulæ several years before their publication by Delambre, and that it was through Gauss' example that they came into general use, it seems to be only fair that they should be called the Gaussian formulæ, although of late the French, and some English writers, call them Delambre's formulæ. These formulæ are frequently used when it would be better and more accurate, I think, to apply the three fundamental equations of spherical trigonometry, with addition and subtraction logarithms.

(7) The method given in articles (55) and (56) for computing the equatorial coordinates of a planet is simple. In the appendix Admiral Davis has added other equations given by Gauss in his first paper on this method, and among these the convenient equation for checking the calculation, into which it will be noticed all the auxiliary quantities enter:

$$\tan i = \frac{\sin b \sin c \sin (C - B)}{\sin a \cos A}.$$

By an error of print the denominator is given as $\sin a \sin A$. This method of computing by rectangular coordinates is objected to by Hansen who prefers polar coordinates as being more accurate. Differential formulæ can be found for changing the Gaussian auxiliaries from one equinox to another, but they do not seem to give any practical advantage.

(8). The continued fractions given in articles 90 and 100 are explained by Gauss in one of his memoirs. [See Runkle's Mathematical Monthly, Vol. III, p. 262.]

(9). A very elegant and satisfactory discussion of Lambert's equation for each of the conic sections will be fonnd in articles 106 to 110. Many of the discussions of this problem given in recent books on astronomy aud rational mechanics are not much more than Gauss' discussion spoiled. In section IV will be found a condensed but pretty complete statement of the most useful relations between several places in space, such as that used by Olbers in computing the orbit of a comet.

(10). I will give one more note, and this on the derivation of the check equations at the end of article 140; the last of which gave me as much trouble as any thing in the whole book notwithstanding Gauss says: "quarum tamen derivationem non ita difficilem brevitatis caussa supprimimus."

If in figure 4 we extend the great circle $B''B$ to meet the circle AA'' in N', and apply equation I, article 140, we shall have

$$0 = \sin(l''-l')\sin N'A - \sin(l''-l)\sin N'A' + \sin(l'-l)\sin N'A''. \quad (1)$$

Proceeding now as in the beginning of article 140, and dividing equation (1) by R'' we have the first of the check equations. For the second equation, transpose and get the value of b from the first, and if we notice that

$$\sin(A''D'-\delta'') = \frac{\sin B \sin BB''}{\sin\epsilon'}; \quad \sin(AD'-\delta) = \frac{\sin B'' \sin BB''}{\sin\epsilon'}:$$

we shall have

$$b = \frac{R'\sin\delta'}{R''\sin\delta''} \cdot \frac{\sin BB''}{\sin(\delta'-\sigma)\sin(AD'-\delta)\sin\epsilon'} \left[\frac{\text{s.}\,\delta\,\text{s.}(l''-l')\,\text{s.}B + \text{s.}\,\delta''\,\text{s.}(l'-l)\,\text{s.}B''}{\sin(l''-l)}\right]$$

But we have

$$\sin N'A \sin N' = \sin\delta \sin B: \quad \sin N'A'' \sin N' = \sin\delta'' \sin B'',$$

and by (1) the factor in the brackets is reduced to

$$\sin N'A' \sin N'.$$

From the last equation of article 110 we have

$$\sin BB'' = \frac{\sin(\alpha''-\alpha)\cos\beta\cos\beta''}{\cos N'},$$

and the value of b becomes

$$b = \frac{R'\sin\delta'}{R''\sin\delta''} \cdot \frac{\cos\beta\cos\beta''}{\sin(\delta'-\sigma)\sin(AD'-\delta)\sin\epsilon'} \cdot \sin(\alpha''-\alpha)\sin N'A'\tan N'.$$

It is required therefore to reduce the factor $\sin(\alpha''-\alpha)\sin N'A'\tan N'$ to the form

$$\tan\beta\sin(\alpha''-l') - \tan\beta''\sin(\alpha-l') = S.$$

For this purpose I put

$$a'' - a = (a'' - l') - (a - l')$$

and expand $\sin(a'' - a)$: From the points B and B'' let fall perpendiculars on the great circle AA'', and denote the points of intersection by a and a''.

We have

$$N'A' = N'a - (a - l'): \; N'A' = N'a'' - (a'' - l'), \qquad (2)$$

$$\tan N' = \frac{\tan\beta}{\sin N'a} = \frac{\tan\beta''}{\sin N'a''}.$$

By means of these equations the factor $\sin(a'' - a)\sin N'A'\tan N'$ takes the symmetrical form

$$+\sin(a''-l')\tan\beta\cos^2(a-l') - \sin(a''-l')\tan\beta\cos(a-l')\sin(a-l')\cot N'a$$
$$-\sin(a-l')\tan\beta''\cos^2(a''-l') + \sin(a-l')\tan\beta''\sin(a''-l')\cos(a''-l')\cot N'a''.$$

If now we change $\cos^2$ to $1 - \sin^2$, we have the value of S and the remaining terms reduce to

$$\sin(a''-l')\sin(a-l')\tan N'.\{\cos[N'a''-(a''-l')] - \cos[N'a-(a-l')]\} = 0,$$

by (2). Hence we have the value given by Gauss

$$b = \frac{R'\sin\delta'}{R''\sin\delta''}\cdot\frac{\cos\beta\cos\beta''.S}{\sin(\delta' - \sigma)\sin(AD' - \delta)\sin\epsilon'}.$$

The preceding reduction is long, and quite likely it may be simplified. It will be seen that the relation

$$\sin(a''-a)\sin N'A'\tan N' = \tan\beta\sin(a''-l') - \tan\beta''\sin(a-l')$$

may be stated as a general theorem in spherical trigonometry, and perhaps some one may give a proof of it.

In reading a book like the Theoria Motus, the student is apt to be discouraged by the feeling that the writer of such a book was a man of wonderful power, and that his own efforts can not have much value. Now while such men were no doubt gifted with great natural ability, yet the history of their lives and their manner of work is full of encouragement to every earnest student. Thus we know that the first method devised by Gauss for computing the orbit of a planet was rude and clumsy compared with the elegant form in which it was published. It was by keeping the the problem steadily before his mind for several years, and carefully working out all its parts, that Gauss brought his solution at last to a form almost perfect. In the case of Laplace we know that at first he had erroneous notions on several subjects, and made mistakes, but he had the good sense and perseverence to correct his own errors, and at last produced the *Mécanique Celeste*.

NOTES ON THE THEORIES OF JUPITER AND SATURN.

BY G. W. HILL, NAUTICAL ALMANAC OFFICE, WASHINGTON, D. C.

[Continued from page 40.]

THE coordinates usually preferred by astronomers are the logarithm of the radius vector, the longitude and the latitude. We suppose that the two last are referred to the plane of maximum areas. Let these coordinates be denoted by the symbols $\log \rho$, λ and β; and let the subscript $({}_0)$ be applied to λ and β when we wish to designate the similar coordinates corresponding to the variables x, y, z, x', y', z'. Then we have

$$\begin{aligned}
\rho\cos\beta\cos\lambda &= r\cos\beta_0\cos\lambda_0 + \varkappa r'\cos\beta'_0\cos\lambda'_0,\\
\rho\cos\beta\sin\lambda &= r\cos\beta_0\sin\lambda_0 + \varkappa r'\cos\beta'_0\sin\lambda'_0,\\
\rho\sin\beta &= r\sin\beta_0 + \varkappa r'\sin\beta'_0 .
\end{aligned}$$

From the first two equations are readily obtained the following two:—

$$\begin{aligned}
\rho\cos\beta\cos(\lambda-\lambda_0) &= r\cos\beta_0 + \varkappa r'\cos\beta'_0\cos(\lambda'_0-\lambda_0),\\
\rho\cos\beta\sin(\lambda-\lambda_0) &= \varkappa r'\cos\beta'_0\sin(\lambda'_0-\lambda_0).
\end{aligned}$$

In the developments in infinite series which follow, the eccentricities of the orbits will be regarded as small quantities of the first order, the squares of the inclinations of the orbits on the plane of maximum areas as quantities of the third order, and $\varkappa$ also as a quantity of the third order. Then all terms, whose order is higher than the sixth, will be neglected. This degree of approximation will be found amply sufficient for the most refined investigations.

Under these conditions, we get

$$\begin{aligned}
\log\rho &= \log r + \tfrac{1}{2}\log\left[1+2\varkappa\frac{r'}{r}s+\varkappa^2\frac{r'^2}{r^2}\right]\\
&= \log r + \varkappa\frac{r'}{r}s + \tfrac{1}{2}\varkappa^2\frac{r'^2}{r^2}(1+2s^2),\\
\lambda &= \lambda_0 + \varkappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda'_0-\lambda_0) - \tfrac{1}{2}\varkappa^2\frac{r'^2}{r^2}\sin 2(\lambda'-\lambda_0),\\
\beta &= \beta_0 + \varkappa\frac{r'}{r}\beta'_0 - \varkappa\frac{r'}{r}s\beta_0 .
\end{aligned}$$

We will write η for $\sin^2\frac{1}{2}i$. Then, to the sufficient degree of approximation,

$$\varkappa\frac{r'}{r}s = -\varkappa\frac{r'}{r}\cos(v-v'+g-g') + 2\varkappa(\eta+\eta')^2\frac{a'}{a}\sin(l+g)\sin(l'+g').$$

In like manner

$$\begin{aligned}
\varkappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda'_0-\lambda_0) &= \varkappa(1+\eta^2-\eta'^2)\frac{r'}{r}\sin(v-v'+g-g')\\
&\quad -\varkappa\eta^2\frac{a'}{a}\sin(3l-l'+3g-g') + \varkappa\eta'^2\frac{a'}{a}\sin(l+l'+g+g').
\end{aligned}$$

The expressions for λ_0 and β_0 in terms of elliptic elements are given by Delaunay.* Log r, as well as the following expressions

$$\frac{r'}{a'}\genfrac{}{}{0pt}{}{\cos}{\sin}(v'+g') = -\tfrac{3}{2}e'\genfrac{}{}{0pt}{}{\cos}{\sin}g'+[1-\tfrac{1}{2}e'^2]\genfrac{}{}{0pt}{}{\cos}{\sin}(l'+g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3]\genfrac{}{}{0pt}{}{\cos}{\sin}(2l'+g')$$
$$+\tfrac{3}{8}e'^2\genfrac{}{}{0pt}{}{\cos}{\sin}(3l'+g')+\tfrac{1}{3}e'^3\genfrac{}{}{0pt}{}{\cos}{\sin}(4l'+g')$$
$$\pm\tfrac{1}{8}e'^2\genfrac{}{}{0pt}{}{\cos}{\sin}(l'-g')\pm\tfrac{1}{24}e'^3\genfrac{}{}{0pt}{}{\cos}{\sin}(2l'-g'),$$

$$\frac{a}{r}\genfrac{}{}{0pt}{}{\cos}{\sin}(v+g) = -[\tfrac{1}{2}e+\tfrac{1}{8}e^3]\genfrac{}{}{0pt}{}{\cos}{\sin}g+[1-e^2]\genfrac{}{}{0pt}{}{\cos}{\sin}(l+g)+[\tfrac{3}{2}e-\tfrac{7}{4}e^3]\genfrac{}{}{0pt}{}{\cos}{\sin}(2l+g)$$
$$+\tfrac{17}{8}e^2\genfrac{}{}{0pt}{}{\cos}{\sin}(3l+g)+\tfrac{71}{24}e^3\genfrac{}{}{0pt}{}{\cos}{\sin}(4l+g)$$
$$\mp\tfrac{1}{8}e^2\genfrac{}{}{0pt}{}{\cos}{\sin}(l-g)\mp\tfrac{1}{12}e^3\genfrac{}{}{0pt}{}{\cos}{\sin}(2l-g),$$

are found in a memoir by Prof. Cayley.† With these data we get

$$\log\rho = \log a+\tfrac{1}{4}e^2+\tfrac{1}{32}e^4+\tfrac{1}{96}e^6+x^2\frac{a'^2}{a^2}$$
$$-[e-\tfrac{3}{8}e^3-\tfrac{1}{64}e^5]\cos l-[\tfrac{3}{4}e^2-\tfrac{11}{24}e^4+\tfrac{3}{64}e^6]\cos 2l$$
$$-[\tfrac{17}{24}e^3-\tfrac{77}{128}e^5]\cos 3l-[\tfrac{71}{96}e^4-\tfrac{129}{160}e^6]\cos 4l$$
$$-\tfrac{523}{640}e^5\cos 5l-\tfrac{899}{960}e^6\cos 6l$$
$$-x\frac{a'}{a}\Big\{[1-e^2-\tfrac{1}{2}e'^2-(\eta+\eta')^2]\cos(l-l'+g-g')$$
$$+[\tfrac{3}{2}e-\tfrac{7}{4}e^3-\tfrac{3}{4}ee'^2]\cos(2l-l'+g-g')+[-\tfrac{3}{2}e'+\tfrac{3}{2}e^2e']$$
$$\times\cos(l+g-g')$$
$$+[-\tfrac{1}{2}e-\tfrac{1}{8}e^3+\tfrac{1}{4}ee'^2]\cos(l'-g+g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3-\tfrac{1}{2}e^2e']$$
$$\times\cos(l-2l'+g-g')$$
$$+\tfrac{17}{8}e^2\cos(3l-l'+g-g')-\tfrac{1}{8}e^2\cos(l+l'-g+g')$$
$$+\tfrac{3}{4}ee'\cos(g-g')-\tfrac{9}{4}ee'\cos(2l+g-g')$$
$$-\tfrac{1}{4}ee'\cos(2l'-g+g')+\tfrac{3}{4}ee'\cos(2l-2l'+g-g')$$
$$+\tfrac{3}{8}e'^2\cos(l-3l'+g-g')+\tfrac{1}{8}e'^2\cos(l+l'+g-g')$$
$$+\tfrac{71}{24}e^3\cos(4l-l'+g-g')-\tfrac{1}{12}e^3\cos(2l+l'-g+g')$$
$$-\tfrac{51}{16}e^2e'\cos(3l+g-g')+\tfrac{3}{16}e^2e'\cos(l-g+g')$$
$$+\tfrac{17}{16}e^2e'\cos(3l-2l'+g-g')-\tfrac{1}{16}e^2e'\cos(l+2l'-g+g')$$
$$-\tfrac{3}{16}ee'^2\cos(3l'-g+g')+\tfrac{9}{16}ee'^2\cos(2l-3l'+g-g')$$
$$-\tfrac{1}{16}ee'^2\cos(l'+g-g')+\tfrac{3}{16}ee'^2\cos(2l+l'+g-g')$$
$$+\tfrac{1}{3}e'^3\cos(l-4l'+g-g')+\tfrac{1}{24}e'^3\cos(l+2l'+g-g')$$

* *Théorie du Mouvement de la Lune.* Tom. I. pp. 56–59.

† *Tables of the Developments of Functions in the Theory of Elliptic Motion.* *Mem. Roy. Astr. Soc.*, Vol. XXIX, p. 191.

$$
\begin{aligned}
& \left.+(\eta+\eta')^2 \cos(l+l'+g+g')\right\} \\
& +\tfrac{1}{2}x^2\frac{a'^2}{a^2}\cos(2l-2l'+2g-2g'),
\end{aligned}
$$

$$
\begin{aligned}
\lambda = {} & l+g+h+[2e-\tfrac{1}{4}e^3+\tfrac{5}{96}e^5]\sin l+[\tfrac{5}{4}e^2-\tfrac{11}{24}e^4+\tfrac{17}{192}e^6]\sin 2l \\
& +[\tfrac{13}{12}e^3-\tfrac{43}{64}e^5]\sin 3l+[\tfrac{103}{96}e^4-\tfrac{451}{480}e^6]\sin 4l \\
& +\tfrac{1097}{960}e^5\sin 5l+\tfrac{1223}{960}e^6\sin 6l \\
& +[-\eta^2-\eta^4+4\eta^2e^2]\sin(2l+2g)+\tfrac{1}{2}\eta^4\sin(4l+4g) \\
& +[-2\eta^2e+\tfrac{27}{4}\eta^2e^3]\sin(3l+2g)+[2\eta^2e-\tfrac{7}{4}\eta^2e^3]\sin(l+2g) \\
& -\tfrac{13}{4}\eta^2e^2\sin(4l+2g)-\tfrac{3}{4}\eta^2e^2\sin 2g \\
& -\tfrac{59}{12}\eta^2e^3\sin(5l+2g)+\tfrac{1}{12}\eta^2e^3\sin(l-2g) \\
& +x\frac{n'}{a}\Big\{[1-e^2-\tfrac{1}{2}e'^2+\eta^2-\eta'^2]\sin(l-l'+g-g') \\
& \quad +[\tfrac{3}{2}e-\tfrac{7}{4}e^3-\tfrac{3}{4}ee'^2]\sin(2l-l'+g-g')+[\tfrac{1}{2}e+\tfrac{1}{8}e^3-\tfrac{1}{4}ee'^2] \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\sin(l'-g+g') \\
& \quad +[-\tfrac{3}{2}e'+\tfrac{3}{2}e^2e']\sin(l+g-g')+[\tfrac{1}{2}e'-\tfrac{3}{8}e'^3-\tfrac{1}{2}e^2e'] \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\sin(l-2l'+g-g') \\
& \quad +\tfrac{17}{8}e^2\sin(3l-l'+g-g')+\tfrac{1}{8}e^2\sin(l+l'-g+g') \\
& \quad +\tfrac{3}{4}ee'\sin(g-g')-\tfrac{9}{4}ee'\sin(2l+g-g')+\tfrac{1}{4}ee'\sin(2l'-g+g') \\
& \quad +\tfrac{3}{4}ee'\sin(2l-2l'+g-g')+\tfrac{3}{8}e'^2\sin(l-3l'+g-g') \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\tfrac{1}{8}e'^2\sin(l+l'+g-g') \\
& \quad +\tfrac{71}{24}e^3\sin(4l-l'+g-g')+\tfrac{1}{12}e^3\sin(2l+l'-g+g') \\
& \quad -\tfrac{51}{16}e^2e'\sin(3l+g-g')-\tfrac{3}{16}e^2e'\sin(l-g+g') \\
& \quad +\tfrac{17}{16}e^2e'\sin(3l-2l'+g-g')+\tfrac{1}{16}e^2e'\sin(l+2l'-g+g') \\
& \quad +\tfrac{3}{16}ee'^2\sin(3l-g+g')+\tfrac{9}{16}ee'^2\sin(2l-3l'+g-g') \\
& \quad -\tfrac{1}{16}ee'^2\sin(l'+g-g')+\tfrac{3}{16}ee'^2\sin(2l+l'+g-g') \\
& \quad +\tfrac{1}{3}e'^3\sin(l-4l'+g-g')+\tfrac{1}{24}e'^3\sin(l+2l'+g-g') \\
& \quad \left.-\eta^2\sin(3l-l'+3g-g')+\eta'^2\sin(l+l'+g+g')\right\} \\
& +\tfrac{1}{2}x^2\frac{a'^2}{a^2}\sin(2l-2l'+2g-2g'),
\end{aligned}
$$

$$
\begin{aligned}
\beta = {} & [2\eta-2\eta e^2+\tfrac{7}{32}\eta e^4]\sin(l+g)-\tfrac{1}{3}\eta^3\sin(3l+3g) \\
& +[2\eta e-\tfrac{5}{2}\eta e^3]\sin(2l+g)-2\eta e\sin g \\
& +[\tfrac{9}{4}\eta e^2-\tfrac{27}{8}\eta e^4]\sin(3l+g)+[\tfrac{1}{4}\eta e^2-\tfrac{1}{24}\eta e^4]\sin(l-g) \\
& +\tfrac{8}{3}\eta e^3\sin(4l+g)+\tfrac{1}{6}\eta e^3\sin(2l-g) \\
& +\tfrac{625}{192}\eta e^4\sin(5l+g)+\tfrac{9}{64}\eta e^4\sin(3l-g) \\
& -\eta^3e\sin(4l+3g)+\eta^3e\sin(2l+3g) \\
& +x\frac{a'}{a}\Big\{\eta\sin(2l-l'+2g-g')+[\eta+2\eta']\sin(l'+g') \\
& \quad +\tfrac{5}{2}\eta e\sin(3l-l'+2g-g')-\tfrac{3}{2}\eta e\sin(l-l'+2g-g')
\end{aligned}
$$

$$-\tfrac{3}{2}\eta e' \sin(2l+2g-g')+\tfrac{1}{2}\eta e' \sin(2l-2l'+2g-g')$$
$$+\tfrac{1}{2}[\eta+2\eta'] \, e \sin(l+l'+g')-\tfrac{1}{2}[\eta+2\eta'] \, e \sin(l-l'-g')$$
$$\left.+\tfrac{1}{2}[\eta+2\eta'] \, e' \sin(2l'+g')-\tfrac{3}{2}[\eta+2\eta'] e' \sin g' \right\}.$$

As written, these expressions give the coordinates of Jupiter. Those of Saturn are obtained by removing the accent from all the accented symbols, and applying it to those which are unaccented, x excepted, for which we have $x'=x$. Also it is to be remembered that we have $h'=h+180°$.

The coordinates of the two planets are obtained by employing in these formulæ, for the quantities involved in them, the values they actually have at the time in question. The latter are determined by the differential equations previously given; but instead of integrating these equations in one step, we may, as Delaunay has done in the lunar theory, divide the process into a series of transformations of the variables involved; each of which must be made not only in the expressions for $\log \rho$, λ, β, $\log \rho'$, λ', β'; but also in R.

As the introduction of l as the independent variable does not appear to be advantageous, we will suppose that the six variables L, L', Γ, l, l', γ are employed and that t is the independent variable.

Delaunay's method, somewhat amplified, amounts to this:— selecting the argument $\theta = il+i'l'+i''\gamma$, suppose, for the moment, that R is limited to the terms

$$-B-A_1 \cos(il+i'l'+i''\gamma)-A_2 \cos 2(il+i'l'+i''\gamma)+\ldots,$$

where B, A_1 &c., are functions of L, L' and Γ only. Then if it is found that the differential equations, corresponding to this limited R, are satisfied by the infinite series

$$\theta = \theta_0(t+c)+\theta_1 \sin\theta_0(t+c)+\theta_2 \sin 2\theta_0(t+c)+\ldots,$$
$$l=(l)+l_0(t+c)+l_1 \sin\theta_0(t+c)+l_2 \sin 2\theta_0(t+c)+\ldots,$$
$$l'=(l')+l'_0(t+c)+l'_1 \sin\theta_0(t+c)+l'_2 \sin 2\theta_0(t+c)+\ldots,$$
$$\gamma=(\gamma)+\gamma_0(t+c)+\gamma_1 \sin\theta_0(t+c)+\gamma_2 \sin 2\theta_0(t+c)+\ldots,$$
$$L=L_0+L_1 \cos\theta_0(t+c)+L_2 \cos 2\theta_0(t+c)+\ldots,$$
$$L'=L'_0+L'_1 \cos\theta_0(t+c)+L'_2 \cos 2\theta_0(t+c)+\ldots,$$
$$\Gamma=\Gamma_0+\Gamma_1 \cos\theta_0(t+c)+\Gamma_2 \cos 2\theta_0(t+c)+\ldots,$$

where c, (l), (l') and (γ) are arbitrary constants, the last three being equivalent to two independent constants, as we have the relation

$$i(l)+i'(l')+i''(\gamma)=0,$$

and all the other coefficients are known functions of three other constants a, a' and e, we can replace

$$L \text{ by } L_0+L_1 \cos(il+i'l'+i''\gamma)+L_2 \cos 2(il+i'l'+i''\gamma)+\ldots,$$
$$L' \text{ by } L'_0+L'_1 \cos(il+i'l'+i''\gamma)+L'_2 \cos 2(il+i'l'+i''\gamma)+\ldots,$$

$$
\begin{aligned}
\Gamma \text{ by } & \Gamma_0 + \Gamma_1 \cos(il+i'l'+i''\gamma) + \Gamma_2 \cos 2(il+i'l'+i''\gamma) + \ldots, \\
l \text{ by } & l \;+ l_1 \sin(il+i'l'+i''\gamma) + \; l_2 \sin 2(il+i'l'+i''\gamma) + \ldots, \\
l' \text{ by } & l' \;+ l'_1 \sin(il+i'l'+i''\gamma) + \; l'_2 \sin 2(il+i'l'+i''\gamma) + \ldots, \\
\gamma \text{ by } & \gamma \;+ \gamma_1 \sin(il+i'l'+i''\gamma) + \gamma_2 \sin 2(il+i'l'+i''\gamma) + \ldots,
\end{aligned}
$$

and will have, for determining the new variables l, l', γ, a, a', e, precisely the same differential equations as we started with, provided we make all these substitutions in the function R, and regard the new variables L, L', Γ as connected with a, a', e by the relations·

$$
\begin{aligned}
L &= L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + \ldots), \\
L' &= L'_0 + \tfrac{1}{2}(\theta_1 L'_1 + 2\theta_2 L'_2 + \ldots), \\
\Gamma &= \Gamma_0 + \tfrac{1}{2}(\theta_1 \Gamma_1 + 2\theta_2 \Gamma_2 + \ldots).
\end{aligned}
$$

It will be perceived that as long as we are dealing with terms of R, whose arguments involve l or l' or both, the second members of the three equations, last written, have values which differ from the elliptic values of L, L' and Γ only by quantities of the second order with respect to disturbing forces. Hence, if we propose to neglect third order terms, until we have reduced R to a function of the argum't γ only, we can assume that L, L' and Γ which are the elements conjugate to the arguments l, l' and γ, are expressed throughout in terms of a, a' and e, in the same way as in the elliptic theory. It may be added that these third order terms are found in experience to be much smaller than those which arise in other ways.

[To be continued.]

SOME RELATIONS DEDUCED FROM EULER'S THEOREM ON THE CURVATURE OF SURFACES.

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LET α = angle of any normal section with line of maximum curvature,
k_α = curvature of this normal section,
k_m = maximum curvature,
k_n = minimum curvature; then by Euler's theorem:

$$k_\alpha = k_m \cos^2\alpha + k_n \sin^2\alpha, \tag{1}$$

$$\therefore\ k_{\alpha+\frac{1}{2}\pi} = k_m \sin^2\alpha + k_n \cos^2\alpha.$$

Adding, we obtain the well known relation:

$$k_\alpha + k_{\alpha+\frac{1}{2}\pi} = k_m + k_n = \text{constant}. \tag{2}$$

But multiplying we have

$$
\begin{aligned}
k_\alpha k_{\alpha+\frac{1}{2}\pi} &= (k_m^2 + k_n^2)\sin^2\alpha\cos^2\alpha + k_m k_n(\cos^4\alpha + \sin^4\alpha) \\
&= \tfrac{1}{4}(k_m + k_n)^2 \sin^2 2\alpha + k_m k_n \cos^2 2\alpha.
\end{aligned} \tag{3}
$$

Place

$$k_{m+n} = \tfrac{1}{2}(k_m + k_n) = \tfrac{1}{2}(k_\alpha + k_{\alpha+\frac{1}{2}\pi}) = k_{\frac{1}{4}\pi}, \tag{4}$$

$$k_{mn} = \surd(k_m k_n), \tag{5}$$

then
$$k_a k_{a+\frac{1}{2}\pi} = k^2_{m+n} \sin^2 2a + k^2_{mn} \cos^2 2a. \tag{3'}$$

From (4) and (3′) we can form the quadratic:

$$k^2 - 2k_{m+n}k + k^2_{m+n}\sin^2 2a + k^2_{mn}\cos^2 2a = 0, \tag{6}$$

whence
$$k_a = k_{m+n} + \cos 2a \surd(k^2_{m+n} - k^2_{mn}). \tag{7}$$

This formula although apparently more complicated than (1) is nevertheless of greater practical value for the following reasons:

The prin'l curvatures k_m and k_n on any surface are given by the quadratic

$$k^2 - \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}k + \frac{rt - s^2*}{1+p^2+q^2} = 0; \tag{8}$$

hence by (4)
$$k_{m+n} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{2(1+p^2+q^2)^{\frac{3}{2}}}, \tag{9}$$

and by (5)
$$k_{mn} = \frac{\surd(rt - s^2)}{1+p^2+q^2}. \tag{10}$$

Thus we see that these two quantities are directly given without solving a complete quadratic. Their geometrical signification is as follows:

We have evidently
$$k_{m+n} = k_{\frac{1}{4}\pi}, \tag{11}$$

also
$$k_{m+n} = \int_0^\pi k_a da \div \int_0^\pi da = \frac{1}{\pi}\int_0^\pi (k_m \cos^2 a + k_n \sin^2 a) da \tag{12}$$

$$= \frac{1}{\pi}\int_0^\pi [k_{m+n} + \cos 2a \surd(k_{m+n}{}^2 - k_{mn}{}^2)] da$$

$$= k_{m+n} + \left[\frac{\surd(k_{m+n}{}^2 - k_{mn}{}^2)}{2\pi}\sin 2a\right]_0^\pi,$$

it is therefore the mean curvature.

Since $k_m k_n$ is the measure of curvature according to Gauss, therefore $1 \div k_{mn}$ is the radius of a sphere of the same curvature (13); also

$$\frac{1}{k_{mn}} = \int_0^\pi \frac{da}{ka} \div \int_0^\pi da = \frac{1}{\pi}\int_0^\pi \frac{da}{k_m \cos^2 a + k_n \sin^2 a}$$

$$= \frac{1}{\pi\surd(k_m k_n)}\int_0^\pi \frac{da \surd(k_n \div k_m)\sec^2 a}{1 + (k_n \div k_m)\tan^2 a}$$

$$= \frac{1}{\pi\surd(k_m k_n)}\left[\tan^{-1}\left\{\sqrt{\left(\frac{k_n}{k_m}\right)}\tan a\right\}\right]_0^\pi$$

$$= \frac{1}{\surd(k_m k_n)}.$$

Thus we see $1 \div k_{mn} =$ mean radius of curvature.

Replacing a by $a + \frac{1}{4}\pi$ in (3′) we have

$$k_{a+\frac{1}{4}\pi} k_{a+\frac{3}{4}\pi} = k^2_{m+n}\cos^2 2a + k^2_{mn}\sin^2 2a.$$

Adding this to (3′) we obtain the very remarkable relation

$$k_a k_{a+\frac{1}{2}\pi} + k_{a+\frac{1}{4}\pi} k_{a+\frac{3}{4}\pi} = k^2_{m+n} + k^2_{mn} = \text{constant}. \tag{15}$$

---

*This is Monge's notation; $p = (dr \div dx)$; $q = (dz \div dy)$; $r = (d^2z \div dx^2)$; $s = (d^2z \div dxdy)$; $t = (d^2z \div dy^2)$.

That is, the sum of the products of the curvatures of one pair of cross sections and of another pair of cross sections bisecting the first is *constant.*

This remarkable theorem, analogous to (2), is but a particular case of a more general one, which I give without proof, which is easy however.

If a pair of cross sections is continually bisected and after N bisections the product of curvatures taken, and if we add to this the product of the curvatures of a system of sections bisecting the first, the sum is constant.

From the examination of some special cases by Mr. Alex. S. Christie of U. S. Coast Surv. it seemed probable that in general the sum of N curvatures evenly distributed around a point; the sum of their products by two, by three . . . , by N—1 is also constant; but that their product depends on the azimuth. To prove this I proceed as follows:

We have by (7), if for brevity we place $k_{m-n} = \sqrt{(k_{m+n}{}^2 - k_{mn}{}^2)}$, (16)

$$k_a = k_{m+n} + k_{m-n} \cos 2a. \qquad (7')$$

We have then $\Sigma_0^{N-1}[k_{a+(v\div N)\pi}] = Nk_{m+n} = \text{constant},$ (17)

$$\begin{aligned}\Sigma_0^{N-1}[k^2_{a+(v\div N)\pi}] &= Nk^2_{m+n} + k^2_{m-n}\Sigma_0^{N-1}\{\cos^2 2[a+(\nu\div N)\pi]\} \\ &= Nk^2_{m+n} + \tfrac{1}{2}k^2_{m-n}\Sigma_0^{N-1}\{1+\cos 4[a+(\nu\div N)\pi]\} \\ &= Nk^2_{m+n} + \tfrac{1}{2}Nk^2_{m-n} = \text{constant}. \qquad (18)\end{aligned}$$

Hence since $\Sigma_0^{N-1}[k^2_{a+(v\div N)p}{}^*] = [\Sigma_0^{N-1}(k_{a+(v\div N)p})]^2 - 2\Sigma_0^{N-1}[k_{a+(v\div N)p}k_{a+(v'\div N)p}],$

we have the sum of the products by two

$$\begin{aligned}\Sigma_0^{N-1}[k_{a+(v\div N)p}k_{a+(v'\div N)p}] &= \tfrac{1}{2}N^2k^2_{m+n} - \tfrac{1}{2}Nk^2_{m+n} - \tfrac{1}{4}N(k^2_{m+n} - k^2_{mn} \\ &= \tfrac{1}{2}N.\tfrac{1}{2}(2N-3)k^2_{m+n} + \tfrac{1}{4}Nk^2_{mn} = \text{const.} \qquad (19)\end{aligned}$$

Again
$$\begin{aligned}\Sigma_0^{N-1}[k^3_{a+(v\div N)p}] &= Nk^3_{m+n} + \tfrac{3}{2}Nk_{m+n}k^2_{m-n} \\ &= \tfrac{5}{2}Nk^3_{m+n} - \tfrac{3}{2}Nk_{m+n}k^2_{mn} = \text{constant}. \qquad (20)\end{aligned}$$

Since
$$\begin{aligned}\Sigma_0^{N-1}[k^3_{a+(v\div N)p}] &= [\Sigma_0^{N-1}(k_{a+(v\div N)p})]^3 - 3[\Sigma_0^{N-1}(k_{a+(v\div N)p})\Sigma_0^{N-1}) \\ &\quad (k^2_{a+(v\div N)p}) - \Sigma_0^N(k^3_{a+(v\div N)p})] - 6\Sigma_0^{N-1}(k_{a+(v\div N)p}k_{a+(v'\div N)p}k_{a+(v''\div N)p} \\ &= -2[\Sigma_0^{N-1}(k_{a+(v\div N)p})]^3 + 6\Sigma_0^{N-1}(k_{a+(v\div N)p})\Sigma(k_{a+(v\div N)p}k_{a+(v'\div N)p}) \\ &\quad + 3\Sigma_0^{N-1}(k_{a+(v\div N)p}) - 6\Sigma_0^{N-1}(k_{a+(v\div N)p}k_{a+(v'\div N)p}k_{a+(v''\div N)p}), \qquad (21)\end{aligned}$$

we infer that the sum of the products by three of the N curvatures is const.

It is easy to see that the sum of the powers, as well as the products by four, five, etc., must be constant, but with this exception, that it does not hold for the sum of the Nth powers nor the product of the N curvatures; for we have

$$\begin{aligned}\Sigma_0^{N-1}[k^N_{a+(v\div N)p}] &= \text{constant} + k^N_{m-n}\Sigma_0^{N-1}\{\cos^N 2[a+(\nu\div N)\pi]\} \\ &= \text{const.} + \frac{1}{2N\text{-}1}k^N_{m-n}\Sigma_0^{N-1}\left\{\cos 2(Na+\nu\pi)\right\} \\ &\qquad + \text{constant} \\ &= \text{const.} + \frac{N}{2N\text{-}1}k^N_{m-n}\cos 2Na; \qquad (22)\end{aligned}$$

therefore also the product of the curvatures is not constant.

*For want of Greek sorts the subscript p is here written for π, and subsc'pt v stands for ν.

A BRIEF ACCOUNT OF THE ESSENTIAL FEATURES OF GRASSMANN'S EXTENSIVE ALGEBRA.*

[Given by the author in Grunert's Archiv, Vol. VI, 1845.]

TRANSLATED BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

I. AIM OF THE EXTENSIVE ALGEBRA AS SUCH.

1. *My Extensive Algebra forms the abstract foundation of Geometry*, i. e., *it is the pure mathematical science divested of all spatial considerations, whose special application to space is Geometry.*

Geometry, inasmuch as it rests upon something given in nature, viz., space, is not a branch of pure mathematics, but an application of the same to nature; yet not a mere application of algebra, even when the algebraic quantity, as in the theory of functions, is regarded as varying continuously; for algebra lacks the notion of different dimensions, which is characteristic of geometry. Hence we need a branch of mathematics which incorporates with the notion of the continuously varying quantity the notion of differences (Verschiedenheiten) corresponding to the dimensions of space. This branch is my extensive algebra.

2. *The theorems of the extensive algebra are not mere translations of geometrical theorems into abstract language; they have a far more general significance, for while geometry is restricted to the three dimensions of space, the abstract science is free from these restrictions.*

In geometry, lines may be generated by the movement of points, surfaces by the movement of lines, volumes by the movement of surfaces, but further geometry cannot go. On the contrary, if we conceive abstract notions independent of space to be introduced in the place of the point and movement, these restrictions disappear.

3. *Hence it follows that the theorems of geometry have a tendency toward generality, which finds no satisfaction in geometry, on account of its restriction to three dimensions, but does find complete satisfaction for the first time in the extensive algebra.*

A couple of examples will make this plain. (1) Two right lines in the same plane intersect in one point, likewise a plane and a right line, two planes intersect in one right line, provided that the right lines, or the plane and the right line, or the two planes, do not coincide, and the intersections at infinity are included. If the point, the right line, the plane, and the solid, be regarded as fields (Gebiete) of the first, second, third, and fourth orders (Stufe) respectively, then in this is suggested the general theorem,

*Prof. Clifford's translation of "Ausdehnungslehre". Tr.

that if a field of the ath order and one of the bth intersect in a field of the cth order, but not in a field of lower order, they will have a field of the $(a+b-c)$th order in common. Geometry, however, can illustrate this theorem only when c is equal to, or less than, 4. (2) The area of a triangle is half that of a parallelogram whose sides are equal in length and parallel, to two sides of the triangle; the volume of a tetraedron is one sixth that of the parallelepiped whose edges are equal in length and parallel, to three edges of the tetraedron meeting in a common point. In this is evidently suggested the theorem, that the space included within n points taken together in a field of the nth order (and not in a field of lower order) is $\frac{1}{1.2.3\ldots(n-1)}$th of the space enclosed by a figure (Gebilde) whose bounding lines (edges) are equal in length and parallel, to the right lines joining one of the n points to all the rest. But even here this theorem does not appear in all its generality. On the contrary, in the extensive algebra—in these two cases, and in all others — the theorems hold true in their most general form. Thus everywhere geometry takes the first step toward generality, but without being able to reach it, strikes against the barriers set up by the idea of space, which barriers only the abstract science of extensive algebra is capable of breaking down.

4. *The figure (Gebilde) in the extensive algebra which corresponds to the line is the aggregate of the elements resulting from a continuous variation of the state of the original element.*

The line may be regarded as the aggregate of the points resulting from a continuous change of position of a point. If we substitute here for the p't, more generally, any object susceptible of a continuous variation of any state in which it is, and then divest it of every other property and every singularity of its state, and call the object, thus divested of every other property, the element, we arrive at the idea set forth.

5. *If now the element continuously varies its state in the same way, so that after passing from an element* a *of the figure (Gebilde) to a second element* b *of the same by one such variation, by a second we pass from* B *to an element* c *of the same figure, we obtain the figure corresponding to the* RIGHT *line, viz., the field of the second order.*

The right line is generated by the point continually varying its position in the same direction; if we substitute for direction mode of variation, we arrive at the idea set forth.*

[To be continued.] to p. [illegible]

*If the right line and the figure corresponding to it are to be infinite both ways, the point (the element) must vary its position in the opposite direction (its state in the opposite manner) also, a thought which for simplicity we have neglected here.

NOTE ON PROF. HALL'S QUERY IN VOL. VII, NO. FOUR.

BY PROF. ORMOND STONE.

As ∇v does not assume eight values at the surface of an attracting body, Todhunter does not give the explanation mentioned by Prof. Eddy in the last number of this journal. What Todhunter does say is that "∇v is an aggregate of three terms, each of which has two values; so that there are in all eight *combinations*, of which one gives the value of ∇v agreeing with that found for an internal particle, and the other gives the value of ∇v agreeing with that found for an external particle; the other six remain without meaning." In other words, there are only *two* values of ∇v, namely, $-4\pi\rho$ and 0. The reason that the remaining combinations are "without meaning" lies in the fact, as I have before stated, that the second differential coefficients of v with regard to x, y, z are *not independent* of one another.

ANSWER TO QUERY (SEE PAGE 63) BY W. E. HEAL.—There are several methods of elimination between equations described in the query.

The following methods are explained in Salmon's Higher Algebra, third edition:—Elimination by Symmetric Functions, Elimination by Greatest Common Divisor, Euler's method, Sylvester's dialytic method, Bezout's method and Caley's statement of Bezout's method.

SOLUTION OF PROB. 338 (SEE P. 31) BY PROF. ORMOND STONE.—Let a and δ be the heliocentric right ascension and declination of the perihelion of the comet's orbit; the comet will evidently approach a point opposite the perihelion, i. e., a point whose right ascension and declination are $a+180°$ and $-\delta$. To find a and δ, we have

$$\tan(a-\Omega) = \cos i \tan(\pi-\Omega),$$
$$\sin\delta \qquad = \sin i \sin(\pi-\Omega),$$

where i is the inclination of the orbit to the equator and $\pi-\Omega$ the distance of the perihelion from the node.

NOTE BY PROF. E. B. SEITZ.—Eq. (4) of Mr. Heal's solution of 334, p. 60, is wrong. From (1) and (2) we see that the tangents of the angles bet. the tang't lines and the axis of x are $-b\cos\theta\div a\sin\theta$ and $-b\cos\varphi\div a\sin\varphi$; hence by the formula for the tangent of the diff of two angles

$$\tan a = \left(\frac{b\cos\varphi}{a\sin\varphi}-\frac{b\cos\theta}{a\sin\theta}\right) \div \left(1+\frac{b^2\cos\theta\cos\varphi}{a^2\sin\theta\sin\varphi}\right) = \frac{ab(\sin\theta\cos\varphi-\cos\theta\sin\varphi)}{a^2\sin\theta\sin\varphi+b^2\cos\theta\cos\varphi}.$$

SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From Prof. L. G. Barbour, 340; Prof. W. P. Casey, 340, 341; G. E. Curtis, 339, 340; Prof. E. J. Edmunds, 340; Geo Eastwood, 340, 345; Prof. A. B. Evans, 340; W. E. Heal, 340; Prof. E. W. Hyde, 343; Wm. Hoover, 339, 340; Prof. D. J. Mc Adam, 340; C. H. Metcalf, 340; Prof. E. B. Seitz, 344; E. Vansickel, 340; R. S. Woodward, 339, 340, 345.

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339. "Required the average distance from the center of a circle to all points in the surface of a sector."

SOLUTION BY G. E. CURTIS, YALE COLLEGE, CONN.

Let the arc of the sector be denoted by a and the distance from the centre to any point of the surface by ρ. Then the sum of the distances from the center to all points of the sector will be

$$\int_0^a\int_0^r \rho.\rho d\rho d\theta = \tfrac{1}{3}r^3a$$

Hence the average distance is $\frac{1}{3}r^3a \div \frac{1}{2}r^2a = \frac{2}{3}r$

340. "Integrate $\dfrac{dx}{\sin x + \cos x}$."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

$$\int\frac{dx}{\sin x+\cos x} = \int\frac{\sin x\,dx}{\sin^2 x+\sin x\cos x} = \int\frac{dz}{1-z^2+(1-z^2)^{\frac{1}{2}}z}, \text{ if } z=\cos x.$$

By integrating, reducing and restoring the value of z, we find

$$\int\frac{dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right).$$

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141. "Show that 'Every even number is the sum of two prime numbers, and every odd number is the sum of three prime numbers.' Barlow's Theory of Numbers, page 259."

SOLUTION BY PROF. E. J. EDMUNDS, SOUTH'N. UNIV., N. ORLEANS, LA.

It is well known that every prime number is of the form $6x\pm1$, x being any integer. Hence

$$6x+1+6x-1 = 12x$$

which is an even number. We have also, putting $x = n$,

$$2n+1 = 6x+1+6x-1+1 = \text{an odd number}.$$

342. No solution received.

343. "If E^2 be the sum of the squares of the edges of a tetrahedron, F^2 the sum of the squares of the areas of the faces and V the volume, show that the principal semi axes of the ellipsoid inscribed in the tetrahedron, touching each face in the center of gravity and having its center at the center of gravity of the tetrahedron, are the roots of

$$k^6-\frac{E^2}{2^4.3}k^4+\frac{F^2}{2^4.3^2}k^2-\frac{V^2}{2^6.3}=0."$$

SOLUTION BY PROF. E. W. HYDE.

Let the equation of the ellipsoid be $S\rho\varphi\rho=1$, in which

$$\varphi\rho=\alpha S\alpha'\rho+\beta S\beta'\rho+\gamma S\gamma'\rho.$$

Let α, β, γ and δ be the vectors from the center of gravity of the tetrahedron, taken as the origin, to its vertices; then $\delta=-(\alpha+\beta+\gamma)$.

The vector from the origin to the c. g. of face opposite α is $-\frac{1}{3}\alpha$, and the perpendicular on this face from the origin is

$$[\varphi(-\tfrac{1}{3}\alpha)]^{-1}=[V(\beta\gamma+\gamma\delta+\delta\beta)]^{-1}S\beta\gamma\delta=[V(\alpha\beta-3\beta\gamma+\gamma\alpha)]^{-1}S\alpha\beta\gamma.$$

$$\therefore\ \varphi\alpha=3V(3\beta\gamma-\gamma\alpha-\alpha\beta)S^{-1}\alpha\beta\gamma;$$

and similarly
$$\varphi\beta=3V(3\gamma\alpha-\alpha\beta-\beta\gamma)S^{-1}\alpha\beta\gamma;$$

$$\varphi\gamma=3V(3\alpha\beta-\beta\gamma-\gamma\alpha)S^{-1}\alpha\beta\gamma.$$

Now for the axes $\varphi\rho$ must coincide in direction with ρ; $\therefore$ make

$$\varphi\rho=-k^{-2}\rho,\ (\varphi+k^{-2})\rho=0.$$

Operating by $S.\rho$ this gives for the value of k, $k^2=T^2\rho$. It may be easily shown that the discriminating cubic is

$$S(\varphi+k^{-2})\alpha(\varphi+k^{-2})\beta(\varphi+k^{-2})\gamma=0;$$

or expanding

$$k^{-6}+\frac{S(\alpha\beta\varphi\gamma+\beta\gamma\varphi\alpha+\gamma\alpha\varphi\beta)}{S\alpha\beta\gamma}.k^{-4}+\frac{S(\alpha\varphi\beta\varphi\gamma+\beta\varphi\gamma\varphi\alpha+\gamma\varphi\alpha\varphi\beta)}{S\alpha\beta\gamma}.k^{-2}$$
$$+\frac{S\varphi\alpha\varphi\beta\varphi\gamma}{S\alpha\beta\gamma}=0.$$

The coefficient of k^{-4} gives on substituting its value for $\varphi\gamma$, etc.,

$$\frac{3}{S^2\alpha\beta\gamma}\Big[3(V^2\alpha\beta+V^2\beta\gamma+V^2\gamma\alpha)-2(S.\beta\gamma V\alpha\beta+S.\gamma\alpha V\beta\gamma+S.\alpha\beta V\gamma\alpha)\Big]$$
$$=\frac{3}{4S^2\alpha\beta\gamma}\Big[V^2(\beta-\alpha)(\gamma-\alpha)+V^2(\beta-\alpha)(\delta-\alpha)+V^2(\delta-\alpha)(\gamma-\alpha)$$
$$+V^2(\beta-\delta)(\gamma-\delta)\Big]=\frac{-4F^2}{3V^2}.$$

For the coefficient of k^{-2} we find

$$-\frac{72}{S^2\alpha\beta\gamma}.S(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha)=-\frac{9}{S^2\alpha\beta\gamma}\Big[(\alpha-\beta)^2+(\beta-\gamma)^2$$
$$+(\gamma-\alpha)^2+(\alpha-\delta)^2+(\beta-\delta)^2+(\gamma-\delta)^2\Big]=\frac{2^2E^2}{V^2}.$$

And finally

$$\frac{S\varphi a\varphi\beta\varphi\gamma}{Sa\beta\gamma}=\frac{-2^4.3^3}{S^2a\beta\gamma}=-\frac{2^6.3}{V^2}.$$

Substi uting these values and multiplying through by $-V^2k^6\div 2^6.3$ we have the required equation,

$$k^6-\frac{E^2}{2^4.3}k^4+\frac{F^2}{2^4.3^2}k^2-\frac{V^2}{2^6.3}=0.$$

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344. "Through each of two points, taken at random within a circle, a random chord is drawn; find (1) the prob'y that the chords will intersect; and (2) if a third random chord be drawn through a 3d random p't, find the probabilities that the 3 chords will intersect in 0, 1, 2, 3 points."

SOLUTION BY PROF. E. B. SEITZ.

1. Let M and N be two random points within the circle whose center is O, and AB, CD random chords drawn through them. Draw the radii OH, OK perpendicular to AB, CD.

Let $OA=r$, $AM=x$, $CN=y$, $AB=x'$, $CD=y'$, $\angle AOH=\theta$, $\angle COK=\varphi$, $\angle HOK=\mu$, and $\omega=$ the angle OH makes with a fixed radius. Then $x'=2r\sin\theta$, $y'=2r\sin\varphi$; an element of the circle at M is $r\sin\theta d\theta dx$, at N it is $r\sin\varphi d\varphi dy$, and for elemental changes in the directions of AB and CD we have $d\omega$ and $d\mu$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and θ and $\frac{1}{2}\pi$; of μ, $\theta-\varphi$ and $\theta+\varphi$, when $\varphi<\theta$, and $\varphi-\theta$ and $\varphi+\theta$, when $\varphi>\theta$, and the result doubled; of ω, 0 and 2π; of x, 0 and x'; and of y, 0 and y'. Hence, since the whole number of ways the two chords can be drawn is π^4r^4, the requir'd prob'ty is

$$p=\frac{2}{\pi^4r^4}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{x'}\left\{\int_0^{y'}\left[\int_0^{\theta}\int_{\phi-\theta}^{\phi+\theta}r\sin\varphi\,d\varphi\,d\mu\right.\right.$$

$$\left.\left.+\int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}r\sin\varphi\,d\varphi\,d\mu\right]dy\right\}r\sin\theta\,d\theta\,d\omega\,dx$$

$$=\frac{1}{3}+\frac{5}{2\pi^2}.$$

[Our space will not permit the insertion of the integration in detail as given by Prof. Seitz, nor of the second part, at present, but we will insert the analysis of the the second part in a future number.]

345. "The great circle from $A\ (\varphi_1, \lambda_1)$ to $B\ (\varphi_2, \lambda_2)$ passes north of the parallel of latitude φ_0; what is the longitude λ_0 of the point P on this parallel so that the course APB shall be the shortest course from A to B which does not pass north of this parallel?"

SOLUTION BY R. S. WOODWARD, DETROIT, MICH.

Since the course APB is to be a minimum, AP and PB must be arcs of great circles. Designate them by s_1 and s_2 respectively. Then

$$\cos s_1 = \sin \varphi_1 \sin \varphi_0 + \cos \varphi_1 \cos \varphi_0 \cos (\lambda_0 - \lambda_1) \tag{1}$$

$$\cos s_2 = \sin \varphi_2 \sin \varphi_0 + \cos \varphi_2 \cos \varphi_1 \cos (\lambda_2 - \lambda_0) \tag{2}$$

Since $(s_1 + s_2)$ is to be a minimum with respect to λ_0, (1) and (2) give

$$\frac{d(s_1+s_2)}{d\lambda_0} = \frac{\cos \varphi_1 \cos \varphi_0 \sin (\lambda_0 - \lambda_1)}{\sin s_1} - \frac{\cos \varphi_2 \cos \varphi_0 \sin (\lambda_2 - \lambda_1)}{\sin s_2} = 0,$$

whence

$$\frac{\cos \varphi_1 \sin (\lambda_0 - \lambda_1)}{\cos \varphi_2 \sin (\lambda_2 - \lambda_0)} = \frac{\sin s_1}{\sin s_2}. \tag{3}$$

Call the angles at P between PA and the meridian of λ_0 and between the latter and PB, θ_1 and θ_2 respectively; then

$$\frac{\cos \varphi_1 \sin (\lambda_0 - \lambda_1)}{\cos \varphi_2 \sin (\lambda_2 - \lambda_1)} = \frac{\sin s_1 \sin \theta_1}{\sin s_2 \sin \theta_2}; \tag{4}$$

(3) and (4) give

$$\sin \theta_1 = \sin \theta_2 \text{ or}$$

$$\theta_1 = \theta_2 \text{ or } (180^\circ - \theta_2).$$

Now the two spherical triangles whose common side is $(90^\circ - \varphi_0)$ give

$$\cot \theta_1 = \frac{\tan \varphi_1 \cos \varphi_1 - \sin \varphi_0 \cos (\lambda_0 - \lambda_1)}{\sin (\lambda_0 - \lambda_1}$$

$$= \pm \cot \theta_2 = \pm \frac{\tan \varphi_2 \cos \varphi_0 - \sin \varphi_0 \cos (\lambda_2 - \lambda_0)}{\sin(\lambda_2 - \lambda_0)},$$

from which λ_0 may be readily found.

[Mr. George Eastwood has given an extended discussion of, and demonstration of the affirmation contained in, the Query at the foot of p. 31, but the space at our command will not permit its publication in this number; we hope, however to be able to present it to our readers in a future number.

We take this opportunity to notify our readers that, in consequence of our intended absence from home during the fore part of June, No. 4 will probably not be mailed to subscribers until about the 10th of July.—Ed.]

PROBLEMS.

346. *By Prof W. W. Hendrickson.*—Chords of the parabola $y = 4ax$ are drawn through the fixed point (h, k); required the locus of the intersection of normals drawn at the extremities of the chord.

347. *By Prof. A. Hall.*—Given $z = a\sin(x+\alpha) + b\sin(y+\beta)$, reduce z to the form

$$z = D\sin\tfrac{1}{2}(x+\alpha+y+\beta+\delta).$$

348. *By R. S. Woodward, Detroit, Mich.*—Show how to determine the values of x and z which will render

$$\begin{aligned} u = &+2a_1\cos(\ qz+\tfrac{1}{2}qx+\beta_1)\sin\tfrac{1}{2}qx \\ &+2a_2\cos(2qz+\ qx+\beta_2)\sin\ qx \\ &+2a_3\cos(3qz+\tfrac{3}{2}qx+\beta_3)\sin\tfrac{3}{2}qx \\ &+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \\ &+2a_n\cos(nqz+\tfrac{n}{2}qx+\beta_n)\sin\tfrac{n}{2}qx, \end{aligned}$$

a max. or min., a_1, a_2, etc., β_1, β_2, etc. and q being constants.

349. *By Prof. W. W. Johnson.*—From any point B of a circle, whose radius is ϱ, a perpendicular BR is drawn to a fixed straight line whose distance from the centre is b; and from R a perpendicular RD is drawn to the tangent at B. Produce RD to P making $DP = RD$. Find the rectangular equation of the locus of P, and of the evolute of this locus.

[This is a re-statement of 321, as the solution of that problem by Prof. Casey indicates that it was not understood as Prof. Johnson intended.]

350. *By Request.*—A series of circles touching each other at a point are cut by a fixed circle; show (by third Book of Euclid) that the intersections of the pairs of tangents to the latter, at the points where it is cut by each of the other circles, lie in a straight line.

351. *By Marcus Baker, U. S. Coast Surv., Washington, D. C.*—In a plane triangle ABC, a line from C perpendicular to AC meets AB in M, and another from C perpendicular to BC meets AB in N; knowing the sides a and b and the intercept $MN = m$, it is required to determine the triangle.

352. *By Artemas Martin, M. A., Erie, Pa.*—Two chords of equal but unknown lengths are drawn at random in a given circle; find the chance of their intersection.

353. *By William Hoover, Wapakoneta, Ohio.*—Required the average area of the circles described on the focal chords of a given ellipse as diameters.

353. *By Prof. H. T. Eddy, Cincinnati, Ohio.*—A cube slides down an inclined plane with four of its edges horizontal. The middle point of its lowest edge comes in contact with a small fixed obstacle and is reduced to rest. Find the direction of the impulsive reaction of the obstacle, and show that it is independent of the velocity of the cube and of the inclination of the plane. Determine also the limiting velocity that the cube may be on the point of overturning.

QUERY BY PROF. A. HALL.—"Observations on the motions of the sun-spots have also established the fact that the sun is not strictly a fixed body, around which the earth revolves, but that it has a motion of its own thro' space." *Physiography*, by T. H. Huxly, F. R. S., 2nd Ed., p. 365.

How can the above fact be determined by observations of the sun-spots ?

QUERY BY PROF. W. W. JOHNSON.—Let $u = \dfrac{\sin ax}{a}$.

Now if $a = \infty$, $u = 0$ independently of the value of x, therefore we should have $\dfrac{du}{dx} = 0$ when $a = \infty$. But we find $\dfrac{du}{dx} = \cos ax$ which is essentially indeterminate when $a = \infty$. What is the explanation of this paradox ?

NOTE BY WILLIAM HOOVER.—In Todhunter's *Plane Trigonometry*, p. 142, Third Edition, 1864, we have the following problem:

Eliminate θ from the equations

$$(a+b)\tan(\theta-\varphi) = (a-b)\tan(\theta+\varphi),$$
$$\cos 2\varphi + b\cos\theta = c.$$

The coefficient of the first term of the left member of the second equation is omitted. The cofficient of $\cos 2\varphi$ is a.

This erratum is pointed out as Todhunter's mathematical works are remarkably free from typographical errors.

PUBLICATIONS RECEIVED.

Meteorological Researches by WILLIAM FERREL. Part II. On Cyclones, Tornadoes and Water-spouts. Appendix No. 10—Report for 1878 of the Superintendent of the United States Coast and Geodetic Survey. Quarto. 95 pages and six plates. 1880.

American Journal of Mathematics, Vol. III, No. 3.

The papers in this No. are, A Method of Developing the Perturbative Function, by Simon Newcomb; On De Morgan's Extension of the Algebraic Processes, by Miss Christine Ladd; On the Motion of a Perfect Incompressible Fluid when no Solid Bodies are Present, by Henry A Rowland; and, On certain Possible Cases of Steady Motion in a Viscous Fluid, by Thomas Craig.

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AN INVESTIGATION OF THE MATHEMATICAL RELATIONS OF ZERO AND INFINITY.

BY PROF. C. H. JUDSON, GREENVILLE, S. C.

"WE have", says Pascal, "three principal objects in the study of truth:— One to discover it; another to demonstrate it; and a third to discriminate it from the false, when it is examined".

The object of the present article is to *discriminate the true from the false* with respect to mathematical infinity—the *infinitely great* and the *infinitely small*. We shall endeavor to show that these terms admit of *exact* definition, and consequently of a *strictly logical* calculus.

Let it be premised however that we set up no claim to originality, as we shall employ these terms in the sense in which they have been used by such accurate thinkers as Locke, Sturm, Du Hamel, De Morgan and others: and further, lest it be objected that because the discussion contains little that is new or original, it is therefore useless, we shall call attention to the misuse of these terms in some of our best Text-books, and to the confusion and error arising therefrom.

We begin by asking the reader to admit the following Postulates:—

(1). A number, or other quantity, as a distance, may be conceived as increasing without limit. We mean that we can conceive of no number so great but that we can readily conceive a greater. We cannot conceive of space as limited, beyond which limit there is no space; and we can not conceive a number beyond which there is no number. Let it be granted then that number and space are *infinite*, in the sense of *unlimited*.

(2). Let x be a variable. That is, let x be a symbol which may represent any and every number. Then by (1) x may be increased without limit.

(3). Since x may be increased *without limit*, it may be considered as greater than any *constant* whatever, i. e., greater than any *assignable* num-

ber. The value of x being wholly arbitrary, the instant we assign a value to any other symbol, as a, we may suppose $x > a$.

(4). We employ the symbol ∞ to represent a *variable* which increases indefinitely, and which, by reason of its indefiniteness, may be considered greater than any *assignable* number. The expression, $x = \infty$, may be read "when x increases without limit"; or, "when x increases indefinitely"; or even, "when x is infinite" (is unlimited), not "when x is equal to infinity". As, however, we require a name for this symbol, ∞, we shall call it infinity.

(5). *Infinity*, then, is a *variable*, considered as greater than any *assignable* number, and as increasing *without limit*.

According to this definition *infinity* is not a value that can be "reached", or that can be "approached"; nor is it a *limit* of any value, or values.

In Davies and Peck's Mathematical Dictionary, article Infinity, we find the following very singular statement:

"Mathematically considered, *infinity* is always a limit of a variable quantity". If now we turn to the word Limit, we find that " a limit is a quantity towards which a varying quantity may approach within less than any assignable quantity but which it cannot pass". Did these writers really mean to assert that they had any notion of infinity as a value that might be "approached within less than any assignable quantity"? They appear to reach the notion in this way. Consider the relation

$$t = \frac{a}{m-n}.$$

If a is a constant and n is very nearly equal to m then t is very great; and "finally, when n equals m, t is infinite". Just here is the source of all the confusion and error. The quantity t is not infinite, they hold, so long as $m-n$ differs from zero; but "finally when $m = n$" or when $m - n = 0$, then $t = \infty$. Observe that the symbol 0 is put for $m - m$, it is the absolute nought. If, however, we turn to the article *Zero* we find that when

$$\frac{a}{\infty} = 0,$$

"this kind of 0 differs analytically from the absolute nought, obtained by subtracting a from a; $a - a = 0$". "*It is in consequence of confounding the* 0 *arising from dividing a by* ∞ *with the absolute* 0 *that so much confusion has been created in the discussions that have grown out of this subject*". (Italics mine.) It appears then, that when we write

$$\frac{a}{0} = \infty \text{ and } \frac{a}{\infty} = 0$$

we have two different kinds of nought. No wonder there is confusion. However, when we turn to the article Nothing, we are consoled by the

statement that "*Nothing* is fast falling into disuse as a mathematical term, and the proper term zero (= infinitesimal) is acquiring its true place in the mathematical vocabulary". Absolute zero is worth nothing; we will therefore throw it away. Are we then, when the mathematical millenium shall have arrived, to write $a-a =$ infinitesimal?

To complete the "confusion", we are told that "In arithmetic *infinity* is the last term of the series of natural numbers". There is then a last term of the series 1, 2, 3, . . . which cannot be passed!

Professor Hackly (see Algebra, pp. 175–6) admits that $+\infty$ is the limit of increasing magnitudes and $-\infty$ the limit of decreasing magnitudes; and further that if we have the relation

$$x = \frac{m}{n-z}$$

and z increases from a value less than n to a value equal to n x becomes $+\infty$, while if z decreases from a value greater than n to a value equal to n x becomes equal to $-\infty$. *Minus* infinity therefore differs from *plus* infinity only as *minus* zero differs from *plus* zero.

We think the foregoing references are sufficient to show that the subject demands reinvestigation. We now return from this digression.

(6). If one variable x, may increase without limit, then its reciprocal $1 \div x$, will decrease without limit. If we put $z = 1 \div x$ then as x increases indefinitely z decreases indefinitely. As x never ceases to increase, z never ceases to decrease; or, in other words, as x never reaches any limit of increase, so z can never reach any limit of decrease.

(7). *A variable which decreases indefinitely and which, by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinites'l. Thus, read when $x = \circ$, when x decreases indefi'ly, or when x is an infinites'l.

(8). If a be a constant, the expressions

$$\frac{a}{\circ} = \infty \text{ and } \frac{a}{\infty} = \circ$$

are rigidly exact. The first asserts that if the numerator of a fract. is const. and the denominator decreases indefinitely the value of the fraction increases without limit. The second asserts that if the denominator increases without limit the value of the fraction decreases indefinitely. Though the value approaches nought as a limit, it decreases *without limit* since it can never become nought by any increase of the denominator.

(9). The *limit* of a variable is a *constant* which the variable indefinitely approaches.

Cor. I. The variable can never reach its limit, otherwise the approach would not be indefinite.

[Though the foregoing Proposition and Corollary are true when referred to the particular variable alluded to, and many others, yet they can not be accepted as generally true. For instance, the limits of a variable sine, or cosine of an arc, are zero and unity, constants, either of which the variable may reach.—Ed.]

Cor. II. Since the symbols ∞ and $\circ$ represent values wholly indeterminate and unassignable, it is evident that we may have the following relat'ns,

$$\infty \pm a = \infty; \ \infty \times a = \infty; \ \infty \div a = \infty.$$

The symbols ∞ in different members of these equations do not represent the same values, as they never represent definite values at all.

Also $\circ \times \infty = 0 \div 0$ is wholly indeterminate. We cannot write $a \pm \circ = a$ nor can we write $a \pm \circ = c$, if c is a constant.

(10). By the Binomial Theorem we may demonstrate that, limit

$$\left(1+\frac{1}{x}\right)^x = 2.7181, \text{ and } \left(1-\frac{1}{x}\right)^x = \frac{1}{2.7181}, \text{ when } x = \infty.$$

But all powers of 1 are 1; $\therefore$ $\circ$ is something real, but different from 0.

Since $1 + \circ$ differs from $1 - \circ$ it is not true that "an infinitesimal must be rejected as having no value in comparison with a finite quantity".

(11). If $a \div 0$ does not properly represent infinity, what interpretation are we to give this expression? We reply that it is the symbol of *impossibility*, not of *quantity*. This will be evident from the discussion of questions which give rise to this symbol. The eq'n $t = a \div (m - n)$ given in (5) is the answer to the problem of the couriers; "In what time will A overtake B?" (m being the rate at which A travels and n that of B.) If $m = n$ or $m - n = 0$, then it is impossible for A to overtake B. To say it will require an infinite time, is to give an affirmative paradoxical form to the negative proposition, A can *never* overtake B.

(12). We have in Trigonometry $\sin x \div \cos x = \tan x$. If $x = \frac{1}{2}\pi$, then $\sin x = 1$, $\cos x = 0$, and we have $1 \div 0 = \tan \frac{1}{2}\pi$. But when $x = \frac{1}{2}\pi$ the secant is parallel to the tangent and hence cannot meet it. Therefore the tangent of $\frac{1}{2}\pi$ is *impossible*, since by its definition it is *terminated* by the secant. The symbol $1 \div 0$ represents this impossibility. The true and correct statement is $\tan(\frac{1}{2}\pi \mp \circ) = \pm \infty$. That is, as the arc indefinitely approaches $\frac{1}{2}\pi$, the tangent increases without limit; if the arc is less than $\frac{1}{2}\pi$ the tan. is positive, if greater than $\frac{1}{2}\pi$ the tangent is negative; but when the arc $= \frac{1}{2}\pi$ the tangent *vanishes*.

(13). Again, in the problem of the lights,

$$x = \frac{a\sqrt{m}}{\sqrt{m} \pm \sqrt{n}},$$

m and n representing the intensities at a unit's distance and x the distance from the brighter light to the point equally illuminated.

If $m=n$ the first value of x is $\frac{1}{2}a$; the second is $a\div 0$, showing that there is no second point of equal illumination. This is the exact truth.

To say that there is a second point of equal illumination infinitely remote in one direction but none in the other, is to make a false statement. The two intensities being equal, there is as much reason to assume a second point infinitely remote to the right as to the left. But if $a\div 0$ indicates impossibility we have one point only.

Again, if $m>n$ and $a=0$, then $x=0$. This case has always presented a difficulty; since there ought to be no point of equal illumination when the intensities are unequal. Now when $x=0$ and $a=0$ the expressions for the intensities become

$$\frac{m}{0^2} \text{ and } \frac{n}{(0-0)^2},$$

which are symbols of impossibility. The true solution therefore follows the right interpretation of the symbol $a\div 0$; i. e., there is *no* point of equal illumination.

Prof. De Morgan says that he dates his first clear conception of mathematical infinity from the time when he rejected the relation $a\div 0=\infty$.

(14). When a student I was taught that $a\div 0=\infty$ by the following reasoning:—Division is a short method of subtraction. To divide a number by 5, we subtract 5 from the number, and then 5 from the remainder and so on until the number is exhausted. The number of subtractions is the quotient. Now to divide 8 by nought we first subtract nought and then subtract nought from the remainder, which is 8, and again subtract nought from each successive remainder and so on for ever, without exhausting the dividend. Therefore the number of subtractions is unlimited and the quotient is infinite.

This reasoning is as falacious as it is specious. To subtract nothing is not subtract any thing. If five books are lying upon the table, how many times can you take away no book? The question is without meaning. To divide by nothing is meaningless if it does not mean not to divide at all. Hence we conclude that $a\div 0$ is not a symbol of value, or symbol of quantity.

(15). Dr. Whewell lays down the following axiom (?) of limits, which has been adopted by Davies and Peck and many American authors:—

"Whatever is true *up to* the limit is true *at* the limit". Let us test this by Trigonometry.

If $x<\frac{1}{2}\pi$, sec x meets tan x. This is true up to the limit $\frac{1}{2}\pi$. Hence

according to the axiom, the secant of $\frac{1}{2}\pi$ meets the tangent which is parallel to it. Again, if $x < \frac{1}{2}\pi$ tan x is positive up to the limit $\frac{1}{2}\pi$; $\therefore$ tan $\frac{1}{2}\pi$ is positive. If $x > \frac{1}{2}\pi$ and decreasing, tan x is negative up to the limit; $\therefore$ tan $\frac{1}{2}\pi$ is negative. Hence also two parallel lines meet in opposite directions and inclose a space; all of which is absurd.

Again, let us test the axiom by analytical geometry. Trace the curve whose equation is

$$y = \sqrt{\frac{x^3}{x-a}}.$$

If x is less than a, y is imaginary. This is true up to the limit, $x = a$; therefore when $x = a$, y is imaginary. Secondly, let x be greater than a, and y is real. This is also true as x decreases up to the limit $x = a$; $\therefore$ when $x = a$ y is real. The results are contradictory, hence the so-called axiom cannot be true.

The true analysis is, if $x < a$, y is *imaginary*, and if $x = a$, y is *impossible;* i. e., the ordinate does not meet the curve. If $x > a$, y is *real* and meets the curve. If $x = a + \circ$, y meets the curve at an indefinitely great distance; hence the line $x = a$, parallel to the axis of y is an asymptote; all of which is rigidly exact.

(16). As a further illustration of absurd conclusions arising from the assumption that $a \div \infty = 0$, let us inscribe a regular polygon of n sides in a given circle. If A, B, C, &c. represent the angular points of the polygon and a, one of the equal angles, the sum of all the angles, $na = 2(n-2)$. (The right angle being the unit angle.) Hence $a = 2 - (4 \div n.)$

Let the number of sides become infinite, then $4 \div n = 0$ and $a = 2$. But if two lines, AB, BC, meet so as to form an angle equal to two right angles, then AB, BC form one straight line. The same is true of BC, CD &c.; hence the entire perimeter is a straight line. Therefore, since the polygon "coincides with the circle", the circumference of a circle is a straight line!

Assumptions from which such conclusions are logically deduced *must be erroneous.*

[We dissent to this conclusion of Prof. Judson. The equation is manifestly true for all finite lines, AB, BC, &c.; but when the number of sides is infinite the lines are reduced to points, which are without length, and therefore can have no curvature: Just as in the motion of a projectile whose initial direction is above the horizon; we know there is a moment of time in which the projectile neither ascends nor descends but has uniform horizontal motion; $\therefore$ during that moment its track is not curved, and yet every finite portion of its path is a smooth curve.—Ed.]

(17). Again, the foundation of the Differential Calculus is often laid on the assumption that an infinitesimal, when added to a finite quantity, must be rejected as zero.

The results of (10), which all accept, are a sufficient refutation of this assumption.

Mr. Price in his very valuable Treatise on the Infinitesimal Calculus (Oxford) devotes some ten pages of his introduction to a discussion of the terms infinite and infinitesimal, and the logic of his work is greatly marred by reason of his inexact notion and use of these terms.

Thus, he says that if a grain of aloetic acid be added to five pounds of pure water, it imparts a crimson color to the whole volume. "The grain of acid is divided into thirty-five millions of parts which are so small as to be beyond the limits of our vision; . . . they are *infinitesimal*, though the sum of them is finite; and as they are so small there must be an *infinity* of them." Infinity it would seem depends on the perfection of our organs of vision!

Again, he says, "the distance of the star Capella is 20 billions of miles; but as it is determinable it is finite; *though on the verge of the infinite.* The distance of the stars which have no paralax, he holds, is *infinite.* Here again the infinite is made to depend on the perfection of our instruments of measurement and that of our organs of vision. Surely mathematical science should aim at greater exactness.

Nought, he holds, is a relative term, like small, or great; and one nothing may be infinitely smaller than another. If so, then nought cannot represent the absence of value, as $5-5=0$. We need another word and another symbol for the *relative* nought.

(18). Messrs. Thomson and Quinby give the following illustration of a false interpretation of $a \div 0 = \infty$. (Algebra, Art. 348, p. 146.) "Given $x^2+xy=10$ (1), and $xy+y^2=15$ (2), to find x and y.

Let $x=zy$. Then, from (1),

$$y^2=\frac{10}{z^2+z}\ (5),\ \text{and from (2)}\ y^2=\frac{15}{z+1}\ (6).$$

From (5) and (6) $10z+10=15z^2+15z$ (7); $\therefore z=\frac{2}{3}$ or -1.

Substituting -1 for z in (5) or (6) we have $y=\pm\infty$, $\therefore x=\mp\infty$; hence $\infty^2-\infty^2=15$ and $\infty^2-\infty^2=10$".

That these results are incorrect is manifest; for if we eliminate y between (1) and (2) we have an equation of the second degree, which should have two roots only. But if $\pm\infty$ be roots, then an equation of the second deg. may have four roots. Adding (1) and (2), $x^2+2xy+y^2=25$, $\therefore x+y=\pm 5$. By substituting in (1) $\pm 5x=10$, or $x=\pm 2$. Substituting in (2), $\pm 5y=15$, $\therefore y=\pm 3$; and these are the only roots.

The correct interpretation of this example is, since when $z = -1$, $y^2 = 15+0$, $\therefore z = -1$ is an impossible value for (1) and (2.)

Prof. Loomis admits $\pm \infty$ as roots of x and y in simultaneous equations, as in $x^5+y^5 = a$, $x+y = b$. But the first is divisible by the second, and the resulting eq'n is of the 4th degree. The direct solution (see Young's Algebra) gives 4 roots for x and 4 for y. Can we have 6 roots? If the equations were not simultaneous, we admit that if y increases without limit x will be negative and increase without limit. In simultaneous and independent equations the values of x and y are not indeterminate, and we may by elimination obtain a single equation with one unknown quantity.

Can infinity be a root of an equation of the form $ax^n+bx^{n-1} + \ldots + bx + m = 0$? The theory of equations is utterly at variance with an affirmative answer. We conclude that *infinity is never a root of simultaneous equations.*

(19). Whatever may be the estimate of the value of Dr. Davies' contributions to the educational literature of our country, it must be confessed that his mind never seemed to settle down upon anything as satisfactry, even to himself, in relation to zero and infinity. We search his works in vain for any consistent or rational exposition of this subject. In his Logic of Mathematics, § 305, he says, "The terms zero and infinity are employed to designate the *limits* to which decreasing and increasing quantities may be made to approach nearer than any assignable quantity". But (page 302), "The science of mathematics employs no definition which may not be clearly comprehended".

In his view, an infinitesimal is that which has no appreciable value; infinity is that which exceeds our appreciation (Logic of Math., p. 282); and demonstration is that which is free from *"appreciable error"*.

"The common impression", says Dr. Davies, "that mathematics is an exact science founded on axioms too obvious to be disputed and carried forward by a logic too luminous to admit of error, is certainly erroneous in regard to the Infinitesimal Calculus". This frank acknowledgment will be admitted so far as it relates to his own expositions.

It is highly important that teachers require of their pupils greater accuracy of expression, as inaccuracy of language leads to inaccuracy of thought. Thus, the phrases, "*at* infinity", "continued to infinity", "when we reach infinity", and the like, should be wholly discarded Instead of saying "the tangent is infinite when x equals 90°" we may say the tangent becomes infinite as x approaches 90°. In tracing curves, if $y = x^2 \div (a-x)$, we should say, y becomes infinite as x approaches a; and not "when x equals a".

Again, if $y = ax+b$, and $y = a'x+b'$ be two lines the tangent of their included angle is

$$\tan A = \frac{a - a'}{1 + aa'}.$$

If this is a right angle then $1 + aa' = 0$; for the tangent of a right angle is impossible; not, the tangent is infinite. Again, log $\circ = -\infty$, not log $0 = -\infty$. The log of an infinites'l is negative and increases without limit.

Locke (Book 2, Chapt. 17) clearly discriminates between infinite space and a space infinite. "The idea of infinity", he says, "consists in a suppos'd *endless progression*", . . . "our idea of infinity being, as I think, an *endless growing idea*"; . . . "an endless progression of thought". Again he says "there is nothing more evident than the absurdity of the actual idea of an infinite number".

Is not the theological use of the term, infinite, very closely allied to the mathematical? When it is affirmed that the attributes of the Deity are *infinite*, as his love, wisdom, knowledge, power, is it meant that these may be "indefinitely approached", or simply that they are without limitation, as are duration, space and number?

The writer would be glad to meet with a single example in pure or app'd mathematics in which the view here imperfectly set forth does not afford a rational and consistent solution.

Reply to Criticism of Editor, p. 108.—The word *limit* is employed in mathematics in two distinct senses — (1) as marking the terminus which cannot be passed; (2) as a constant which may be indefinitely approached.

(1). In the equation of a circle, $x^2 + y^2 = r^2$, the limits of x are $\pm r$. These limits may be reached, as there is no *indefinite approach*.

(2). But if z is a function of x, and x increases or decreases without limit, then z can never reach its limit. "Limit" in Cor. 1. is that defined in (9).

Page (110). Is a curve line "one no part of which is straight", or is it "composed of an infinite number of infinitesimal straight lines"? If the latter then the tang't coincides with the curve for an infinitesimal distance, and we must abandon Euclid and define an infinite number and an infinitesimal distance, as *constants*.

The notion of a line as composed of an infinite number of consecutive p'ts is not mathematically exact. By what process can the sides of a polygon be reduced to points "without length"? Certainly not by increasing the number of the sides without limit. What is meant by the phrase, "when the number of sides is infinite"? And how do we know that the equa ion is not true then? My solution, of course, is that $4 \div n$ is not zero, but an infinitesimal when n increases without limit.

[We admit the difficulty of constructing a line, which has length, of points which are without length; but we cannot perceive that Prof. Judson's treatment of the subject obviates that difficulty.—Ed.]

A BRIEF ACCOUNT OF THE ESSENTIAL FEATURES OF GRASSMANN'S EXTENSIVE ALGEBRA.

[Given by the author in Grunert's Archiv, Vol. VI, 1845.]

TRANSLATED BY PROF. W. W. BEMAN, ANN ARBOR, MICH.

[Continued from page 97.]

6. *If all the elements of a field of the* n*th order be subjected to one and t same mode of variation, leading to new elements (not contained in that fielc then the aggregate of the elements producible by this mode of variation and opposite is called a field of the* (n+1)*th order; the field of the third order co responds to the plane, that of the fourth to space in general.*

If the points of a right line all move in one and the same directio leading to new points (not contained in that right line), then is the aggr gate of the points producible by this motion and its opposite, the plane, ar if we proceed in the same way with the points of the plane, we get all spac If we substitute here for the spatial notions the abstract ones given abov and keep the transition from one order to the next higher general, the id just mentioned is obtained.

II. Bearing of the Calculus Employed in My Extensive Algebra upon Geometry Explained.

7. *In my extensive algebra there appears a peculiar calculus, which, tran ferred to geometry, is of inexhaustible fertility, and here (in geometry) consi in subjecting spatial figures (points, lines, etc.) immediately to calculation.*

For example, the right line drawn through two points, is, with respect its position and magnitude, regarded as the join (Verknüpfung) of those p' and, indeed, as a peculiar kind of multiplication (see No. 15 below); lik wise the triangle included by three points, with respect to its area and t position of its plane, as the product of three points, so that this product zero, when the area of the triangle is zero, i. e., when the three points l in a right line; further, in a sense to be explained more fully later (see N 22 and Prob. 18), the point of intersection of two right lines is regarded their product.

8. *The effect of the application of this calculus to geometry is to unite t synthetic and analytic methods,* i. e., *to transplant the advantages of each in the soil of the other, while side by side with every construction we have a si ple analytical operation, and conversely.*

For illustration, take the following example. As is well known, the ve tex γ of a variable triangle, whose other two vertices α, β move in fix right lines A and B, and whose sides pass through three fixed points a, b,

describes a conic section. If a, b, c, are the fixed points through which pass the sides opposite the vertices α, β, γ, respectively, then we see (No. 7) that γaB represents the vertex β, $\gamma aBcA$, the vertex α, and, since the points α, b, γ, lie in one right line, their product being zero, we have the equation

$$\gamma aBcAb\gamma = 0$$

as the equation of a conic section described by γ. We see that this equation is of the second degree with respect to γ, and in this we already have a presentiment of an important law applicable to all algebraic curves.

III. Simplest Rules of Operation for the New Analysis.

The combinations which occur in this part of the extensive algebra are addition, subtraction, combinatory multiplication, combinatory division.

9. *For all kinds of addition and subtraction, the ordinary processes hold good.*

10. *For all kinds of multiplication and division, the following law holds good: Instead of multiplying or dividing an aggregate of terms by a signless expression in any way whatever, we can, without changing the final result, multiply or divide the separate terms in the same way* by this expression, and unite the separate products or quotients into an aggregate by placing before each one the sign of that term by whose multiplication or division it was obtained: further a numerical factor associated with any factor of the product may be associated with any other, or with the product: finally $A \div A$ is always* 1, *when A is not zero.*

11. *A product* a . b . c . . . *I call a combinatory product, when in addition to law No.* 10, *the following law holds good, that if two consecutive factors of the product* a . b . c . . . . *be interchanged, the product takes the opposite value; and I call* a, b, c, . . . *and their sums or differences in that case factors of the first order* (Ordnung).

For example, according to this, $a \, . \, b \, . \, c \, . \, d = -a \, . \, c \, . \, b \, . \, d$.

12. *If in a combinatory product two factors of the first order are equal to each other, the product is zero.*

For example, $a \, . \, b \, . \, b \, . \, d = 0$ (as is seen at once if b and c are made equal in the example of No. 11). The following problems will serve to elucidate this method of multiplication:

Prob. 1. To develop the combinatory product $(\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_3\varepsilon_3) \, . \, (\beta_1\varepsilon_1 + \beta_2\varepsilon_2 + \beta_3\varepsilon_3) . (\gamma_1\varepsilon_1 + \gamma_2\varepsilon_2 + \gamma_3\varepsilon_3)$, where $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$; $\gamma_1, \gamma_2, \gamma_3$ represent numerical quantities, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$, combinatory factors of the first order. By applying Rules (9–12) we finally obtain the expression

*This expression refers not only to the method of combination in general, but also to the position of the factor in the product.

$$(\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1+\alpha_2\beta_3\gamma_1-\alpha_2\beta_1\gamma_3)\varepsilon_1.\varepsilon_2.\varepsilon_3.$$

Prob. 2. To solve three equations of the first degree, involving three unknown quantities, by the rules of combinatory multiplication.

Let the three equations be

$$(1)\qquad \begin{cases} \alpha_1 x+\beta_1 y+\gamma_1 z=\delta_1, \\ \alpha_2 x+\beta_2 y+\gamma_2 z=\delta_2, \\ \alpha_3 x+\beta_3 y+\gamma_3 z=\delta_3. \end{cases}$$

Multiply the three equations respectively by three combinatory factors of the first order ε_1, ε_2, ε_3, whose product is not zero, add, and assume

$$(2)\qquad \begin{cases} \alpha_1\varepsilon_1+\alpha_2\varepsilon_2+\alpha_3\varepsilon_3=a, \\ \beta_1\varepsilon_1+\beta_2\varepsilon_2+\beta_3\varepsilon_3=b, \\ \gamma_1\varepsilon_1+\gamma_2\varepsilon_2+\gamma_3\varepsilon_3=c, \\ \delta_1\varepsilon_1+\delta_2\varepsilon_2+\delta_3\varepsilon_3=d; \end{cases}$$

we then get the equation

$$(3)\qquad xa+yb+zc=d.$$

Multiply this equation combinatorily by $b.c$, and we get, since $b.b.c$ and $c.b.c$ are zero (No. 12), the equation

$$x.a.b.c=d.b.c.$$

$$\therefore\ x=\frac{d.b.c}{a.b.c};$$

and in a similar way we find y and z, and get

$$(4)\qquad x=\frac{d.b.c}{a.b.c},\qquad y=\frac{a.d.c}{a.b.c},\qquad z=\frac{a.b.d}{a.b.c}.$$

These expressions (in which the laws of combinatory multiplication allow no cancellation of the separate combinatory factors) are extremely convenient in analysis. If we wish the unknown quantities expressed in the ordinary form, we have only to substitute from equation (2), to develop according to Prob. 1, and cancel $\varepsilon_1\,\varepsilon_2\,\varepsilon_3$ (No. 10) in numerator and denominator; e. g. we find

$$(5)\quad x=\frac{\delta_1\beta_2\gamma_3-\delta_1\beta_3\gamma_2+\delta_3\beta_1\gamma_2-\delta_3\beta_2\gamma_1+\delta_2\beta_3\gamma_1-\delta_2\beta_1\gamma_3}{\alpha_1\beta_2\gamma_3-\alpha_1\beta_3\gamma_2+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1+\alpha_2\beta_3\gamma_1-\alpha_2\beta_1\gamma_3}.$$

We see not only how this process is in general applicable to n equations of the first degree involving n unknown quantities, but also how we can write out the final result with considerable facility as soon as the n equations are given.

IV. Elementary Notions of the Different Magnitudes and Modes of Combination in Geometry.

13. *The spatial magnitudes of the first order are simple or multiple points, and right lines of definite length and direction.*

If A and B are two points, I represent the right line from A to B, so far as length and direction are fixed in it but nothing more, by $B-A$; I say therefore that $B-A$ can be equal to B_1-A_1 then and then only, when the right lines from A to B and from A_1 to B_1 have the same length and direction.

14. *The spatial magnitudes of the* n*th order arise from combinatory multiplication of* n *magnitudes of the first order* (Stufe), *which are taken as factors of the first order* (Ordnung).

In this case if the factors of the first order are likewise magnitudes of the first order, I call the multiplication an *outer* one (aeussere).

15. *If A, B, C, D, are points, then we mean by*

(1) $A \,.\, B$, *the line, which has A and B as extremities, regarded as a definite part of the infinite right line determined by A and B,*

(2) $A \,.\, B \,.\, C$, *the triangle, whose vertices are A, B, C, regarded as a definite part of the infinite plane determined by A, B, C,*

(3) $A \,.\, B \,.\, C \,.\, D$, *the tetraedron, whose vertices are A, B, C, D, regarded as a definite part of infinite space.*

That is, we put $A \,.\, B = A_1 \,.\, B_1$, when both products represent equal p'ts, with like signs,* of the same right line; further

$$A \,.\, B \,.\, C = A_1 \,.\, B_1 \,.\, C_1,$$

when both triangles are equal parts, with like signs, of the same plane; and finally

$$A \,.\, B \,.\, C \,.\, D = A_1 \,.\, B_1 \,.\, C_1 \,.\, D_1,$$

when both tetraedrons have equal volumes, with like signs.

16. *If* a, b, c, *are lines of definite length and direction, then we mean by*

(1) a . b, *the parallelogram, whose sides are equal and parallel to* a *and* b, *regarded as an area of definite magnitude and aspect of plane.*†

(2) a . b . c, *the parallelepiped, whose edges are equal and parallel to* a, b, c, *regarded as a volume of definite magnitude.* That is, we put

$$a \,.\, b = a_1 \,.\, b_1,$$

when the parallelograms, which are represented by these products lie in parallel planes, and have the same areas with like signs;

$$a \,.\, b \,.\, c = a_1 \,.\, b_1 \,.\, c_1,$$

when the parallelepipeds represented by these products have equal volumes with like signs.

17. *The side (right or left) toward which a spatial magnitude is constructed, determines its positive or negative value.*

*Two magnitudes have like signs, when they have either both a positive, or both a negative value.

†I say that two parallel planes have the same aspect.

(1) Two parts of the same line, $A.B$ and $A_1.B_1$, we regard as having the same sign, when B lies on the same side of A, as B_1 of A_1.

(2) Two parts of the same plane $A.B.C$ and $A_1.B_1.C_1$, we regard as having the same sign, when C lies on the same side of $A.B$, as C_1 of $A_1.B_1$; or, more plainly, when C is on the same side of one standing at A and looking toward B, as C_1 of one standing at A_1 and looking toward B_1.

(3) Two parts of the same solid, $A.B.C.D$, and $A_1.B_1.C_1.D_1$, we regard as having the same sign, when D lies on the same side of $A.B.C$, as D_1 of $A_1.B_1.C_1$; or, more plainly, if the point D lies on the same side of a human body whose head is directed toward A, feet toward B, and eye toward C, as D_1 is of a body whose head is directed toward A_1, feet toward B_1, and eye toward C_1.

(4) Two parallel surfaces $a.b$ and $a_1.b_1$ we regard as having the same sign when the direction b lies on the same side of direction a, as b_1 of a_1.

(5) Two solids $a.b.c$ and $a_1.b_1.c_1$ we regard as having the same sign when the direction c lies on the same side of $a.b$, as c_1 of $a_1.b_1$, i. e. when the direction c lies on the same side of a human body in which the direction a leads from the feet to the head, and whose eyes look forward in the direction b, as the direction c_1 of a body, etc.

18. *There are seven classes of spatial magnitudes separated into four orders* (Stufen):

Ord. I.	(1)	*Simple or multiple points.*
	(2)	*Right lines of definite length and direction.*
Ord. II.	(3)	*Definite parts of given infinite right lines.*
	(4)	*Plane surfaces of definite magnitude and aspect of plane.*
Ord. III.	(5)	*Definite parts of given infinite planes.*
	(6)	*Definite volumes.*
Ord. IV.	(7)	*Definite volumes.*

Here volumes appear twice, once as magnitudes of the third order, again as magnitudes of the fourth order, according as they are regarded as the product of three right lines of definite direction and length, or as the product of four points.

19. *Parts, with like signs, of one and the same whole have as their sum a part, with the same sign, of the same whole, which part is as great as these two combined.*

For example, if $A.B$ and $A_1.B_1$ are parts with the same direction, of the same infinite right line, they have as their sum, a part, with the same direction, of the same right line, which is as great as these two combined.

20. *Any two magnitudes of the same order, but only such, can be added; the meaning of the addition of such magnitudes can always be determined, if we adhere to the previously given signification of these magnitudes and apply the rules of III.*

Prob. 3. To add two points, A and B.

If we put $A+B=2S$, we get $B-A=2(S-A)$, i. e. S is the mean between A and B. Thus the sum of two points is twice the mean bet. them.

Prob. 4. To add two multiple points, αA and βB, when the coefficients, α and β, are positive, i. e. to find the point S, which satisfies the equation

$$\alpha A+\beta B = (\alpha+\beta)S.$$

If this equation is satisfied, we must have

$$\beta(B-A) = (\alpha+\beta)(S-A)$$

and, conversely, from the latter we get the former. But from the latter is obtained this construction:

Take on the line AB from A toward B the part $\beta\div(\alpha+\beta)$ [or from B toward A the part $\alpha\div(\alpha+\beta)$]; then the termination of this part is the p't S.—Hence "the sum of two multiple points with positive coefficients, is a point, multiplied by the sum of the coefficients, which so lies in the line between the two points that its distances from these two points are inversely proportional to the coefficients belonging to these points."*

Prob. 5. To add a point A and a right line of definite length and direction $C-B$.

Construct a right line from A equal in length to $C-B$ and having the same direction. Let this be $D-A$; then is D the sum required; for since $C-B = D-A$, we have

$$A+(C-B)=A+(D-A)=D.$$

Hence "the sum of a point A and a right line of definite length and direction is the termination of this line, when A is its origin."

Prob. 6. To add a multiple point αA and a right line of definite length and direction $C-B$.

Construct a right line from A having the same direction as $C-B$, but only $\frac{1}{\alpha}$th as long. Let this be $D-A$, then is αD the sum required. For since $C-B=\alpha(D-A)$, we have

$$\alpha A+(C-B)=\alpha A+\alpha(D-A)=\alpha D.$$

Prob. 7. To add two right lines of definite length and direction, $B-A$ and $D-C$.

Make $E-B = D-C$; then

$$(B-A)+(D-C)=(B-A)+(E-B)=E-A.$$

*It is easily seen that this point is the center of gravity, when the coefficients represent weights.

Hence "two right lines of definite length and direction can be added by putting the origin of the second at the termination of the first, without changing length and direction, when the right line from the origin of the first to the termination of the second is the sum required."

Prob. 8. To add n right lines of definite length and direction.

The repeated application of the solution of Prob. 7 leads at once to the solution of this problem. Hence "n right lines of definite length and direction can be added by so joining the lines in a continuous series, without changing length and direction, that where one terminates, the next following begins; then is the right line from the origin of the first to the termination of the last the sum required."

Prob. 9. To find the sum of n points, A_1, A_2, A_3, ... A_n, i. e. to find the point S, which satisfies the equation

$$A_1 + A_2 + A_3 + \dots A_n = S.$$

If we subtract from both members of the equation nR, where R is any point, we get

$$(A_1 - R) + (A_2 - R) + (A_3 - R) + \dots (A_n - R) = n(S - R).$$

Now, since the first equation may be deduced again from this, we have the following: "To add n points, draw from any point R right lines to the n points, join them continuously to one another without changing length and direction, beginning with the origin of the first at R, connect R with the termination of the last by a right line, and divide this right line into n equal parts; then is the first point of division from R the point S, n times which is the sum required."

Prob. 10. To add any number of multiple points, αA, βB, γC . . . . . , when the sum of the coefficients $\alpha + \beta + \gamma + \dots$ is not zero.

Put $$\alpha A + \beta B + \gamma C + \dots = (\alpha + \beta + \gamma \dots) S,$$
and subtract from both members $(\alpha + \beta + \gamma \dots) R$, where R is any point; then we get

$$\alpha(A - R) + \beta(B - R) + \gamma(C - R) + \dots = (\alpha + \beta + \gamma + \dots)(S - R).$$

Now, since the first equation may be deduced again from this, we have the following: "To find the sum of any number of multiple points αA, βB, γC - - -, the sum of whose coefficients is not zero, draw from any point R lines to A, B, C - - -, multiply these by α, β, γ - - -, respectively,* join the lines so obtained continuously to one another without changing direction and length, beginning with the origin of the first at R, connect R with the termination of the last by a right line, and take upon this line from R a

*In such multiplication by a numerical quantity α the direction is not changed when α is positive, as is easily seen, while the length is changed in the ratio $1 : \alpha$; if α is negative the direction becomes the opposite.

distance equal to the $\frac{1}{\alpha+\beta+\gamma+\ldots}$th part; then is the termination of this distance multiplied by $(\alpha+\beta+\gamma+\ldots)$ the sum required.

Prob. 11. To find the sum of multiple points, αA, βB, ... when $\alpha+\beta+\gamma+\ldots=0$.

Subtract from the sum $\alpha A+\beta B+\gamma C+\ldots$ the expression $(\alpha+\beta+\gamma+\ldots)R$; then since this subtracted quantity is zero, the value of the sum is not changed and we have

$$\alpha A+\beta B+\gamma C+\ldots=\alpha(A-R)+\beta(B-R)+\gamma(C-R)+\ldots$$

Hence "the sum of multiple points, the sum of whose coefficients is zero, is a right line of definite length and direction, which may be obtained by drawing from any point R right lines to the given points, multiplying them by the coefficients belonging to these points and adding the products."

Prob. 12. To add two parts $A.B$ and $C.D$ of lines which intersect in E."

Make $E.F=A.B$, and $E.G=C.D$; then $A.B+C.D=E.F+E.G=E.(F+G)=2E.S$, if S is the mean of F and G (Prob. 3).

Hence "To find the sum of two parts of lines which intersect; take the point of intersection of these lines as the origin; then twice the right line from the intersection to the middle point of the two extremities is the sum required."*

Prob. 13. To add two parts $A.B$ and $C.D$ of parallel lines, when they are of unequal length, and have opposite directions.

If $A.B$ and $C.D$ are parallel, $D-C$ must be equal to $a(B-A)$, where a is any positive or negative number. Now, since $A.B$ is equal to $A.(B-A)$, because $A.A=0$ (No. 12), we have

$$\begin{aligned} A.B+C.D &= A.(B-A)+C.(D-C) \\ &= A.(B-A)+aC.(B-A) \\ &= (A+aC)(B-A). \end{aligned}$$

If the sum $A+aC=(1+a)S$ (see Prob. 4), then the last expression

$$=S.(1+a)(B-A)=S.(B-A+D-C)$$

from which a simple construction of that sum may be obtained.

Prob. 14. To add two parts of lines $A.B$ and $C.D$, when they are of equal length and have opposite directions.

If both lie in the same right line, their sum is zero. If this is not the case, we have, since $D-C=-(B-A)$,

$$\begin{aligned} A.B+C.D &= A.(B-A)+C.(D-C) \\ &= A.(B-A)-C.(B-A) \\ &= (A-C)(B-A). \end{aligned}$$

*This is the diag'l of the parallelogram which has these parts of lines as its sides, whence we see that the sum of the parts of lines is the resultant when the parts of lines repr't forces.

The sum therefore is an area of definite magnitude and aspect of plaue.

Prob. 15. To add two surfaces, $a \, . \, b$ and $c \, . \, d$, of definite magnitude and aspect of plane.

If the planes are parallel, they can be added according to (No. 19); if they are not, both planes will have a direction in common. Let e be a right line having this direction, and $a \, . \, b = e \, . f$, $c \, . \, d = e \, . \, g$; then

$$a \, . \, b + c \, . \, d = e \, . f + e \, . \, g = e \, . \, (f+g).$$

Prob. 16. To add two parts $A \, . \, B \, . \, C$ and $D \, . \, E \, . \, F$ of definite planes which are not parallel.

If the planes are not parallel they will intersect. Let $G \, . \, H$ be a part of their line of intersection, and let $A \, . \, B \, . \, C = G \, . \, H \, . \, J$, $D \, . \, E \, . \, F = G \, . \, H \, . \, K$. Then

$$\begin{aligned} A \, . \, B \, . \, C + D \, . \, E \, . \, F &= G \, . \, H . J + G \, . \, H \, . \, K \\ &= G \, . \, H \, . \, (J+K) = 2G \, . \, H \, . \, S, \end{aligned}$$

if S be the mean between J and K. Hence "To add two parts of planes not parallel, represent them as triangles whose common base lies in the intersection of the two planes; then twice the triangle having the same base, whose vertex is the middle point between the vertices of these triangles is the sum required."

Prob. 17. To add two parts, $A \, . \, B \, . \, C$ and $D \, . \, E \, . \, F$, of parallel planes.

If the planes are parallel, we can make $(E-D) \, . \, (F-D) = a \, . \, (B-A) . (C-A)$, where a is a numerical quantity. Hence

$$\begin{aligned} A \, . \, B \, . \, C + D \, . \, E \, . \, F &= A \, . (B-A) . (C-A) + D . (E-D) . (F-D) \quad \text{(No. 12)} \\ &= (A + aD) . (B-A) . (C-A) \\ &= S \, . \, (1+a) . (B-A) . (C-A), \end{aligned}$$

if $(1+a)S$ is the sum $A + aD$. The last expression

$$= S[(B-A) . (C-A) + (E-D) . (F-D)],$$

in which a simple construction is again manifest. If, however, $a = -1$, i. e. if both figures are equal in area but have opposite signs, then $A + aD$ is a right line of definite direction and length (Prob. 11). Let this equal $H-G$. Then

$$A \, . \, B \, . \, C + D \, . \, E \, . \, F = (H-G) . (B-A) . (C-A),$$

and hence the sum is a volume.

Prob. 18. To add a part $A \, . \, B \, . \, C$ of a definite plane, and a volume $(D-A) . (B-A) . (C-A)$.

$$\begin{aligned} A.B.C + (D-A).(B-A).(C-A) &= A.(B-A).(C-A) + (D-A).(B-A).(C-A) \\ &= D.(B-A).(C-A), \end{aligned}$$

from which the meaning of this addition is easily seen.

21. *A combinatory product whose factors of the first order* (Ordnung) *are magnitudes of the* $(n-1)$th *order* (Stufe), *but all of which lie in one and*

the same field of the nth *order, I call a regressive* (eingewandtes) *product taken with reference to that field*, e. g. a combinatory product of parts of lines in the plane, or of parts of planes in space.

22. If henceforth outer multiplication be designated by simply writing the factors together, regressive multiplication by a point placed between the factors, *we understand by the regressive product* $AB.AC$, *where* A, B, C, *are any magnitudes, the product* $ABC.A$, *in which* ABC *is treated as a coefficient belonging to* A, *provided that the product be referred to the field of lowest order in which* A, B *and* C *lie at the same time.*

Prob. 19. To find the product with reference to the plane ABC of two parts of lines $AB.AC$.

According to No. 22, this is equal to $ABC.A$; i. e. "the product of two parts of lines which intersect is their point of intersection combined with a part of the plane as a coefficient." If we regard a part of the plane as unity, the parts of planes by which the points are multiplied will be actual numerical quantities, and the products will appear as multiple points; all the magnitudes to be compared must then lie in the same plane, to which the products refer (as is always the case in plane geometry).

Prob. 20. To find the product of three segments of lines AB, AC, BC, with reference to the plane ABC.

Sol. $$AB.AC.BC \doteq ABC.ABC = (ABC)^2.$$

Prob 21. To find the regressive product of two parts of planes ABC and ABD (with reference to the volume).

Sol. $$ABC.ABD = ABCD.AB.$$

Prob. 22. To find the regressive product of three parts of planes ABC, ABD, ACD.

Sol. $$ABC.ABD.ACD = ABCD.ABCD.A = (ABCD)^2.A.$$

Prob. 23. To find the regressive product of four parts of planes ABC, ABD, ACD, BCD.

Sol. $$ABC.ABD.ACD.BCD = (ABCD)^2.$$

Note. The product of two parts of planes is therefore a part of a line, of three a point, but the part of the line and the point still have a volume or a product of volumes as coefficients, and if we make a definite volume a unit, these coefficients become actual numerical quantities.

These are perhaps the most essential conceptions which appear in the first part of my extensive algebra. But it is impossible in this place to give any thing more than a superficial idea of the infinite fruitfulness of this new method for the treatment not only of geometry but, in general, of all sciences which are based upon spatial relations. As little could I in this place

give the proofs that the rules of operation found in III. are applicable to the forms of combination here set forth, but here also I must refer to my extended treatise in which these proofs are drawn out with all requisite exactness; and where at the same time the development advances in such a way that every thing, which seems to be arbitrary in the presentation of the different ideas, vanishes.

GEOMETRICAL DETERMINATION OF THE SOLIDITY OF THE ELLIPSOID.

BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLND.

LET $ABCD$ be a right cylinder bounding a regular prism having an infinite number of sides, and let $EFGH$ represent one of these sides. Draw FEI tangent to the cylinder at F and meeting the diameter AB produced in I. From I let fall the perpendicular IK to meet the lower diameter CD produced in K. From K draw KGH a tangent to the cylind. at H.

From B to K pass a plane with cutting edge parallel to the lower base, and it will produce the ellipse $LMNB$ containing a polygon having MN for one of its infinite number of sides.

LB, the continuation of KL, will be the transverse axis of the ellipse, while the conjugate axis will be equal to the diameter AB of the cylinder.

From the centre O of the upper base let fall the perpendicular OO' and it will pass thro' the centre O' of the ellipse.

From O draw OE and OF, and from O', $O'M$ and $O'N$. From the similar triangles BOO' and BIK, we have $BO : BO' :: OI : O'K$. Since BO is half of AB and BO' is half of BL, $BO =$ semi-conjugate diameter $= B$, and $BO' =$ semi-transverse $= A$; $\therefore OI : OK :: B : A$. But OI is the subtangent corresponding to the tangent line FEI of the circle, while $O'K$ is the subtangent crresponding to the tangent line KMN of the ellipse.

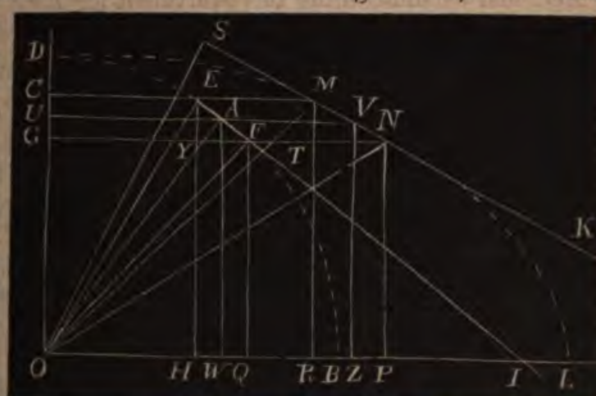
From the construction of the solid it will be seen that the triangle $O'MN$ is directly beneath the triangle OEF, and every triangle that can be drawn to the sides of the polygon in the circle will have its correlative triangle directly beneath it drawn to the sides of the polygon formed in the ellipse, and it can easily be shown that, for all such correlative triangles, the subtangent of the triangle in the circle is to the subtangent of the triangle in the ellipse as B to A.

It is also evident from the construction that any line drawn from the circumference of the circle $AEFB$ perpendicular to the diameter AB must be equal in length to the line that is directly beneath it drawn from the circumference of the ellipse perpendicular to the the transverse axis BL.

Lines dropt from the vertices E and F perpendicularly on the diameter AB must be equal to lines dropt from vertices M and N perpdndicularly on the transverse axis BL; and these equal perpendiculars indicate the corresponding vertices of the correlative triangles.

Let $DMNL$ represent the ellipse, OL one half the transverse axis, and OD one half the conjugate axis.

With OD as a radius inscribe the circle $DEFB$, and let EF represent one of the infinite number of sides of the inscribed regular polygon. Draw OE and through E and F the vertices of the triangle EFO, draw CEM and GFN parallel to OL, and intersecting the ellipse at M and N. From E and F let fall on OL the perpendiculars EH and FQ, also from M and N let fall on OL the perpendicul's MR and NP. Then, because they lie between parallel lines, EH and FQ in the circle are respectively equal to MR and NP in the ellipse; ∴ M and N are the vertices of the triangle that is correlative to OEF.



Join OM and ON; the triangles OMN and EFO being correlative their subtangents reckoning from O are to each other as A to B.

Draw MNK a tangent to the ellipse at M intersecting OL produced in K, and draw EFI tangent to the circle at I and intersecting OB produced in I; then OI is to OK as B to A.

Prolong the tangent KNM to meet OS drawn perpendicular to it,

then from the similar triangles NMT and KOS we have the proportion

$$MT : SO :: MN : OK;$$

$\therefore$ $MT \times OK = SO \times MN$. But $SO \times MN$ equals double the area OMN;
$\therefore$ $MT \times OK =$ equals double the area OMN.

From A the middle of EF let fall the perpendicular AO. Then from the similar triangles EYF and OAI we have

$$EP : AO :: EF : OI;$$

$\therefore$ $EY \times OI = AO \times EF$. But $A\ O \times\ EF$ equals double the area EFO;
$\therefore$ $EY \times OI =$ equals double the area EFO.

Therefore $MNO : EFO :: MT \times KO : EY \times IO :: A : B$ (because MT and EY are between parallels and are therefore equal, and OK is to OI as A to B); $\therefore$ area $OMN = (A \div B) \times$ area OEF.

Through the middle of EF draw UAV paralell to OK and bisecting MN in V, and from A and V let fall the equal perpendiculars, AW and VZ, on the axis OK.

The solid generated by revolving the triangle OMN around the axis OK is represented by area $OMN \times \frac{2}{3}\pi . VZ$. Or, since $MNO = (A \div B) \times OEF$, and $OEF = EF \times \frac{1}{2} AO$, the solid is $\frac{1}{3}\pi . EF . AW . AO \times (A \div B)$.

From the similar triangles EYF and AWO we find $EF \times AW = HQ \times AO$; and by substituting this equivalent for $EF \times AW$ in the above expression we have $\frac{1}{3}\pi . AO^2 . HQ \times (A \div B)$. When the number of sides of the inscribed polygon is infinite, $AO = OD = B$, and hence the above expression for the solid generated by revolving the triangle MNO about the axis OK, is $\frac{1}{3}\pi . A . B \times HQ$. As this expression is true for all values of HQ it is true for their sum; hence, substituting for the several values of HQ their sum, $OB = B$, we get, for the solidity ot half the prolate ellipsoid $\frac{2}{3}\pi A . B^2$. Hence the solidity of the prolate ellipsoid is $\frac{4}{3}\pi . A . B^2$.

In a manner precisely similar to the foregoing it may be shown that the solidlty of the oblate ellipsoid is $\frac{4}{3}\pi . A^2 . B$.

GENERAL SOLUTION OF THE PROBLEM OF ANY NUMBER OF BODIES.

BY R. J. ADCOCK, ROSEVILLE, ILL.

THE masses of any number of bodies being given, together with their positions and motions at any given time with reference to any three rectangular axes, to find their positions and motions at any other time.

Let k_1 = the mutual attraction between two units of mass at a unit's distance, $m_1, m_2, m_3, \ldots$ the given masses, (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), ... their coordinates at the time $t = 0$, $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, ... their initial axial velocities, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), ... their coordinates and (u_1, v_1, w_1), (u_2, v_2, w_2), (u_3, v_3, w_3),.. their axial velocities at any other time t.

Let the initial distances between m_1 and m_2, between m_1 and m_3, &c., be $d_1, d_2, \ldots$ &c.; and the same for the time t be $\delta_1, \delta_2 \ldots$ &c. Then

$$d_1^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 + (b_1 - b_3)^2,$$
$$d_2^2 = (a_1 - a_3)^2 + (b_1 - b_3)^2 + (c_1 - c_3)^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\delta_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$
$$\delta_2^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and the attractions between m_1 and m_2, m_1 and m_3, &c., are

$$\frac{k_1 m_1 m_2}{\delta_1^2}, \quad \frac{k_1 m_1 m_3}{\delta_2^2}, \ldots \text{\&c.}$$

The sums of the axial components of the attractions of each of the bodies for m_1, give the three equations of motion for m_1:

$$\frac{d^2 x_1}{dt^2} = -\frac{k_1(x_1 - x_2)m_2}{\delta_1^3} - \frac{k_1(x_1 - x_3)m_3}{\delta_2^3} - \text{\&c.}, \tag{1}$$

$$\frac{d^2 y_1}{dt^2} = -\frac{k_1(y_1 - y_2)m_2}{\delta_1^3} - \frac{k_1(y_1 - y_3)m_3}{\delta_2^3} - \text{\&c.}, \tag{2}$$

$$\frac{d^2 z_1}{dt^2} = -\frac{k_1(z_1 - z_2)m_2}{\delta_1^3} - \frac{k_1(z_1 - z_3)m_3}{\delta_2^3} - \text{\&c.} \tag{3}$$

Since every astronomical variable is a function of the same independent variable t, therefore by Maclaurin's theorem

$$x_1 = a_1 + u_1 t + \left(-\frac{k_1 m_2(a_1 - a_2)}{d_1^3} - \frac{k_1 m_3(a_1 - a_3)}{d_2^3} - \text{\&c.}\right)\frac{t^2}{1.2} + \text{\&c.}$$

$$y_1 = b_1 + v_1 t + \left(-\frac{k_1 m_2(b_1 - b_2)}{d_1^3} - \frac{k_1 m_3(b_1 - b_3)}{d_2^3} - \text{\&c.}\right)\frac{t^2}{1.2} + \text{\&c.},$$

$$z_2 = c_1 + w_1 t + \left(-\frac{k_1 m_2(c_1 - c_2)}{d_1^3} - \frac{k_1 m_3(c_1 - c_3)}{d_2^3} - \text{\&c.}\right)\frac{t^2}{1.2} + \text{\&c.}$$

In the same manner may the other variables, velocities, &c., be found in an approximating series according to the ascending powers of t with coefficients containing the initial constants.

Discussion of a Proposition by R. J. Adcock.—*Proposition.* The Earth is in the Moon's path.

Let ABC be the elliptic orbit, nearly circular, of the moon described about the earth at E, when unaffected by the sun's attraction at S. Draw CD at right ang's to ES, AR and BO making the angles NEB and NEA each equal to about 54° 44′.

As the moon moves in the same plane with ES, it is shown in elementary astronomy that the radial component of the sun's disturbing action is zero at O, A, B, R, that it acts towards the earth while the moon moves from O to A and from B to R, and from the earth from A to B and from R to O. And that the component at right angles to the radius vector accelerates the moon's angular motion from C to N and from D to M, and retards it from N to D and from M to C.

If the tangential component of the sun's disturbing force act upon the moon for one entire revolution, while the moon is constrained to move in its instantaneous fixed ellipse the angular momentum or area described by the radius vector in a unit of time would be the same at the end as at the beginning of the revolution. But when not thus constrained, the radial component will cause a greater decrease of the radius vector in the second half of each of the angles OEA and BER than in the first. And a greater increase in the second half of AEB and of REO than in the first. Therefore the tangential component will make a decrease of angular momentum in each of the four angles OEA, AEB, BER, REO. Therefore the angular momentum of the moon will decrease at every revolution causing it to descend to the earth.

[Granting Mr. Adcock's position with respect to the sun's disturbing influence on the elliptic orbit of the moon about the earth, it follows that the moon would, during the lapse of a sufficient period of time, descend in a spiral path to the earth, or sufficiently near to cause disruption of the moon and the descent of a part of it to the earth, yet we cannot perceive that the above proposition, that "The Earth *is* in the Moon's path", has been established. But, as the problem of the moon's motions is by no means a simple one, perhaps we do not understand the proposition as Mr. Adcock intended it to be understood.—Ed.]

ANSWERS TO PROF. JOHNSON'S QUERY IN No. 3.—"QUERY. Let $u = \frac{\sin ax}{a}$. Now if $a=\infty$, $u=0$ independently of the value of x, therefore we should have $\frac{du}{dx}=0$ when $a=\infty$. But we find $\frac{du}{dx}=$ cosine ax which is essentially indeterminate when $a=\infty$. What is the explanation of this paradox?"

BY R. J. ADCOCK.—When $u=0$ independently of x it is not a function of x, and therefore cannot be differentiated with respect to x. Therefore the value of $du \div dx$ is $\cos ax$ independently of the value of a.

BY PROF. JUDSON.—If $u=0$, independently of x, then u is not a function of x, and $du \div dx$ is without meaning.

If a is a constant, then a cannot $=\infty$. If a is a variable, independent of x, and $a=\infty$, i. e. a increases without limit, then $\frac{\sin ax}{a}=$an infinitesimal (not $=0$), and u is therefore indeterminate; $du \div dx=\cos ax$ is also indeterminate, and there is no paradox.

BY PROF. BARBOUR. Let $u=\frac{\sin 360x}{360}$; $\frac{du}{dx}=\cos 360x$. Now for $x=1°$, or $2°$ or any other integral number of degrees, $u=0$; $\frac{du}{dx}=\cos 0°=1$. Hence it is clear that u may be equal to 0, and yet $du \div dx=1$.

SOLUTIONS OF PROBLEMS.

342. *By Prof. Kershner.*—"Prove Schlömilch's Theorem: If $D_a, D_b, \ldots D_n$ are divisors of 10^k+1, so that $N_a=\frac{10^k+1}{D_a}$, $N_b=\frac{10^k+1}{D_b}$, $N_n=\frac{10^k+1}{D_n}$ the k digits or figures of the whole numbers D_a-1, D_b-1, D_n-1 are the k first figures of the circulator or period of $\frac{1}{N_a}, \frac{1}{N_b}, \frac{1}{N_n}$, respectively.

SOLUTION BY PROF. J. SCHEFFER, HARRISBURGH, PA.

A well known principle relating to circulating decimals is as follows:

If $1 \div M$ produces a period of 2k *dicimals, the remainder after the* kth *digit is $(M-1)$, and the following decimals can be obtained by subtracting each digit from* 9.

Now let $10^k+1=MN$, where M and N may be prime factors or the product of such; and denote by x the number which represents the first k

decimals, then we have obviously according to the above principle:

$$\frac{1}{M}=\frac{x}{10^k}+\frac{M-1}{10^k M}, \text{ whence } x=\frac{(10^k+1)-M}{M},$$

but $10^k+1=MN$, $\therefore x=(MN-M)\div M=N-1$, which proves the theorem.

SOLUTIONS of problems in No. 3 have been received as follows:

From R. J. Adcock, 352; Marcus Baker, 351; Prof. W. P. Casey, 346, 349, 350, 351; Prof. H. T. Eddy, 354; Prof. W. W. Hendrickson, 346; O. L. Mathiot, 351; Prof. E. B, Seitz, 349, 351, 352, 353; Prof. J. Scheffer, 347, 348, 351; R. S. Woodward, 347, 348.

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346. "Chords of the parabola $y^2=4ax$ are drawn through the fixed point (h, k); required the locus of the intersection of normals drawn at the extremities of the chord."

SOLUTION BY PROF. W. W. HENDRICKSON.

Let the equation to the chord be $y_1-k=m_1(x_1-h)$ .... (1), and the equation to the parabola $y_1^2=4ax$ .. (2); combining (1) and (2) we have

$$y_1^2 m_1-4ay_1+4ak-4am_1h=0.$$

Let the roots of this equation be α and β, then $\alpha+\beta=\frac{4a}{m_1}$, $\alpha\beta=\frac{4a(k-m_1h)}{m_1}$

Let m_2, m_3 be the direction ratios of the normals, then

$$m_2=\frac{-\alpha}{2a},\ m_3=\frac{-\beta}{2a},\ m_2m_3=\frac{k-m_1h}{am_1},\ m_2+m_3=\frac{-2}{m_1}.$$

Taking the origin at $(2a, 0)$, the equation to the normal is $y=mx-am^3$; Denote the roots of this equation by m_2, m_3, m_4; then $m_2+m_3+m_4=0$, but $m_2+m_3=\frac{-2}{m_1}$, hence $m_4=\frac{2}{m_1}$, also $m_2m_3m_4=\frac{-y}{a}=\frac{k-m_1h}{am_1}\cdot\frac{2}{m_1}$, or $y=\frac{2(m_1h-k)}{m_1^2}$. Substituting this value of y in the equation to the normal we find $x=h+\frac{4a-km_1}{m_1^2}$; or moving the origin again to $(h, 0)$ we have

$$y=\frac{2(m_1h-k)}{m_1^2}, \qquad x=\frac{4a-km_1}{m_1^2}.$$

Finally eliminating m_1 between these two equations, the equation to the required locus is

$$(4ah-k^2)(ky+2hx)=2(2ay+kx)^2.$$

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347. "Given $z=a\sin(x+\alpha)+b\sin(y+\beta)$, reduce z to the form

$$z=D\sin\tfrac{1}{2}(x+\alpha+y+\beta+\delta)."$$

SOLUTION BY R. S. WOODWARD, U. S. LAKE SURVEY, DETROIT, MICH.

Put $a = c+d$ and $b = c-d$. Then

$$z = c[\sin(x+a)+\sin(y+\beta)]+d[\sin(x+a)-\sin(y+\beta)]$$
$$= 2c\sin\tfrac{1}{2}(x+a+y+\beta)\cos\tfrac{1}{2}(x+a-y-\beta)$$
$$+\ 2d\cos\tfrac{1}{2}(x+a+y+\beta)\sin\tfrac{1}{2}(x+a+y+\beta).$$

Now put $2c\cos\frac{1}{2}(x+a-y-\beta) = D\cos\frac{1}{2}\delta$,

$2d\sin\frac{1}{2}(x+a-y-\beta) = D\sin\frac{1}{2}\delta$, and we get

$$z = D\sin\tfrac{1}{2}(x+a+y+\beta+\delta).$$

348. "Show how to determine the values of x and z which will render

$$\begin{aligned} u = &+2a_1\cos(\ qz+\tfrac{1}{2}qx+\beta_1)\sin\tfrac{1}{2}qx \\ &+2a_2\cos(2qz+\ qx+\beta_2)\sin\ qx \\ &+2a_3\cos(3qz+\tfrac{3}{2}qx+\beta_3)\sin\tfrac{3}{2}qx \\ &+\ \ldots\ldots\ldots \\ &+2a_n\cos(nqz+\tfrac{n}{2}qx+\beta_n)\sin\tfrac{n}{2}qx, \end{aligned}$$

a max. or min., a_1, a_2, etc., β_1, β_2, etc. and q being constants."

SOLUTION BY R. S. WOODWARD.

For brevity this expression may be written

$$u = 2\Sigma_{r=1}^{r=n} a_r\cos(rqz+\tfrac{1}{2}rqx+\beta_r)\sin\tfrac{1}{2}rqx.$$

Hence for a max. or min.

$$(1)\qquad \frac{du}{dx} = 2q\sum_{r=1}^{r=n}\left\{\begin{matrix} ra_r\cos(rqz+\frac{1}{2}rqx+\beta_r)\cos\frac{1}{2}rqx\ - \\ ra_r\sin(rqz+\frac{1}{2}rqx+\beta_r)\sin\frac{1}{2}rqx \end{matrix}\right\}$$
$$= 2q\Sigma_{r=1}^{r=n} ra_r\cos(rqz+\ rqx+\beta_r) = 0.$$

$$(2)\qquad \frac{du}{dz} = -2q\sum_{r=1}^{r=n} ra_r\sin(rqz+\tfrac{1}{2}rqx+\beta_r)\sin\tfrac{1}{2}rqx = 0.$$

Subtracting twice (2) from (1) there results

$$(3)\qquad 2q\Sigma_{r=1}^{r=n} ra_r\cos(rqz+\beta_r) = 0.$$

The last eq'n will give the critical values of z. Denote them by z_1, z_2, $\ldots z_m$. Then the critical values of x corresponding to z_1 will be 0, (z_2-z_1), $(z_3-z_1), \ldots (z_m-z_1)$, since z_1 and either of these values of x will satisfy (1). Collectively the critical values of x corresponding to any critical value of z are shown in the following table:

Crit. val's of z	Corresponding critical values of x			
z_1	0,	(z_2-z_1),	(z_3-z_1),	$\ldots (z_m-z_1)$
z_2	(z_1-z_2),	0,	(z_3-z_2),	$\ldots (z_m-z_2)$
z_3	(z_1-z_3),	(z_2-z_3),	0,	$\ldots (z_m-z_3)$
..	. .	. .	. .	. .
z_m	(z_1-z_m),	(z_2-z_m),	. . . .	0

It may be remarked that the series u, expresses in general terms the correction for periodic error to the observed value of an angle measured on a circle read by q equidistant microscopes, z being the reading of either microscope and x the angle observed, or the difference between the means of the microscope readings in their two positions. When for any instrument the values of the constants α_1, α_2, etc., β_1, β_2, etc., are known, it may be important to know what values of z and x will render u a max. or min.

The practical application of equation (3) presents no special difficulty, since the roots z_1, z_2, etc., are not generally required with any great precision. By computing for each term of (3) its value for a few equidistant intervals throughout its period, the curve represented by

$$\Sigma_{r=1}^{r=n} r\alpha_r \cos(rqz + \beta_r) = y, \text{ say,}$$

may be plotted and the values of z, making $y = 0$, readily detected.

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349. "From any point B of a circle, whose radius is a, a perpendicular BR is drawn to a fixed straight line whose distance from the centre is b; and from R a perpendicular RD is drawn to the tangent at B. Produce RD to P making $DP = RD$. Find the rectangular equation of the locus of P, and of the evolute of this locus."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $AFA'F'$ be the given circle, and GG' the given straight line. Draw the diameter AA' perpendicular to GG', and FF' parallel to GG'.

Let B, C be two consecutive points in the circumference, and P, Q the corresponding points in the locus, CS being drawn perpendicular to GG', and SQ perpendicular to the tangent at C. Produce PB to K in OA, and QC to T in BK, and draw PM and TN perpendicular to OA.

Let $OA = a$, $OH = b$, $OM = x$, $PM = y$, $ON = x'$, $TN = y'$, $\angle AOB = \theta$, and arc $BC = i$.

Then since $\angle CBR = 90° - \theta = BCT$, $CQ \doteq CS = BR - i \sin\theta = BP - i \sin\theta$, and $CT = BT + i \sin\theta$; hence $CQ + CT = BP + BT$, or $QT = PT$. Therefore T is the point in the evolute of the curve corresponding to P.

We have $BP = a\cos\theta - b$, $BK = OK = \frac{1}{2}a \sec\theta$, $MK \,.\, \sin MKP$

$= PM \sin MPK$, or $(x - \frac{1}{2}a \sec\theta)\sin 2\theta = y \cos 2\theta$, whence

$$x \sin 2\theta - y \cos 2\theta = a \sin\theta. \qquad (1)$$

We also have $MK \cos MKP + PM \cos MPK = PK$, or

$$(x - \tfrac{1}{2}a \sec\theta)\cos 2\theta + y \sin 2\theta = \tfrac{1}{2}a \sec\theta + a\cos\theta - b, \text{ whence}$$

$$x \cos 2\theta + y \sin 2\theta = 2a\cos\theta - b. \qquad (2)$$

The sum of the squares of (1) and (2) gives

$$3a^2\cos^2\theta - 4ab\cos\theta - (x^2+y^2-a^2-b^2) = 0.$$

The sum of (1) multiplied by sin 2θ, and (2) multiplied by cos 2θ gives

$$2a\cos^3\theta - 2b\cos^2\theta - (x-b) = 0. \qquad (4)$$

Subtracting (3) multiplied by $2\cos\theta$, from (4) multiplied by $3a$ we have

$$2ab\cos^2\theta + 2(x^2+y^2-a^2-b^2)\cos\theta - 3a(x-b) = 0. \qquad (5)$$

Subtracting (3) multiplied by $2b$, from (5) multiplied by $3a$, we have

$$[6(x^2+y^2-a^2-b^2)+8b^2]\,a\cos\theta - [9a^2(x-b) - 2b(x^2+y^2-a^2-b^2)] = 0. \quad (6)$$

Subtracting (5) multiplied by $x^2+y^2-a^2-b^2$, from (3) multiplied by $3a(x-b)$, we have

$$[9a^2(x-b) - 2b(x^2+y^2-a^2-b^2)]a\cos\theta - [12a^2b(x-b) + 2(x^2+y^2-a^2-b^2)^2] = 0. \quad (7)$$

From (6) and (7) we find

$$4(x^2+y^2-a^2)(x^2+y^2-a^2-b^2)^2 + 36a^2b(x-b)(x^2+y^2-a^2-b^2)$$
$$+ 32a^2b^3(x-b) - 27a^4(x-b)^2 = 0,$$

the equation of the locus of P.

Since the angle $PTQ = 2BOC$, we have $BC : \frac{1}{2}BC \sin BCT :: BO : BT$, or $i : \frac{1}{2}i\cos\theta :: a : BT$, whence $BT = \frac{1}{2}a\cos\theta$. Hence we have

$$KT = \tfrac{1}{2}a\sec\theta - \tfrac{1}{2}a\cos\theta = \tfrac{1}{2}a\sin^2\theta\sec\theta,\ TN = KT \sin TKN, \text{ or}$$

$$y' = \tfrac{1}{2}a\sin^2\theta\sec\theta\sin 2\theta = a\sin^3\theta. \qquad (8)$$

We also have $KN = KT\cos TKN = \frac{1}{2}a\sin^2\theta\sec\theta\cos 2\theta$, and

$$x' = \tfrac{1}{2}a\sec\theta + \tfrac{1}{2}a\sin^2\theta\sec\theta\cos 2\theta = \tfrac{3}{2}a\cos\theta - a\cos^3\theta. \qquad (9)$$

From (8) $\sin\theta = \sqrt[3]{(y' \div a)}$. Squaring (9), and substituting the value **of** $\sin\theta$, **we** find $4x'^2 + 4y'^2 - a^2 = 3\sqrt[3]{(a^4y'^2)}$, whence

$$(4x'^2 + 4y'^2 - a^2)^3 = 27a^4y'^2,$$

the equation of the required evolute.

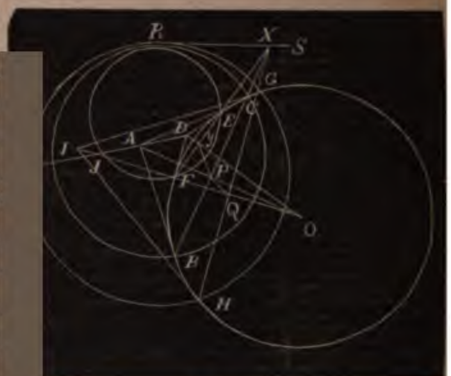
The locus of P consists of the two branches LIL' and $LI'L'$, forming cusps at L and L'. The evolute consists of the two equal branches EFE' and $EF'E'$, forming cusps at E and E'; the parts EL and EL' giving the branch LIL' of the involute, and the parts $E'L$ and $E'L'$ the branch $LI'L'$.

It will be noticed by the equation that the evolute is independent of b, or the distance of the given line from thr center of the circle.

350. "A series of circles touching each other at a point are cut by a fixed circle; show (by third Book of Euclid) that the intersections of the pairs of tangents to the latter, at the points where it is cut by each of the other circles, lie in a straight line."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let RGH, RCB, REP, &c. be the series of circles touching each other at the point R, and O the fixed one intersecting them in the points G, H; C, B; E, F; &c.; I, A, D, &c., the intersections of the tang'ts from these points. Then will I, A, D, &c., be in a straight line.



For if not, allow DA when produced not to pass through I. Join OD, OA and OI intersecting DA in J, and draw the tangent RS. Join FE and produce it to meet RS in x; BC and HG will also pass through x. Join YP and PQ. The figure $OQPYX$ is inscriptible in a circle, the angles OQX, OPX and OYX being right angles, and $\therefore$ $\angle OYP = \angle PQF$; but $OYP = DAP$, because $ADYP$ is also inscriptible in a circle, as $DO\times OY = AO\times OP$, each being equal to R^2. Therefore $\angle PQF = DAP$, whence the figure $APQJ$ is inscriptible in a circle, therefore $AO\times OP = JO\times OQ = R^2 = OI\times OQ$, whence $OI = OJ$ which is impossible; $\therefore$ D, A, I are in the same straight line.

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351. "In a plane triangle ABC, a line from C perpendicular to AC meets AB in M and another from C perpendicular to BC meets AB in N; knowing the sides a and b and the intercept $MN = m$, it is required to determine the triangle."

SOLUTION BY PROF J. SCHEFFER, HARRISBURGH, PA.

Denoting the angles of the triangle ABC lying opposite the sides a and b by A and B, respectively, and the third side AB, by x, we have $CN = a\tan B$, $CM = b\tan A$. Also $CN\sin B + CM\sin A = m$, therefore

$$a\tan B\sin B + b\tan A\sin A = m, \text{ or}$$

$$\frac{a\sin^2 B}{\cos B} + \frac{b\sin^2 A}{\cos A} = m.$$

Subst'ng for $\sin B$, $\cos B$, &c., their values, found from the sides a, b, x,

$$x^5 + mx^4 - 2(a^2+b^2)x^3 + (a^2-b^2)^2x - m(a^2-b^2)^2 = 0.$$

352. "Two chords of equal but unknown lengths are drawn at random in a given circle; find the chance of their intersection."

[Prof. Seitz obtains for answer to this problem $2 \div \pi$, while Mr. Adcock gets $\frac{1}{4}\pi$. The difference in these results arises from different conceptions of the problem. In both solutions the first chord is supposed to be drawn from any fixed point to every other point in the semi circumf., and in Mr. Seitz's solution the intersections are supposed to occur at equidistant points on the first chord; while in Mr. Adcock's solution the second chord is supposed to be drawn from equidist. points in the arc subt'd by the first chord. As these two methods give different results, and no good reason is apparent why one should be adopted rather than the other, we dismiss the quest. for the present and until further discussed by our contributors.

Solutions of 353 and 354 (incorrectly printed 353 at p. 104) will be published in the Sept. No.—Ed.]

PROBLEMS.

355. *By Benj. Headley, Dillsborough, Ind.*—The length of a garden, in the form of a parallelogram, is one rod greater than the breadth. Within the garden is a fountain; and a gravel walk extends diagonally across the garden, from corner to corner, and the distance from the fountain to one end of said walk is three rods, and to the other end four rods; and from this end of the walk, along one end of the garden, to the next corner, and from thence to the fountain, is eight rods. Required the area of the garden.

356. *By Prof. Casey.*—In a triangle ABC, BD is perpendicular to the base AC, and O is the center of gravity of the triangle. Join AO, DO & CO. Given the base AC and the angles AOD, AOC to construct the triangle ABC.

357. *By Prof. De Volson Wood.*—An elastic string without weight and of given length, has one end fixed in a perfectly smooth horizontal plane, and the other to a point in the surface of a sphere, the string being unwound. The sphere is projected on the plane from the fixed point with a linear velocity v and an angular velocity ω, winding the string on the circumference of a great circle; required the elongation of the string when fully stretched, and the subsequent motion of the sphere.

358. *By R. S. Woodward.*—Given the angles A, B and C of a plane triangle, and $d \log a$, $d \log b$ and $d \log c$; a, b, c being the sides respectively. What are the corresponding values of dA, dB and dC expressed in seconds of arc?

Query by Prof. L. G. Barbour.—If it be an axiom that the shortest distance between any two given points is measured on the straight line connecting them, do the writers on the Calculus of Variation really *prove* the same truth?

Note by Prof. Casey.—In reference to the Note on Todhunter's Trigonometry in Analyst, No. 3, p. 104, the equation should read $a\cos 2\varphi + b\cos 2\theta = c$, and not $a\cos 2\varphi + b\cos\theta = c$. Mr. T. has had his Trigonometry freed from both typographical errors long ago.

PUBLICATIONS RECEIVED.

A Treatise on Trigonometry, by Profs. Oliver, Wait and Jones of Cornell University. 102 p. 8vo. Ithaca: Finch and Apgar. 1881.

The names of the authors are a sufficient guarantee of the value of this work and of its adaptation to the wants of students in trigonometry. We extract the following from the Preface.

"It is designed as a drill-book for class use; its leading features are:

The general definitions of the trigonometric functions in terms applicable to all angles, without regard to sign or magnitude,

The expression of the functions of all angles in terms of the functions of positive angles less than a right angle, by direct reference to the definitions.

The graphical representation of functions.

The general proof of the formulæ for the functions of the sum and difference of two angles, of double angles, half-angles, etc.

The differentiation of trigonometric functions, their development thereby into series, and the computation of the trigonometric canon by means of these series." &c.

An Analysis of Relationships. By A. Macfarlane, M. A., D. Sc., F. R. S. E. From the Philosophical Magazine for June, 1881. Pamphlet. 10 p. 8vo.

The Endowment of Scientific Research. From the Annual Address of the President of the California Academy of Sciences, Prof. George Davidson, A. M., D. Ph. Pamphlet.

The Mathematical Visitor, No. 6. Artemas Martin, A. M., Editor, Erie, Pa. This No. completes Vol. I, and contains 48 pages and Index to Vol. I.

ERRATA.

On page 103, line 2, for $y = 4ax$, read $y^2 = 4ax$.
" " 104, " 1, for 353 read 354.
" " 105, " 10, from bottom, insert of, after "conceived".
" " 109, " 11, " " for falacious read fallacious.
" " 111, " 18, for paralax read parallax.

THE ANALYST.

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ON THE ELEMENTARY THEORY OF ERRORS.

BY E. L. DE FOREST.

OUR methods of estimating the error in the result of any combination of independent observations rest upon a well known elementary theorem, demonstrated in some such manner as this. Denote by x_1 and x_2 the true values of two independently observed quantities of the same kind, measured lines for example, and let their sum or difference, taken positively or negatively at pleasure, be

$$X = \pm x_1 \pm x_2. \tag{1}$$

If a single measurement of each gives x_1 and x_2 with the errors $\varDelta_1$ and $\varDelta_2$, which may happen to be either positive or negative, these will produce in X an error $\varDelta$ such that

$$X+\varDelta = \pm(x_1+\varDelta_1) \pm (x_2+\varDelta_2). \tag{2}$$

Subtracting (1) from (2) and squaring the result, we get

$$\varDelta^2 = \varDelta_1^2+\varDelta_2^2 \pm 2\varDelta_1\varDelta_2, \tag{3}$$

where the doubtful sign is + or — according as the signs of x_1 and x_2 in (1) are like or unlike. If n observations of x_1 and x_2 are taken, we have n such equations, and adding them all together and dividing their sum by n, using the brackets [ ] to denote summation, we get

$$\frac{[\varDelta^2]}{n} = \frac{[\varDelta_1^2]}{n} + \frac{[\varDelta_2^2]}{n} \pm 2\frac{[\varDelta_1\varDelta_2]}{n}. \tag{4}$$

Here the first three terms are the squares of the *quadratic mean errors* of X, x_1 and x_2. Denote these q. m. errors by M, μ_1 and μ_2. It may be presumed that the mean of the squares of the errors of a single quantity will not vary much, whatever the number of observations may be, provided it is large, and we may suppose that the number n is very large, so that M, μ_1 and μ_2 have their limiting values. In other words, μ_1 for instance will

represent the square root of the mean of the squares of all the values which the accidental error Δ_1 can possibly have, from the nature of the method of observation employed, each possible value being taken a number of times proportional to the probability of its occurrence.

The two systems of possible errors Δ_1 and Δ_2 are not supposed to be necessarily alike, for x_1 and x_2 may have been measured by two distinct methods. The last term in (4) will disappear if we make the plausible assumption that the errors are distributed symmetrically in either direction, so that positive and negative errors of x_1 of given amount are equally likely to occur; and so also in the case of x_2. Then each positive product $\Delta_1 \Delta_2$ is offset by a negative one equal in amount and equally likely to occur, so that

$$[\Delta_1 \Delta_2] = 0, \tag{5}$$

and (4) is reduced to

$$M^2 = \mu_1^2 + \mu_2^2, \tag{6}$$

a result which has been compared to the geometrical proposition respecting the "square on the hypothenuse", owing to its similar form and its important character as a basis of further investigations.* It holds good approximately even when the number n of observations is not very large, and not the same for x_1 as for x_2. If the observed quantities are three in number, so that

$$X' = \pm x_1 \pm x_2 \pm x_3,$$

the q. m. error of X' may be regarded as that of the sum or difference of the two independently observed quantities X and x_3, and is therefore given by the formula

$$\begin{aligned} M'^2 &= M^2 + \mu_3^2 \\ &= \mu_1^2 + \mu_2^2 + \mu_3^2. \end{aligned} \tag{7}$$

The theorem can thus be extended to four or any number of quantities. Each observed x may also be multiplied by a known coefficient. Any actual error of a_1x_1 is a_1 times the actual arror of x_1, it being understood that the quantity observed is x_1 and not a_1x_1, so that in n observations, the q. m. error of a_1x_1 is $a_1\mu_1$, that of a_2x_2 is $a_2\mu_2$, and so on. Hence, if we have any linear function u of independently observed quantities x_1, x_2 &c.,

$$u = a_1x_1 + a_2x_2 + a_3x_3 + \&c., \tag{8}$$

where a_1, a_2 &c., may be essentially either + or —, the q. m. error μ of u will be found from the q. m. errors μ_1, μ_2 &c. of x_1, x_2 &c., by the formula

$$\mu^2 = (a_1\mu_1)^2 + (a_2\mu_2)^2 + (a_3\mu_3)^2 + \&c. \tag{9}$$

*"Dieser Satz, welcher ausserliche Aehnlichkeit mit dem Pythagoraischen Satz der Geometrie hat, ist der wichtigste Satz der ganzen Ausgleichungsrechnung". (Jordan, *Vermessungskunde*, p. 10.)

The exactitude of the foregoing demonstration evidently depends on the correctness of the assumption that the possible true errors of any two of the observed quantities are so distributed as to satisfy the condition (5). All writers on the subject, so far as I know, have regarded the errors as true errors, or deviations from the true value of the quantity, and assumed that in any observation, positive and negative errors of equal amount are equally likely to occur. The mode of demonstration I have given is in substance the one usually followed. (See for instance Chauvenet, *Astronomy*, Vol. II, p. 497; also Helmert, *Ausgleichungsrechnung*, p. 43.) Airy adopts a different method. (*Theory of Errors of Observations*, pp. 28 to 33.) Assuming that the errors of the two observed quantities x_1 and x_2 in (1) follow the exponential law of facility

$$y = ce^{-h^2x^2},$$

while their q. m. errors μ_1 and μ_2 are in general unequal, so as to give different values to h, and consequently to c, in the two cases, it is then proved that the law of facility for the errors of X is of the same exponential form, only their squared q. m. error M^2, is equal to the sum of μ_1^2 and μ_2^2. This mode of proof, like the other, evidently presupposes that + and — errors of equal amount are equally probable, since the exponential curve is symmetrical on either side of its origin. Hence the condition (5) must be satisfied, in order to prove the theorem (6) by either method.

Some recent investigations, however, have enabled us to demonstrate a similar theorem in a manner which is free from this restriction.

Let all the possible true errors of the observed quantities, errors in the lengths of the lines x_1 and x_2 for example, be expressed by multiples of a single unit of measure Δx, which may be taken as small as we please, and let m be a whole number, so large that $m\Delta x$ is a limit which the greatest error, positive or negative, will not exceed. Any observed value, that of x_1 for instance, is here supposed to be obtained either by a single measurement or by taking the arith. mean of several measurements.

Let the probabilities of the occurrence of the various possible errors in the observed values of x_1 and x_2, from $-m\Delta x$ to $+m\Delta x$, be represented by values of p' and p'', ranging from p'_{-m} to p'_m for x_1 and from p''_{-m} to p''_m for x_2. Write these probabilities as coefficients, and the corresponding numbers of units of error as exponents of z, in the polynomials

$$p'_{-m}z^{-m} + \ldots + p'_{-1}z^{-1} + p'_0 + p'_1z + \ldots + p'_mz^m, \tag{10}$$
$$p''_{-m}z^{-m} + \ldots + p''_{-1}z^{-1} + p''_0 + p''_1z + \ldots + p''_mz^m. \tag{11}$$

If the observed values of x_1 and x_2 are to be added together to find X, so that

$$X = x_1 + x_2,$$

the error in this value of X may occur by the algebraic addition of any possible error in the value of x_2 to any possible error in the value of x_1, and the probability that any two particular possible errors of x_1 and x_2 will thus occur in combination is the product of their separate probabilities. The probability that the sum of the two errors that do occur will be a given amount, is the sum of the probabilities of all the combinations which would severally produce that amount. Hence all the possible errors in the sum of x_1 and x_2 will be represented by the exponents, and their probabilities will be the coefficients, in the product of the polynomials (10) and (11), which we denote by

$$q_{-2m}z^{-2m}+\ldots+q_{-1}z^{-1}+q_0+q_1z+\ldots+q_{2m}z^{2m}. \tag{12}$$

The probability that the error of the sum X will be a given quantity $s\Delta x$, is the coefficient q_s of z^s in this product.

To determine the quadratic mean error of X from the q. m. errors of x_1 and x_2, we employ two general properties of polynomials which, with their application to a class of questions in probability, were set forth by me in the ANALYST during 1880, and had never been published before so far as I know.

In (10) for example, let the coefficients p' represent the weights of material points ranged along an imponderable straight line or axis of X, at intervals equal to Δx, and let this axis be a lever turning about the place of p'_0 as a fulcrum. The distance from the fulcrum to the centre of gravity of the system of weights is the lever arm of the system, and its length is

$$i\Delta x=(-mp'_{-m}-\ldots-1p'_{-1}+0p'_0+1p'_1+\ldots+mp'_m)\Delta x. \tag{13}$$

The sum of the products of the weights on one side of the centre of gravity, into their distances from that centre, is equal to the sum of the products on the other side. Hence the lever arm $i\Delta x$ is the arithmetical mean of all the possible true errors in the observed value of x_1, each possible error being taken with a weight proportional to the probability of its occurrence. Likewise in (11) we denote by $j\Delta x$ the lever arm of the system of weights p'' about the place of p''_0, and this arm is the arith. mean of all the possible errors in the observed value of x_2. Of course i and j are essentially $+$ or $-$ according as the centres of gravity lie on the $+$ or $-$ side of the fulcrum. By one of the general properties of polynomials, since (12) is the product of (10) and (11), the lever arm $I\Delta x$ of its coefficients q about the place of q_0 is the algebraic sum of the arms of the two factors, or

$$I\Delta x=(i+j)\Delta x. \tag{14}$$

This lever arm is the arith. mean of all the possible errors in the value of X. Now suppose that the systems of weights in (10), (11) and (12)

revolve around their respective centres of gravity, and let their radii of gyration be e_1, e_2 and E. By a second general property of polynomials, the square of the radius of gyration for the product is equal to the sum of the squares of the radii for the two factors, or

$$E^2 = e_1^2 + e_2^2, \tag{15}$$

a relation of the same form as that in (6). It also has a similar meaning, when we regard e_1, e_2 and E as quadratic mean errors in this modif'd sense, that they are the q. m. deviations of x_1, x_2 and X, not from their true values, but from the arith. means of all their possible values. These arithmetical and quadratic means are formed as already indicated, counting each possible value, and each deviation, a number of times proportional to the probability that such value and deviation will occur. In (10) for example, the sum of all the coefficients being necessarily unity, e_1^2 is the sum of the products formed by mutiplying each weight p' into the square of its dist. from the centre of rotation, and these distances are the possible errors of x_1, in the modified sense. They are analogous to and yet distinct from what are known as residual errors, or deviations from the arith. mean of a number of observations which is very much less than the whole number of possible errors. For want of a name, the mean we are dealing with might be called the *ultimate* arith. mean, and the deviations, ultimate errors. By a well known mechanical theorem, the moment of inertia, and consequently the radius of gyration, of a system, is a minimum when the axis of rotation passes through the centre of gravity. (See for instance *Weisbach's Mech.*) The ultimate mean, or arith. mean of all the possible values of an observed quantity, is equal to the true value of that quantity, plus the lever arm or arith. mean of all its possible true errors. Hence the ultimate mean is the most probable value of the observed quantity, in the sense that it is the value which renders the quadratic mean of all the possible deviations from it a minimum. (Compare Chauvenet, *Astronomy*, II. p. 476.) In accordance with (14), the most probable error $I\Delta x$ of X is the sum of the most probable errors $i\Delta x$ and $j\Delta x$ of x_1 and x_2, and the most probable value of X is the sum of the most probable values of x_1 and x_2.

Suppose now that X is the difference, instead of the sum, of x_1 and x_2, or

$$X = x_1 - x_2.$$

The possible errors of $-x_2$ are the same as those of $+x_2$, but with contrary signs. Their probabilities remain unchanged, so that the errors are represented by the exponents, and their probabilities are the coefficients in the polynomial

$$p''_m z^{-m} + \ldots + p''_1 z^{-1} + p''_0 + p''_{-1} z + \ldots + p''_{-m} z^m, \tag{16}$$

which is (11) with its coefficients in reversed order. The lever arm of the coefficients about the place of p''_0 is the same as before, but with contrary sign. In the case of (11) it was

$$(-mp''_{-m}-\ldots-1p''_{-1}+0p''_0+1p''_1+\ldots+mp''_m)\Delta x=\quad j\Delta x, \qquad (17)$$

while for (16) it is

$$(-mp''_m-\ldots-1p''_1+0p''_0+1p''_{-1}+\ldots+mp''_{-m})\Delta x=-j\Delta x. \qquad (18)$$

The lever arm for the product of (10) and (16) is the algebraic sum of the arms of the factors, or

$$I\Delta x=(i-j)\Delta x, \qquad (19)$$

so that the arith. mean of all the possible values of X is now the difference instead of the sum, of the arith. means of all the possible values of x_1 and x_2. The radius of gyration of the coefficients in (16), about their centre of gravity, remains the same as in (11), for since the distances of each weight p'' from that centre are unchanged except in sign, their squares are entirely unchanged. Thus the radius for the product of (10) and (16) is expressed by the same relation (15) already found for the product of (10) and (11). In other words, the q. m. error of the difference of x_1 and x_2 is the same as that of their sum. And it is evident that by changing the sign of either x_1 or x_2 as in (1), we shall simply change the sign of the lever arm $i\Delta x$ or $j\Delta x$, without affecting the radius of gyration e_1 or e_2.

The reasoning by which these results are obtained may be extended to the sum or difference of three or more observed quantities, by regarding each new quantity as the subject of a second independent observation to be added to or subtracted from the total of the others, as when we derived (7) from (6). The errors in the total, and their probabilities, will be exponents and coefficients in the continued product of the three or more polynomials which represent the probabilities of error in the three or more quantities. Each quantity x may have a known coefficient a, and the q. m. error of ax is a times that of x. Thus the q. m. error e of the linear function

$$u = a_1x_1+a_2x_2+a_3x_3+\&c.,$$

whose coefficients a may be either + or —, is connected with the q. m. errors e_1, e_2 &c. of x_1, x_2 &c. by the relation

$$e^2=(a_1e_1)^2+(a_2e_2)^2+(a_3e_3)^2+\&c. \qquad (20)$$

The arith. means of all the possible true errors of a_1x_1, a_2x_2, a_3x_3 &c. are represented by the lever arms

$$a_1i\Delta x,\qquad a_2j\Delta x,\qquad a_3k\Delta x,\qquad \&c.,$$

and the arith. mean of all the possible errors of u is the sum of the above, or the lever arm

$$I\Delta x=(a_1i+a_2j+a_3k+\&c.)\Delta x, \qquad (21)$$

so that the most probable deviation of u from its true value is the algebraic sum of the most probable deviations of a_1x_1, a_2x_2, &c. from their true values.

The result (20) is more general than (9). Having been obtained without introducing any condition such as (5), it will hold good when the elementary or possible errors of the observed quantities are distributed in any manner, unrestricted by that relation to each other. The q. m. errors μ_1, μ_2, &c., in (9) represent deviations from the true values of x_1, x_2 &c., while the modified or ultimate q. m. errors e_1, e_2 in (20) represent deviations from the arith. means of all the possible values of x_1, x_2 &c., these values being weighted according to the probabilities of their occurrence. The true value of an observed quantity is usually unknown and undiscoverable. We may reasonably assume that the arith. mean of a large number of observed values of x_1 for instance, is an approximation to the arith. mean of all the possible values of x_1, when each of the latter values is weighted for the probability of its occurrence. But the mean of all the possible values is not the true value, unless the distribution of the possible true errors is restricted so that their arith. mean, the lever arm $i\Delta x$ in (13), becomes zero. There is some *a priori* reason to believe that they will be so restricted in most cases, for + and — errors of equal amount are equally likely to occur for aught we know to the contrary, at least there is usually no reason to expect a preponderance on one side rather than on the other. The difference between the demonstrations of (9) and (20) is interesting chiefly from the theoretical point of view. I think that the new theorems respecting the lever arm and radius of gyration in polynomials and their products, which have been employed here, enable us to demonstrate the principles of the arith. mean and q. m. error with greater generality, clearness and exactness than heretofore, and help to justify the common use of the quadratic mean in preference to any other mean for the purpose of estimating the amount of error, and the universal practice of taking the arith. mean of a number of observations as the standard value of the observed quantity, from which deviations or errors are to be reckoned. The arith. mean of all the possible values is always the most probable value, in the sense of being the one which renders the quadratic mean of all the possible deviations from it a minimum. It is not necessarily the most probable in the sense of being the most likely to occur in a single observation, although it is approximately the most likely to occur as the arith. mean of a large number of observations. (Lagrange, *Oeuvres*, ed. of 1868, Vol, II. p. 199; also ANALYST, Nov. 1880.)

If for example the coefficient p'_2 in (10) were zero, it would be impossible

for the error $2\Delta x$ to occur in any single observation, though it might be the arith. mean of all the possible errors.

In the probability curve

$$y = \frac{hdx}{\sqrt{\pi}}e^{-h^2x^2}, \qquad h^2 = \frac{1}{2kr^2}, \tag{22}$$

the function y, as I have heretofore shown, is in general the limit of the series of coefficients in the expansion of any polynomial to the k power, when k is a large number, or infinite. The interval Δx between successive coefficients is represented by dx at the limit, and this is a unit of measure for the distance x of any coefficient y from the origin, which is at the cent. of grav. of the series, and for the radius of gyration r of the coeff. in the first power, about their c. g. Thus hdx and h^2x^2 are abstract numbers. The curve does not necessarily represent the law of facility of deviation of a single observation from its most probable value, but it does represent the law of facility of deviation of the arith. mean of a large number k of observations of equal weight, from the most probable value of the mean. For y is the probability of a deviation $x = idx$ in the sum of the observations, i being an integer, and hence it is also the probability of a deviation $x \div k$ in their mean. In order that the distances of the coefficients from their centre of gravity may represent deviations in the mean rather than in the sum of the observations, we have only to bring the coefficients or ordinates y closer together, so that the interval or unit Δx or dx is reduced to $\Delta x \div k$ or $dx \div k$. The most probable value of the mean of any number of observations, that is, the value which renders the q. m. deviation of the mean a minimum, is the arith. mean of all the values the mean can have under the given system of elementary or possible errors, and is equal to the arith. mean of all the possible values of a single observation, each possible value being weighted for the probability of its occurrence. The probabilities of the possible errors of a single observation are usually quite unknown, but it is natural to presume that they will generally follow some such law of distribution as the probabilities of error in the sum or the mean of a large number of observations do. In other words, the sum or the mean of a large number k of equally good observations of one quantity may be regarded as the result of a single complex observation, and its proved exponential law of facility of deviation from its most probable value may be taken to represent the most plausible form of the unknown law of facility for any class of observations. It is in this way, as it seems to me, that a proof that the curve (22) represents the limit of the expansion of a polynomial, affords evidence of the general validity of this exponential law of probability. The fact that the origin and vertex of the curve is located at the centre of gravity of the

coefficients in the expansion, shows that the arithmetical mean of all the possible results of the complex observation is not only the most probable result in the sense of rendering the quadratic mean of all the possible deviations from it a minimum, but also in the sense of being the result most likely to occur. We thus reach as a plausible conclusion, a property of the arith. mean which is virtually taken for granted in the well known Gaussian demonstration of the exponential law of probability, which has been so generally followed in elementary treatises. One objection to that demonstration has been that its assumed axiom, that the arith. mean of a number of observed values is the most probable in the sense of being the one most likely to occur, is not really self evident, but stands in need of proof. (Merriman, *Least Squares*, p. 196.)

The limiting form of the expansion of an entire polynomial, when its coefficients are all positive and their sum is unity, is so important in the theory of probability as to make it desirable to have the simplest possible demonstration of it, omitting those considerations which, in my former proof (ANALYST, Sept. '79), arose from the possibility of negative coefficients. Any such polynomial may be written

$$\lambda_{-m}z^{-m}+\ldots+\lambda_{-1}z^{-1}+\lambda_0+\lambda_1 z+\ldots+\lambda_m z^m, \tag{23}$$

by adding terms with zero coefficients if required. Its expansion to the k power may likewise be written

$$l_{-km}z^{-km}+\ldots+l_{-1}z^{-1}+l_0+l_1 z+\ldots+l_{km}z^{km}. \tag{24}$$

From the relation

$$(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m)^k=l_{-km}z^{-km}+\ldots+l_{km}z^{km},$$

$$\therefore\quad k\log(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m)=\log(l_{-km}z^{-km}+\ldots+l_{km}z^{km}),$$

which holds good for all values of z, we get by differentiation with respect to z,

$$k(-m\lambda_{-m}z^{-m-1}-\ldots+m\lambda_m z^{m-1})(l_{-km}z^{-km}+\ldots+l_{km}z^{km})$$
$$=(\lambda_{-m}z^{-m}+\ldots+\lambda_m z^m)(-kml_{-km}z^{-km-1}-\ldots+kml_{km}z^{km-1}.) \tag{25}$$

Forming the coefficient of z^{i-1} in the polynomial product in each member, and equating the two to each other by the principle of indeterminate coefficients, we have

$$k(-m\lambda_{-m}l_{i+m}-\ldots+m\lambda_m l_{i-m})=(i+m)\lambda_{-m}l_{i+m}+\ldots+(i-m)\lambda_m l_{i-m}.$$

In the second member, let that part which does not have the coefficient i be transferred to the first member; then

$$-m\lambda_{-m}l_{i+m}-\ldots+m\lambda_m l_{i-m}=\frac{i}{k+1}(\lambda_{-m}l_{i+m}+\ldots+\lambda_m l_{i-m}). \tag{26}$$

This expresses the relation between the $2m+1$ coeff's λ of the given polynomial, and any group of $2m+1$ coefficients l in the expansion to the k

power, the rank of the middle l of this group, reckoned from l_0, being i. Let the coefficients λ and l be now represented by two series of ordinates referred to a common origin or place of λ_0 and l_0, so that taking the constant interval Δx between ordinates as a unit of measure, the distance of any coefficient from the origin is equal to the corresponding exponent of z. At the limit, k being very large, every l becomes an ordinate y to the limiting curve, and supposing them to be set very near together, Δx is reduced to dx, and the abscissa corresponding to $y = l_i$ will be

$$x = idx. \tag{27}$$

The whole expansion will occupy the interval $(2km+1)dx$, and this will be extended over the whole infinite axis of X if we make k an infinity of the second order. Since a finite number $2m + 1$ of consecutive ordinates y will occupy but an infinitesimal interval along the axis of X, we may regard the curve within this interval as sensibly a straight line coinciding with the tangent at its middle point, and the group of coefficients

$$l_{i-m}, \dots . l_i, \dots . l_{i+m}$$

will be represented by the group of ordinates

$$y - mdy, \dots y, \dots y + mdy,$$

so that (26) may be written

$$\begin{aligned} &- m\lambda_{-m}(- y - mdy) - \dots . + m\lambda_m(- y + mdy) \\ &= \frac{-i}{k+1}\left\{ \lambda_{-m}(y + mdy) + \dots . + \lambda_m(y - mdy) \right\}. \end{aligned}$$

Collecting separately the coefficients of y and dy, using α and β as auxiliary letters, remembering that $\Sigma\lambda = 1$, and giving i its value from (27), we get

$$\left.\begin{aligned} \alpha = -m\lambda_{-m} - \dots + m\lambda_m, \quad \beta = (-m)^2\lambda_{-m} + \dots + m^2\lambda_m, \\ -\alpha y + \beta dy = \frac{-xdx}{(k+1)(dx)^2}(y - \alpha dy). \end{aligned}\right\} \tag{28}$$

This last is the differential equation of the limiting curve, and α, β and $(k + 1)(dx)^2$ are constants, the two first being abstract numbers, while the third is a finite area. If we now regard the coefficients λ as the weights of a series of material points ranged along the imponderable axis of X at intervals equal to the unit of measure Δx, or dx at the limit, then αdx is the lever arm of this system about the place of λ_0, and $\beta(dx)^2$ is the square of its radius of gyration about the same point. If λ_0 is at the centre of gravity of the system, α is zero. If some other λ is the place of the c. g., it may be made the origin or coefficient of z^0 by adding or subtracting a constant integer in all the exponents of z in (23), a change which does not alter the coefficients in the expansion. Any λ may be made the middle of a new group of $(2m+1)$ coefficients, the value of m being changed so as to include

the given coefficients and a certain number of others which are zero. For example, in the polynomial of 5 terms,

$$\tfrac{1}{20}(10z^{-2}+4z^{-1}+3+2z+z^2),$$

if we add 1 to each exponent, and prefix two zero terms, we get the polynomial of 7 terms

$$\tfrac{1}{20}(0z^{-3}+0z^{-2}+10z^{-1}+4+3z+2z^2+z^3),$$

where the form of (23) is preserved, but the origin or place of the coeff. of z^0 has been transferred from the 3 to the 4. The significant coefficients, that is, those which are not terminal zeros, remain the same and in the same order, and hence the two expansions to the k power will have their significant coefficients alike, and the limiting curves will be the same, differing only in position relatively to the origin. In any given case, if the origin is placed at any λ we please, (28) is the diff. equation of the limiting curve, provided the lever arm adx and radius of gyration $\beta(dx)^2$ are understood to refer to this adopted origin, as the fulcrum and axis of rotation. By the law of continuity, if the origin is placed at any point intermediate between two consecutive coefficients in (23), by adding or subtracting a fractional constant in all the exponents of z, a change which does not alter the coefficients in the expansion, then reckoning a and β with reference to this new origin, (28) still holds good. Here we have the means of causing a to disappear, by transferring the origin or place of λ_0 and z^0 to the centre of gravity of the series of coefficients λ. Then in the expansion the centre of gravity remains at this new place of z^0, so that it becomes the point for which $x=0$ in the limiting curve, the diff. equation reducing to

$$\frac{dy}{y} = \frac{-xdx}{(k+1)\beta(dx)^2},$$

or by taking a new constant $h^2 = 1 \div 2(k+1)\beta(dx)^2$, (29)

$$\frac{dy}{y} = -2h^2xdx.$$

Hence by integration

$$y = ce^{-h^2x^2}. \tag{30}$$

The sum of the coefficients in the expansion is unity, so that

$$\frac{1}{dx}\int_{-\infty}^{+\infty} ydx = 1,$$

which, as is well known, gives c in terms of hdx,

$$c = \frac{hdx}{\sqrt{\pi}},$$

and the final equation of the limiting curve stands as in (22), for since k is large, or infinite, we may write k for $k+1$ in (29). The squared radius of gyration of the coefficients about their centre of gravity is represented by

$$r^2 = \beta(dx)^2$$

for the given polynomial and becomes

$$kr^2 = k\beta(dx)^2$$

for the expansion to the k power, enabling us to compute the value of h in any given case. Owing to the change of $k+1$ to k, (29) gives

$$kr^2 = 1 \div 2h^2,$$

in precise agreement with the well known result

$$\varepsilon^2 = 1 \div 2h^2,$$

where ε is the q. m. error or radius of gyration as found from the curve by integration. (ANALYST, May '79, p. 69.) The article cited discusses the form of the expansion of the binomial $(p+q)^\infty$, or what is the same thing, the form of the series of coefficients in the expansion of $(p+qz)^\infty$. This becomes a special case under our present method, if we suppose all the coefficients in (23) to be zero except $\lambda_0 = p$ and $\lambda_1 = q$.

Likewise in the equation of the probability surface

$$z = \frac{h_1 h_2 dxdy}{\pi} e^{-(h_1^2 x^2 + h_2^2 y^2)}, \qquad h_1^2 = \frac{1}{2kr_1^2}, \qquad h_2^2 = \frac{1}{2kr_2^2}, \tag{31}$$

the function z represents, as I have shown, the limit of the series of coefficients in the expansion of a polynomial of two variables to a power whose exponent k is large, when the free axes of the system of coefficients are taken as axes of X and Y. (ANALYST, March, '81.)

While the surface does not necessarily represent the law of facility of deviation of a single observed point in a plane, from its most probable position, it does represent the law of facility of deviation of the centre of gravity of a large number k of similarly observed points, from its most probable place. The function z is the probability that the sums of the deviations in the X and Y directions will be $x = idx$ and $y = jdy$ respectively, i and j being integers, and this is the probability that their arith. mean will be $x \div k$ and $y \div k$. If the ordinates z are set closer together, so that dx and dy are reduced to $dx \div k$ and $dy \div k$, they will represent the law of facility of deviation of the centre of gravity of the k observed points, from its most probable place. The most probable position of the centre of gravity is the centre of gravity of all the positions it can possibly occupy under the given system of elementary or possible errors, and the same as the c. g. of all the possible positions of a single observed point; possible positions being always weighted for the prob'ty of their occurrence. Practically, in the absence of any definite knowledge respecting the probabilities of the various possible errors of a single observation of a point, it seems most natural to presume that their distribution will resemble that of the prob's of deviation in the c. g. of a large number of similarly observed points of error. In this way, as it seems to me, the surface (31) may be held to represent the most plausible law of facility of error in the observed position of a point in a plane.

A DEMONSTRATION OF MACLAURIN'S THEOREM.

BY J. S. HAYES, HENDERSON, KENTUCKY.

LET the following be the statement of this Theorem:

$$u = f(x) = f(0) + xf'(0) + \frac{x^2}{1.2}f''(0) + \ldots \frac{x^n}{n!}f^n(0) + \ldots,$$

where $f'(x) = \frac{du}{dx}$, $f''(x) = \frac{d^2u}{dx^2}$, &c., and $f(0)$, $f'(0)$, &c. mean $f(x)$, $f'(x)$, &c., when $x = 0$.

In the demonstrations usually given of this Theorem it is necessary to know the remainder after $n + 1$ terms, to be certain that the series is convergent, that is, if

$$f(x) = f(0) + xf'(0) + \ldots \frac{x^n}{n!}f^n(0) + \frac{x^{n+1}}{n+1!}f^{n+1}(\theta x),$$

we must know that $\frac{x^{n+1}}{n+1!}f^{n+1}(\theta x)$ can be made as small as we please by sufficiently increasing n, if we would be certain that the series will approximate a correct result. This necessity makes it difficult in some cases to determine the validity of the expansion. Lagrange's and Laplace's Theorems are instances. "It must be remembered", says Mr. Todhunter, "that in quoting Maclaurin's Theorem, which serves as the foundation for those of Lagrange and Laplace, we ought strictly to have used it in the form given in Art. 95, with an expression for the remainder after $n + 1$ terms. That expression for the remainder, however, becomes so complicated in this case that we have not referred to it. The investigation of Lagrange's and Laplace's Theorems must be confessed to be imperfect, since the tests of the convergence of these series, which alone can justify our use of them as arithmetical equivalents, for the functions they profess to represent are of too difficult a character for an elementary work." (Dif. Cal., p. 115.)

The demonstration here to be given furnishes tests of the convergence of these series independent of any remainder and is therefore easily applied.

Let $f(x) = L = f(0) + L_1 = f(0) + xf'(0) + L_2 = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + L_3 = f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{n-1!}f^{n-1}(0) + L_n = f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{n-1!}f^{n-1}(0) + \frac{x^n}{n!}f^n(0) + L_{n+1}$, L_1, L_2, being func's of x. (1)

Prop. 1. To determine the conditions under which L_{n+1} and $f^{n+1}(0)$ have the same sign.

Let x have a certain positive value x_1 and suppose x to increase gradually in value from 0 to x_1.

From (1), $L_n = \frac{x^n}{n!}f^n(0)+L_{n+1}$. Suppose that L_{n+1} is positive when $x = x_1$; then $L_n > \frac{x^n}{n!}f^n(0)$. But $L_n = \frac{x^n}{n!}f^n(0)$ when $x = 0$, each being equal to zero; therefore L_n has increased more with the increase of x than $\frac{x^n}{n!}f^n(0)$. It therefore either began to increase faster, began to increase at the same rate, or began to increase more slowly and afterwards increased so much faster that its increment became greater than that of $(x^n \div n!)f^n(0)$.

Suppose that it began to increase faster. Then the first small increase of L_n is greater than that of $(x^n \div n!)f^n(0)$. Let ΔL_n be the small increase of L_n; $\Delta\frac{x^n}{n!}f^n(0)$, that of $\frac{x^n}{n!}f^n(0)$. Then $\Delta L_n > \Delta\frac{x^n}{n!}f^n(0)$; $\therefore$

$$\frac{\Delta L_n}{\Delta x} > \frac{\Delta(x^n \div n!)f^n(0)}{\Delta x},$$

or, passing to the limit,

$$\frac{dL_n}{dx} > \frac{d}{dx}\frac{x^n}{n!}f^n(0); \therefore \frac{dL_n}{dx} > \frac{x^{n-1}}{n-1!}f^n(0).$$

But, differentiating eq. (1), $f'(x) = f'(0)+\ldots+\frac{dL_n}{dx}$; $\therefore \frac{dL_n}{dx} = 0$, when $x = 0$; $\therefore \frac{dL_n}{dx} = \frac{x^{n-1}}{n-1!}f^n(0)$ when $x = 0$. Therefore $\frac{dL_n}{dx}$ increases faster than $[x^{n-1} \div (n-1!)]f^n(0)$ with the first inc. of x. Suppose we have found that $\frac{d^rL_n}{dx^r} > \frac{x^{n-r}}{n-r!}f^n(0)$, r being less than n. Differentiating (1) r times, $f^r(x) = f^r(0)+xf^{r+1}(0)+\ldots+\frac{d^rL_n}{dx^r}$; $\therefore \frac{d^rL_n}{dx^r} = 0 = \frac{x^{n-r}}{n-r!}f^n(0)$ when $x = 0$. Therefore $d^rL_n \div dx^r$ increases faster than $[x^{n-r} \div (n-r!)]f^n(0)$ with the first small increase of x; or the first small increase of $d^rL_n \div dx^r$ is greater than that of $[x^{n-r} \div (n-r!)]f^n(0)$. Let $\Delta(d^rL_n \div dx^r)$ be the small increase of $d^rL_n \div dx^r$, $\Delta[x^{n-r} \div (n-r!)]f^n(0)$ that of $[x^{n-r} \div (n-r!)]f^n(0)$.

Then
$$\Delta\frac{d^rL_n}{dx^r} > \Delta\frac{x^{n-r}}{n-r!}f^n(0).$$

Dividing by Δx and passing to the limit,

$$\frac{d^{r+1}L_n}{dx^{r+1}} > \frac{x^{n-r-1}}{n-r-1!}f^n(0).$$

But we have found that

$$\frac{dL_n}{dx} > \frac{x^{n-1}}{n-1!}f^n(0); \therefore \frac{d^2L_n}{dx^2} > \frac{x^{n-2}}{n-2!}f^n(0), \frac{d^3L_n}{dx^3} > \frac{x^{n-3}}{n-3!}f^n(0)\ldots\frac{d^nL_n}{dx^n}$$

$> f^n(0)$. Differentiating eq. (1) n times, $d^nL_n \div dx^n = f^n(x)$. Therefore $f^n(x) > f^n(0)$ on the first small increase of x. But $f^n(x) = f^n(0)$ when $x = 0$,

therefore $f^n(x)$ increases with the first small increase of x, or $f^n(x)$ begins as an increasing function of x. Therefore $f^{n+1}x$, at the beginning, when $x=0$, is pos., or $f^{n+1}(0)$ is pos. Therefore L_{n+1} and $f^{n+1}(0)$ have the same sign.

Suppose that L_n and $(x^n \div n!)f^n(0)$ begin to increase at the same rate. Then by a similar method of proof $f^n(0) = 0$, and may for all practical purposes be considered positive.

Suppose that L_n begins to increase more slowly than $(x^n \div n!)f^n(0)$. Then, by a similar method of proof, $f^n(0)$ is negative and L_n and $f^n(0)$ have different signs. But, in this case, L_n begins to grow less than $(x^n \div n!)f^n(0)$, and afterwards becomes greater. It must therefore pass through ∞ or the value of $(x^n \div n!)f^n(0)$. Therefore L_{n+1} and $f^{n+1}(0)$ have the same sign unless in the increase of x from 0 to x_1 L_n passes through ∞ or the value of $(x^n \div n!)f^n(0)$. Observe that L_n cannot pass through ∞ unless $f(x)$ does.

The same result is obtained if we suppose L_n negative.

If $f(x)$ becomes imaginary as x increases, L_{n+1} becomes imaginary also, and the preceding demonstration does not apply. In this case, L_{n+1} is neither positive nor negative.

Therefore *if* f(x) *does not become imaginary or infinite as* x *increases from* 0 *to* x_1, L_{n+1} *and* $f^{n+1}(0)$ *have the same sign unless the value of* L_n *first becomes different from and then equal to that of* $(x^n \div n!)f^n(0)$.

The demonstration applies and the result is equally true when $n = 0$; that is, when $L_n = L = f(x)$ and $(x^n \div n!)f^n(0) = f(0)$.

L, L_2, ... L_n, in eq. (1), may be called the remainders after 1, 2, ... n terms. It is evident that, if each of these remainders is numerically less than the preceding one, the sum of the series in Maclaurin's Theorem continually approaches the true value of $f(x)$ as the number of terms increases. More than that, it approaches that value as long as these remainders are in a numerically decreasing order of progression, and recedes from that value whenever they are in a numerically increasing order.

Now let $f(x) = A + Bx + Cx^2 + \ldots + Rx^{n-1} + Sx^n + Tx^{n+1} + \ldots\ldots$ where $B = f(0)$, $C = f'(0) \div 2$, ... $S = f^n(0) \div n!$, &c.; and let B and L_1, C and L_2, S and L_n, &c, have the same sign under the conditions in Prop. 1.

1°. Let all the terms after Rx^{n-1} have the same sign. By equation (1), $L_{n-1} = Rx^{n-1} + L_n$, $L_n = Sx^n + L_{n+1}$, &c. Therefore L_n is numerically less than L_{n-1}, L_{n+1} less than L_n, &c.; and the sum of the series continually approaches the value of $f(x)$. This general result is equally true if any term, as Rx^{n-1}, vanishes.

2°. Let the terms after Rx^{n-1} have alternate signs. $L_{n-1} = Rx^{n-1} + Sx^n + Tx^{n+1}$. L_{n-1}, Rx^{n-1} and Tx^{n+1} have the same sign, Sx^n a diff. sign.

Therefore if Rx^{n-1} is numerically greater than Sx^n, L_{n+1} is numerically less than L_{n-1}, and *vice versa*. So with the succeeding remainders. Therefore the sum of a decreasing series of this kind, or of a series of this kind while it is decreasing, approaches the value of $f(x)$; that of an increasing series, or a series while it is increasing, recedes from that value.

This result is equally true if any number of terms vanish between each positive and negative sign.

In this case we can determine the nearness of the approximation at any period. For $L_n = Sx^n - L_{n+1}$. Therefore Sx^n is greater than either L_n or L_{n+1}; that is the remainder after n terms is less than Sx^n. If the limit of Sx^n when n is indefinitely large, is zero, the limit of the remainder is also zero, and $f(x)$ is equal to the sum of the series

3°. Let a positive signs be followed by b negative signs, these by c positive signs, these by d negative signs, &c. Let

$$f(x) = A + \ldots + Rx^m + \ldots - Sx^n - \ldots + Tx^r + \ldots - Ux^s - \ldots,$$

the number of terms after Rx^m being $a-1$, after Sx^n, $b-1$, aftar Tx^r, $c-1$, &c. $L_m = Rx^m + \ldots - Sx^n - \ldots + L_r$. Therefore, if $Rx^m + \ldots$ is numerically greater than $Sx^n + \ldots$, L_r is numerically less than L_m and *vice versa*. So if $Sx^n + \ldots$ is numerically greater than $Tx^r + \ldots$, L_s is numerically less than L_n, and *vice versa*. Therefore, if the sum of the succeeding negative or positive terms is numerically less than that of the preceding positive or negative terms, the sum of the series approaches the value of $f(x)$, and *vice versa*.

This result is equally true if some terms vanish. In this case, as in the preceding, we can determine the nearness of the approximation at the point where the signs change, and, if the limit of Sx^n is zero $f(x)$ is equal to the sum of the series indefinitely extended.

When the signs of the terms are the same, we have not proved that $f(x)$ is equal to the sum of the series, for, although L, $L_2 \ldots L_n$, &c., continually decrease, their limit may not be zero. It is evident, however, that $f(x)$ is either equal to, or numerically greater than, the sum of the series. Their equality will be proved hereafter.

If L_n does not become different from Sx^n as x increases, $f(x) = A + Bx + \ldots + Sx^n$. Moreover, as

$$L_n = Sx^n = \frac{x^n}{n!} f^n(0), \quad \frac{d^n L_n}{dx^n} = f^n(0);$$

therefore $f^n(x) = f^n(0)$, a constant, and the succeeding terms vanish.

Suppose now that $f(x)$ becomes neither imaginary nor infinite as x increases from 0 to x_1; then $f(x)$ is continuous and L_n is also continuous; that is, for an indefinite increase of x there is a corresponding indefinite

change of $f(x)$ or L_n. Let x_2 be some value of x such that L_n, L_{n+1}, &c. have become different in value from Sx^n, Tx^{n+1}, &c., as x increases from 0 to x_2 but not equal to them again, and such that $A+Bx_2+$ &c. is a decreasing series and therefore falls under one of the conditions in 1°, 2°, 3°.

It is evident that there is such a value of x unless L_n or some succeeding remainder is always equal to Sx^n or some succeeding term, in which case, as we have seen, the series terminates, and $f(x)$ under any circumstance is equal to the sum of the series. Then $f(x_2)$ is equal to, or numerically greater than $A+Bx_2+$&c.; and L_n is equal to, or numerically greater than Sx_2^n+&c., continued indefinitely. Or, it need not be continued indefinitely; for $f(x)$ is numerically greater than $A+Bx_2+$ &c., continued to a term with the same sign that $f(x)$ has, and L_n is greater than Sx_2^n+&c., continued to a term with the same sign that L_n has. Now suppose that x gradually increases from x_2, under one of the conditions established in 1°, 2°, 3°. L_n will continue equal to, or numerically greater than Sx^n+&c., continued as before till L_n or some succeeding remainder becomes equal to Sx^n or some succeeding term. But L_n gradually changes with the gradual increase of x, and therefore cannot become equal to Sx^n till it first becomes indefinitely near to that value; and so with any succeeding remainder. But, in this case, the equal or smaller quantity Sx^n+ &c., continued as before, becomes, or has become indefinitely near to Sx^n, or the sum of the terms succeeding Sx^n becomes, or has become, indefinitely near zero. Under the conditions in 1°, 2°, 3°, therefore, L_n cannot pass through the value of Sx^n till the sum of the succeeding terms, continued indefinitely or to a term with the sign of L_n, has become indefinitely near to zero.

Now suppose that all the terms after Sx^n have the same sign. Then Sx^n $+$&c. continually increase numerically with the increase of x. It never becomes indefinitely near to Sx^n; therefore L_n never passes through the value of Sx^n. So with any succeeding remainder.

Suppose that the signs alternate, and the series decreases after Sx^n for the value of x_1. L_n cannot pass through Sx^n till $Tx^{n+1}-Ux^{n+2}+Vx^{n+3}+Wx^{n+4}$ &c., or $x^{n+1}(T-Ux)+x^{n+3}(V-Wx)+$&c., becomes indefinitely near to zero. $T-Ux$, $V-Wx$, &c. have the same sign and, after the value x_1 are not indefinitely near to zero. Moreover, for any value of x less than x_1 $T-Ux$, $V-Wx$, &c., are numerically greater than $T-Ux_1$, $V-Wx_1$, &c. On the gradual increase of x from 0 to x_1, then, neither x, $T-Ux$, $V-Wx$, &c., has become indefinitely near to zero. Therefore L_n has not passed through the value of Sx^n. So with any succeeding remainder.

In the same way we can prove that L_n has not passed through the value of Sx^n in case 3°.

This result, it is evident, is equally true if some terms vanish.

In the first part of this demonstration we had the condition that L_n must not pass through the value of Sx^n. But as we have since proved that this cannot be the case under the conditions in 1°, 2°, 3°, we may erase this condition and we have the following

General Result. In the expansion of $f(x)$ by Maclaurin's Theorem, unless $f(x)$ becomes imaginary or infinite as x increases from 0 to x_1 the sum of the series approaches the true value of $f(x)$—

1. If the signs of the terms are the same;
2. If the signs alternate, and the series decreases;
3. If m terms of one sign follow n terms of the other, and the sum of the former is numerically less than that of the latter.
4. If the series terminates.

[To be continued.]

SOME EXAMPLES OF A NEW METHOD OF SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

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EXAMPLE 1. It is required to formulate a method for integrating partial differential equations of the second order, with variable coefficients, that shall facilitate their integration from the terms of the first order. It is also required to apply the method to the integration of the equation

$$\frac{\frac{d^2u}{dt^2}}{t^2}-\frac{\frac{d^2u}{dx^2}}{x^2}=\frac{\frac{du}{dt}}{t^3}-\frac{\frac{du}{dx}}{x^3}+P.$$

Solution.—First, as to the required method. Let u be any function of t and x. Then since

$$\frac{du}{dt}=\left(\frac{du}{dt}\right)+\frac{du}{dx}\cdot\frac{dx}{dt},$$

let it be put in the symbolic form

$$D_tu=d_tu+D_tx.d_xu. \qquad (\alpha)$$

Let equation (α) be differentiated with the symbol Δ_t so that

$$\Delta_tD_tu=\Delta_t(d_tu)+\Delta_t(D_tx).d_xu+D_tx.\Delta_t(d_xu). \qquad (\beta)$$

Now if in equation (α) we place d_tu and d_xu successively for u, we shall find

$$D_{t'}.d_tu = d_{t'}.d_tu + D_tx.d_xd_tu = d_t^2u + D_tx.d_{t'}.d_xu,$$
$$D_{t'}.d_x = d_{t'}.d_xu + D_tx.d_xd_xu = d_{t'}.d_xu + D_tx.d_x^2u.$$

We shall also find, D_t and Δ_t being commutative,

$$\Delta_t(d_tu) = d_t\Delta_tu = d_tD_tu = d_t^2u + D_tx.d_td_xu,$$
$$\Delta_t(D_tx).d_xu = D_{t'}.\Delta_tx.d_xu = \Delta_tx(d_td_xu + D_tx.d_x^2u),$$
$$D_tx.\Delta_t(d_xu) = D_tx.d_x.\Delta_tu = \Delta_{t'}.D_tx.d_xu.$$

Making these substitutions in equation (β), we shall have

$$\Delta_tD_tu = d_t^2u + (D_tx + \Delta_tx)d_td_xu + D_tx.\Delta_tx.d_x^2u + \Delta_tD_tx.d_xu. \qquad (\gamma)$$

Equation (γ) is a very important one; it enables us to solve a certain class of equations with variable coefficients and to eliminate, at one step, all terms of the second order by substituting

$$\Delta_tD_tu - \Delta_tD_tx.d_xu \qquad (\delta)$$

for them. If therefore, in any equation of the second order, with variable coefficients, we substitute the expression (δ) we shall find a new feature of integration-power that will furnish us with the required method.

Secondly, as to the application of the method. Multiply the proposed equation by t^2, then

$$\frac{d^2u}{dt^2} - \frac{t^2}{x^2}\cdot\frac{d^2u}{dx^2} - \frac{1}{t}\cdot\frac{du}{dt} + \frac{t^2}{x^3}\cdot\frac{du}{dx} = Pt^2,$$

which expressed symbolically gives

$$d_t^2u - \frac{t^2}{x^2}.d_x^2u - \frac{1}{t}.d_tu + \frac{t^2}{x^3}.d_xu = Pt^2.$$

From equation (γ) we have

$$\Delta_tD_tu - \Delta_tD_tx.d_xu = d_t^2u + (D_tx + \Delta_tx)d_td_xu + D_tx.\Delta_tx.d_x^2u.$$

Comparing this with the proposed example we find that it will agree with the terms of the second order if we assume

$$D_tx = -\Delta_tx = t \div x.$$

Now these assumptions give, by differentiation and integration,

$$x^2 = t^2 + \xi,$$
$$x^2 = -t^2 + \xi',$$
$$\Delta_tD_tx = \Delta_t\times(t \div x)$$
$$= \frac{1}{x} + \frac{t^2}{x^3}.$$

Hence, by substitution,

$$\Delta_tD_tu - \left(\frac{1}{x} + \frac{t^2}{x^3}\right)d_xu - \frac{1}{t}.d_tu + \frac{t^2}{x^3}.d_xu = Pt^2.$$

$$\therefore \qquad \Delta_tD_tu - \frac{1}{t}\left(d_tu + \frac{t}{x}.d_xu\right) = Pt^2;$$

$$\therefore\ \Delta_t D_t u - (1 \div t) D_t u = Pt^2,$$

$$\because\ D_t u = \left(d_t + \frac{t}{x}.d_x\right) u.$$

Divide by t and integrate with Σ_t,

$$\therefore\ D_t u = t\Sigma_t(Pt) + tf(\xi');$$

Integrate again with S_t,

$$\therefore\ u = S_t t\Sigma_t(Pt) + S_t f(\xi') + F(\xi).$$

But $\qquad S_t tf(\xi') = S_t tf(\xi + 2t^2) = \varphi(\xi + 2t^2) = \varphi(x^2 + t^2);$

$$\therefore\ u = S_t t\Sigma_t Pt + F(x^2 - t^2) + \varphi(x^2 + t^2).$$

EXAMPLE 2. The integral of the equation

$$\frac{d^2u}{dt^2} - a^2\frac{d^2u}{dx^2} + b\left(\frac{du}{dt} + a\frac{du}{dx}\right) = 0,$$

is $u = F(x - at) + \varepsilon^{-bt} f(x + at)$. Show, by a simple method how this integral is obtained.

Solution.—Put the given equation into the following symbolic form, viz.;

$$d_t^2 u - a^2 d_x^2 u + b(d_t u + a d_x u) = 0.$$

Then, since

$$d_t^2 - a^2 d_x^2 = (d_t + ad_x)(d_t - ad_x) = (d_t - ad_x)(d_t + ad_x),$$

we *may* use two sets of independent variables, corresponding to the two equations,

$$D_t u = d_t u + a d_x u,$$
$$\Delta_t u = d_t u - a d_x u.$$

In these equations we *may* assume $D_t x = a$, and $\Delta_t x = -a$, and since the symbols D_t, Δ_t, are respectively equivalent to the compound symbols $d_t + ad_x$, $d_t - ad_x$, the substitution of them in the proposed equation gives

$$D_t \Delta_t u + b D_t u = 0.$$

Hence the following system of simultaneous equations is equivalent to the proposed equation.

$$\left.\begin{aligned} D_t x &= a \\ \Delta_t x &= -a \\ D_t \Delta_t u + b.D_t u &= 0. \end{aligned}\right\} \qquad (1)$$

The first and second of equations (1) give

$$x = at + \xi, \text{ suppose,}$$
$$x = -at + \xi', \text{ "}$$

while the third, being integrated with the symbol S_t, gives

$$\Delta_t u + bu = F(\xi) = F(x - at). \qquad (2)$$

Now the symbols D_t and Δ_t are commutative, because their equations $d_t + ad_x$, $d_t - ad_x$ are so; consequently eq. (1) may be written in the form

$$\Delta_t D_t u + b.D_t u = 0.$$

This equation being integrated with Σ_t gives

$$D_t u = \varepsilon^{-bt} f(\xi') = \varepsilon^{-bt} f(x+at),$$

which added to equation (2) and the sum equated to u, gives

$$u = F(x-at) + \varepsilon^{-bt} f(x+at),$$

for the required integral.

Proof.—That u may be written for $D_t u + \varDelta_t u + bu$ is manifest from the following operations, viz.:—As the proposed equation is represented by

$$D_t \varDelta_t u + bD_t u = 0, \qquad (a)$$

let it be differentiated separately with the symbols D_t and $\varDelta_t$, and also multiplied by the constant b, thus;

$$D_t\varDelta_t \, . \, D_t u + bD_t \, . \, D_t u = 0,$$
$$D_t\varDelta_t \, . \, \varDelta_t u + bD_t \, . \, \varDelta_t u = 0,$$
$$D_t\varDelta_t \, . \, bu \; + bD_t \, . \, bu \; = 0.$$

Hence, since D_t and $\varDelta_t$ are commutative,

$$D_t\varDelta_t(D_t u+\varDelta_t u+bu)+bD_t(D_t u+\varDelta_t u+bu) = 0.$$

This equation is of the same form as equation (a), and we learn from it that $D_t u+\varDelta_t u+bu$ will satisfy the proposed equation as well as u will satisfy it. If therefore $D_t u+\varDelta_t u+bu$ contain the requisite number of arbit'y functions, it is a complete integral of the proposed equation, and, being so, we may write u for it.

EXAMPLE 3. Given the equations

$$d_t^2\psi+(a+b)d_t d_x\psi+abd_x^2\psi = tx, \qquad (a)$$
$$t^2 d_t^2\psi + 2txd_t d_x\psi + x^2 d_x^2\psi = t^a x^b; \qquad (\beta)$$

to find their symbolic integrals.

Solution.—The late Prof. Peirce, in his admirable Integral Calculus, p. 275, proposed these equations as exercises, and solved the first for $P = tx$ by one of his own *unique* methods of solution. A solution, by somewhat different methods, is herewith offered in the hope that it will not be uninteresting to the readers of the ANALYST.

Of the first equation (a) it is well known that it consists of two independent parts; one of them, independent of $P=tx$, is called the absolute part of the integral; the other is dependent on P. These two parts, being independent of each other, may be found separately, and their sum will constitute the complete integral. The absolute part corresponds to the general supposition, $P=0$.

Under the condition $P=0$, equation (a) may take the form

$$(d_t + ad_x)(d_t + bd_x)\psi = 0,$$

in which we are at liberty to assume $D_t = d_t+ad_x$, $\varDelta_t = d_t + bd_x$, and $D_t x = a$, $\varDelta_t x = b$. The two last being integrated and increased by an arbitrary constant for each, give

$$x = at + \xi,\ x = bt + \xi'.$$
$$\therefore \xi = x - at,\ \xi' = x - bt.$$

Hence equation (a) may be written

$$D_t \Delta_t \psi = P + Q, \text{ suppose. } (Q = 0.)$$

Integrating this last equation with S_t and Σ_t, we find

$$\begin{aligned}\psi &= S_t\Sigma_t P + S_t Q + \Sigma_t Q \\ &= S_t\Sigma_t P + F(\xi) + f(\xi') \\ &= S_t\Sigma_t P + F(x-at) + f(x-bt).\end{aligned}$$

Let the function P be expounded by tx; then as $D_t x = a$, and $\Delta_t x = b$, we find

$$D_t(tx) = x + at,$$
$$\Delta_t(tx) = x + bt.$$

If therefore we may put

$$S_t\Delta_t P = \Delta_t S_t P = \frac{(tx)^3}{2.3(x+at)(x+bt)},$$

the integral of equation (a) may be written

$$\psi = \frac{(tx)^3}{2.3(x+at)(x+bt)} + F(x-at) + f(x-bt).$$

Next, dividing equation (β) by t^2 we have

$$[d_t + (x \div t) . d_x]^2 = t^{a-2} x^b,$$
$$D_t = d_t + (x \div t) . d_x.$$

Hence, if we put $D_t^2 \psi = t^{a-2} x^b$, and $\xi = (x \div t)$, we shall have

$$D_t^2 \psi = \xi^b . t^{a+b-2};$$

which being integrated *twice* with S_t, and each integration increased by an arbitrary function of ξ, gives

$$\begin{aligned}\psi &= \frac{t^{a+b}}{(a+b)(a+b-1)} . \xi^b + t F(\xi) + f(\xi) \\ &= \frac{t^a x^b}{(a+b)(a+b-1)} + F\left(\frac{x}{t}\right) + f\left(\frac{x}{t}\right).\end{aligned}$$

This result holds good for any function of $x \div t$ written in the place of a and b.

EXAMPLE 4. It is required to integrate the equation

$$\left(\frac{d\psi}{dx}\right)^2 . \frac{d^2\psi}{dt^2} - 2\frac{d\psi}{dx}\frac{d\psi}{dt} . \frac{d^2\psi}{dt\,dx} + \left(\frac{d\psi}{dt}\right)^2 . \frac{d^2\psi}{dx^2} = P\left(\frac{d\psi}{dx}\right)^3,$$

(1) when $P = \frac{d\psi}{dt} \div \frac{d\psi}{dx}$, and (2) when $P = 0$.

Solution.—This equation includes examples (4) and (5) of Section 228, Vol. II, of Peirce's "Curves Functions and Forces", under a different

notation; and to bring it within the domain of more recent discussions on partial differential equations of the second order, let us denote

$$\frac{d\psi}{dt} \text{ by } p, \ \frac{d\psi}{dx} \text{ by } q, \ \frac{d^2\psi}{dt^2} \text{ by } d_t^2\psi, \text{ and } \frac{d^2\psi}{dx^2} \text{ by } d_x^2\psi.$$

Then, the given equation, expressed symbolically, gives

$$q^2 d_t^2\psi - 2pq d_t d_x \psi + p^2 d_x^2 \psi = pq^2.$$

Divide this equation by q^2, then

$$d_t^2\psi - \frac{2p}{q} d_t d_x \psi + \frac{p^2}{q^2} d_x^2 \psi = p. \qquad (1)$$

If now we bear in mind that x is a function of t, and designate the total differential coefficient of p and q by D_tp and D_tq, when p and q are differentiated with reference to the variables x and t, we shall find

$$D_tp = d_tp + d_xp.D_tx = d_tp + D_tx.d_xp = d_t^2\psi + D_tx.d_td_x\psi, \qquad (\alpha)$$

$$D_tq = d_tq + d_xq.D_tx = d_tq + D_tx.d_xq = d_td_x\psi + D_tx.d_x^2\psi. \qquad (\beta)$$

Multiply equation (β) by D_tx and add the product to equation (α); then

$$D_tp + D_tx.D_tq = d_t^2\psi + 2D_tx.d_td_x\psi + (D_tx)^2.d_x^2\psi. \qquad (\gamma)$$

Comparing equation (1) with equation (γ) we find that

$$D_tx = -p \div q, \text{ and } D_tp + D_tx.D_tq = p.$$

By transposition and substitution, these equations give

$$p + D_tx.q = 0, \text{ and } D_tp - (p \div q).D_tq = p; \ \therefore D_t\psi = p + D_tx.q = 0,$$

$$\frac{qD_tp - pD_tq}{q^2} = \frac{p}{q} = -D_tx.$$

Hence, by integration $\psi = \xi$, an undetermined constant, and

$$p \div q = -\varphi'(t) + F(\xi);$$

whence $D_tx = -p \div q$, gives $-Dx = -\varphi'(t)dt + F(\xi)dt$.

By integration we get

$$-x = -\varphi(t) + tF(\xi) + f(\xi),$$

$$\therefore 0 = x - \varphi(t) + tF(\xi) + f(\xi) = x - \varphi(t) + tF(\psi) + f(\psi). \qquad (2)$$

The condition (2), viz., $P = 0$, is readily fulfilled by making $D_tp + D_tx.D_tq = 0$, from which we obtain

$$\frac{qD_tp - pD_tq}{q^2} = 0.$$

By means of this equation, and the condition $D_t\psi = 0$, we readily find

$$-D_tx = F(\xi); \text{ whence } -x = tF(\xi) + f(\xi);$$

$$\therefore 0 = x + tF(\xi) + f(\xi) = x + tF(\psi) + f(\psi), \qquad (3)$$

a result that might have been obtained by making $\varphi(t) = 0$ in eq'n (2).

Equation (3) is the typical eq'n of a certain class of Ruled Surfaces.

Remarks on the Solution of Prob. 352 by the Editor.—Though the solution of this problem presents no difficulty yet our contributors obtain different results. This discrepancy results, however, from the different assumptions of the manner in which the chords would be drawn.

If it were required to draw all possible chords perpendicular to a given diameter, then, it seems clear to me, the number would be proportional to the *diameter* of the circle; again, if random chords are drawn through a point within a given circle, the rays would evidently be equally dense on the circumference of a circle described about that point as a center; but because, in a given circle, if any one point of the circumference be joined to every other point, by straight lines, no two of these lines will have the same length, and yet some one of them will represent, both in length and direction, every chord that can be drawn in that circle, either through a point or perpendicular to a diameter. I hold, therefore, that when chords of a circle are compared, without reference to any line or point within or without the circle, these lines only should be considered the extremities of which are equidistant on the circumference of the given circle.

Since the note at p. 135 was written, a solution of this problem, by Mr. H. Heaton, has been received which differs from both the other solutions referred to. Mr. Heaton infers from the statement that the chords are of "unknown lengths" that all lengths, from 0 to $2r$ should be regarded as equally probable; whence he obtains $1-(2\div\pi)$ for the required chance.

The answers received are therefore as follows:—

By Mr. Adcock, the required chance $= \frac{1}{4}\pi$.

By Mr. Seitz, " " " $= 2\div\pi$.

By Mr. Heaton, " " " $= 1-2\div\pi$.

The solution, by Mr. Adcock's method may be presented as follows:

With the center at C, describe the given circle AC. Draw any chord as EF, at right angles to the diameter AB, intersecting AB in D, and with CD as radius describe a circle concentric with the given circle.

The chord EF may be denoted by $2\sin\varphi$, and if tangents be drawn at all points of the circumf. of the concentric circle and terminate in the circumf. of the given circle, they will represent all the *equal* chords that can be drawn for any assumed value of φ, and the total number for each value of φ will be 2π, while the number that intersect, for each value of φ, will be 4φ.

Hence, summing for all values of φ from 0 to $\frac{1}{2}\pi$, we get

$$\int_0^{\frac{1}{2}\pi} \frac{4\varphi d\varphi}{2\pi} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \varphi d\varphi = \frac{1}{4}\pi.$$

SOLUTION OF PROBLEMS 353 & 354, *IN NUMBER THREE.*

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353. "Required the average area of the circles described on the focal chords of a given ellipse as diameters."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $\theta =$ the angle which a focal chord makes with the major axis. Then the lengths of the two parts of the chord are
$\dfrac{a(1-e^2)}{1+e\cos\theta}$ and $\dfrac{a(1-e^2)}{1-e\cos\theta}$, and the length of the chord is $l = \dfrac{2a(1-e^2)}{1-e^2\cos^2\theta}$.

Hence the average area of the circle described on the chord is

$$\varDelta = \int_0^\pi \tfrac{1}{4}\pi l^2 d\theta \div \int_0^\pi d\theta = a^2 \int_0^\pi \frac{(1-e^2)^2}{(1-e^2\cos^2\theta)^2} d\theta.$$

Let $\tan\theta = \surd(1-e^2)\tan\varphi$, then

$$\cos^2\theta = \frac{\cos^2\varphi}{1-e^2\sin^2\varphi},\quad \frac{1-e^2}{1-e^2\cos^2\theta} = 1-e^2\sin^2\varphi,$$

$$d\theta = \frac{\surd(1-e^2)\cos^2\theta d\varphi}{\cos^2\varphi} = \frac{\surd(1-e^2)d\varphi}{1-e^2\sin^2\varphi}.$$

$\varphi = 0$, when $\theta = 0$, and $\varphi = \pi$, when $\theta = \pi$; therefore

$$\varDelta = a^2(1-e^2)^{\frac{1}{2}} \int_0^\pi (1-e^2\sin^2\varphi)d\varphi = \pi ab(1-\tfrac{1}{2}e^2).$$

---

354. "A cube slides down an inclined plane with four of its edges horizontal. The middle point of its lowest edge comes in contact with a small fixed obstacle and is reduced to rest. Find the direction of the impulsive reaction of the obstacle, and show that it is independent of the velocity of the cube and of the inclination of the plane. Determine also the limiting velocity that the cube may be on the point of overturning."

SOLUTION BY PROF. H. T. EDDY, CINCINNATI, OHIO.

Let P be the opstacle and v the velocity of impact of the cube whose side is $2a$, whose mass is unity and whose radius of gyration squared $k^2 = \frac{2}{3}a^2$. Also $PC = a\surd 2$.

The velocity along PC is destroyed by the impact, after which C has a velocity u perpendic'r to PC and there is also rotary velocity w about C, such that $u = aw\surd 2$.

Take the moments of momenta about P; $\therefore ua\sqrt{2}-va+k^2w=0$.

Substitute the values of u and k^2 in this equation, $\therefore aw=\frac{3}{8}v$, $\therefore u=v.\frac{3}{8}\sqrt{2}$.

The impulse f which must be combined with the initial momentum v in order that the cube may have the resultant momentum u is

$$f^2=v^2+u^2-2vu\cos\tfrac{1}{4}\pi=(1+\tfrac{18}{64}-\tfrac{6}{8})v^2;\ \therefore f=v.\tfrac{1}{8}\sqrt{34}.$$

To find the inclination a, we have $u^2=v^2+f^2-2vf\cos a$;

$$\therefore \cos a=5\div\sqrt{34},$$

which is *constant*.

That this is the correct result may be seen by showing that f applied at P in the direction a will cause the angular velocity w found above. For take moments about C; $k^2w=fa\sqrt{2}.\sin(\frac{1}{4}\pi-a)=fa(\cos a-\sin a)$;

$$\therefore \tfrac{2}{3}aw=\frac{\sqrt{34}}{8}v\left(\frac{5-3}{\sqrt{34}}\right)=\frac{v}{4};\ \therefore aw=\tfrac{3}{8}v,$$

which is the result previously obtained.

The initial velocity of the cube will be just sufficient to overturn it when the energy of translation and rotation are sufficient to raise the center C to a point directly above P; i. e., when

$$u^2+k^2w=2ga\sqrt{2}.[1-\cos(\tfrac{1}{4}\pi-\beta)];$$
$$\therefore\quad v^2=\tfrac{16}{3}\sqrt{2}.ag[1-\cos(\tfrac{1}{4}\pi-\beta)].$$

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 358; Prof. W. P. Casey, 355, 356, 358; George Eastwood, 356, 358; Prof. Asaph Hall, 358; Prof. E. B. Seitz, 355, 358; R. S. Woodward, 358. We also received a solution of 353, of No. 3, from the proposer, Mr. Hoover, and a solution of 352 from Mr. Heaton.

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355. "The length of a garden, in the form of a parallelogram, is one rod greater than the breadth. Within the garden is a fountain; and a gravel walk extends diagonally across the garden, from corner to corner, and the distance from the fountain to one end of said walk is three rods, and to the other end four rods; and from this end of the walk, along one end of the garden, to the next corner, and from thence to the fountain, is eight rods. Required the area of the garden."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $ABCD$ be the rectangular garden, AC the diagonal walk, and F the fountain.

Let $BC=x$, $AB=x+1$, $FA=3$ rods, $FC=4$ rods. Then since $CB+BF=8$ rods, $BF=8-x$;

$$\cos ABF=\frac{(x+1)^2+(8-x)^2-3^2}{2(x+1)(x-8)}=\frac{x^2-7x+28}{(x+1)(x-8)},$$

$$\cos CBF=\frac{x^2+(8-x)^2-4^2}{2x(8-x)}=\frac{x^2-8x+24}{x(8-x)}.$$

But $\cos ABF=\sin CBF$, whence we find $\cos^2 ABF+\cos^2 CBF=1$.

By substitution and reduction we find

$$x^6-14x^5+153x^4-680x^3+640x^2+768x+576=0.$$

This equation has two positive roots, $x=3$, and $x=4.4236748$. For the first value of x the sides of the garden would be 3 and 4 rods, and the fountain would be situated at D, the corner of the garden. For the second value of x the fountain is situated as represented in the diagram, and the area of the garden $=x(x+1)=23.99257$ square rods.

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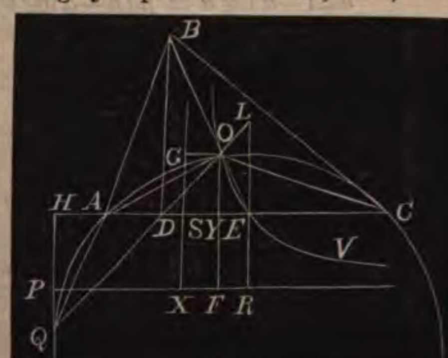
356. "In a triangle ABC, BD is perpendicular to the base AC, and O is the center of gravity of the triangle. Join AO, DO and CO. Given the base AC and the angles AOD, AOC to construct the triangle ABC."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Analysis. Look upon the line AC as being given in position, therefore the points A, C are given, and as the angle AOC is given, therefore the circle $QAOC$ is given. Produce OD to Q, and as the angle AOQ is given, therefore the line AQ is given, and, as A is a given point, therefore Q is a given point. Draw BO and produce it to E and E is a given point, then draw EL perpendicular to AC meeting QO produced in L; and, BO being equal to $2OE$, $\therefore$ $DO=2OL$. The perpendicular QH to CA produced is in position, and $\therefore$ QH is a given line. And as $DL : LO$ so make $HE : ES$; $\therefore$ ES is a given line and S a given point; and as $LD : DO$ so make $QH : HP$; $\therefore$ HP is a given line and P a given point; through P, O and S, draw PR, OG parallel to AC, and OF, GH par'l to BD.



Now $HE : ES :: DL : LO :: DE : EY$; $\therefore HD : SY :: DE : EY$, (i. e.) $ED : DH :: EY : YS$; (1) and, again, $QH : HP :: LD : DO :: LE : OY$; $\therefore LE : QH :: OY : HP$ or $YF :: ED : DH :: EY : YS$, by (1); $\therefore EY : YS :: OY : YF$, and (Euc. VI., 16) the fig. YG = fig. YR; to each add the fig. YX, $\therefore$ the rectangle OX = rectangle EX; but the rectangle EX is given, whence the locus of O is a rectangular hyperbola OEV whose asymptotes are XG, XR. Therefore O is a given p't; whence B is a given point, and $\triangle ABC$ is given. The const'n is evident.

COR. If through a given point a line be drawn intersecting two given lines, and if the portion of this line intercepted by the two given ones be divided in a given ratio, the locus of the point of section is a hyperbola.

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357. [No solution received. A solution by the proposer is invited]

358. "Given the angles A, B and C of a plane triangle, and $d\log a$, $d\log b$ and $d\log c$; a, b, c being the sides respectively.

What are the corresponding values of dA, dB and dC expressed in seconds of arc?"

SOLUTION BY R. S. WOODWARD, DETROIT, MICH.

Since in any plane triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$$d\log\sin A - d\log a = d\log\sin B - d\log b = d\log\sin C - d\log c.$$

Put α = log sin dif. for 1″ corresponing to A,

β = " " " " " " " B,

γ = " " " " " " " C.

Then, dA, dB and dC being expressed in seconds of arc, the above equation becomes

$$\alpha dA - d\log a = \beta dB - d\log b = \gamma dC - d\log c = Q \text{ say.} \qquad (1)$$

Since $A+B+C = 180°$,

$$dA + dB + dC = 0. \qquad (2)$$

From (1)

$$dA = \frac{Q + d\log a}{\alpha},$$

$$dB = \frac{Q + d\log b}{\beta}, \qquad (3)$$

$$dC = \frac{Q + d\log c}{\gamma},$$

which substituted in (2) gives

$$Q = -\frac{\beta\gamma\, d\log a + \gamma\alpha\, d\log b + \alpha\beta\, d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

Therefore by (3)

$$dA = \frac{\beta(d\log a - d\log c) + \gamma(d\log a - d\log b)}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dB = \frac{\gamma(d\log b - d\log a) + \alpha(d\log b - d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dC = \frac{\alpha(d\log c - d\log b) + \beta(d\log c - d\log a)}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

SOLUTION BY PROF. ASAPH HALL, NAVAL OBSERVAT'Y, WASH., D. C.

Projecting the sides b and c of a plane triangle on the side a we have

$$a = b\cos C + c\cos B;$$

and in a similar manner we find two more equations of this kind. Differentiating these, considering all the parts variable and noticing the condition

$$A + B + C = \pi = \text{constant},$$

we have the three symmetrical differential equations of a plane triangle,

$$da = \cos C.db + \cos B.dc + c\sin B.dA,$$
$$db = \cos A.dc + \cos C.da + a\sin C.dB,$$
$$dc = \cos B.da + \cos A.db + b\sin A.dC.$$

The quantities $\cos C.db$ and $\cos B.dc$ are the increments of the sides b and c projected on the side a; and the sum of these applied to da gives the total increment of the side a. Also $c\sin B$ is the perpendicular from the angle A on the opposite side, and the total increment of a divided by $c\sin B$ gives dA. For seconds of arc we must multiply this ratio by 206264.8, the number of seconds in radius. Since

$$da = \frac{a.da}{a} = a.d\log a,$$

we may use the differentials themselves or $d\log a$, &c. One should notice the similarity of these differential equations to those of spherical trigonometry.

SOLUTION OF PART (2), PROB. 344, BY PROF. SEITZ (SEE P. 101).—Let P be the third random point, and EF the random chord, through it.

Draw the radius OL perpendicular to EF. Let p_0, p_1, p_2, p_3 be the respective probabilities that AB, CD, EF will intersect in 0, 1, 2, 3 points.

Let $EP = z$, $EF = z'$, $\angle EOL = \phi$, and $\angle KOL = \rho$. Then $z' = 2r \times \sin\phi$; an element of the circle at P is $r\sin\phi\, d\phi\, dz$, and for an elemental change in the direction of EF we have $d\rho$.

For non-intersection, the chord CD being between AB and EF, the limits of θ are 0 and π; of φ, 0 and θ; of ψ, 0 and φ; of μ, $\varphi-\theta$ and $\theta-\varphi$; and of ρ, $\psi-\varphi$ and $\varphi-\psi$. The result of these integrations must be multiplied by 6 to allow for the interchange of the chords.

For non-intersection, the chord EF having its extremities in the arc BD, the limits of θ are 0 and π; of φ, 0 and θ; of ψ, 0 and $\theta-\varphi$; of μ, $2\psi-\theta+\varphi$ $(=\alpha)$ and $\theta-\varphi$; and of ρ, $\varphi+\psi$ and $\theta+\mu-\psi$ $(=\beta)$. This result must be multiplied by 4 to allow for the positions of EF, in which its extremities are in the arc AC, and for the positions of CD and EF in which their extremities are in the arc AGB.

The limits of ω are 0 and π; of x, 0 and x'; of y, 0 and y'; and of z, 0 and z'. Hence, since the whole number of ways the three chords can be drawn is $(\pi r^2)^3.\pi^3=\pi^6r^6$, we have

$$p_0=\frac{6}{\pi^6r^6}\int_0^\pi\int_0^\theta\int_0^\phi\int_{\phi-\theta}^{\theta-\phi}\int_{\psi-\phi}^{\phi-\psi}\int_0^\pi\int_0^{x'}\int_0^{y'}\int_0^{z'}r^3\sin\theta\sin\varphi\sin\psi\,d\theta d\varphi d\psi d\mu \times d\rho d\omega dx dy dz$$
$$+\frac{4}{\pi^6r^6}\int_0^\pi\int_0^\theta\int_0^{\theta-\phi}\int_\alpha^{\theta-\phi}\int_{\phi+\psi}^{\beta}\int_0^\pi\int_0^{x'}\int_0^{y'}\int_0^{z'}r^3\sin\theta\sin\varphi\sin\psi\,d\theta d\varphi d\psi \times d\mu d\rho d\omega dx dy dz$$
$$=\frac{1}{3}-\frac{5}{\pi^2}+\frac{105}{4\pi^4}.$$

We will next find the probability that the chords CD and EF will both intersect AB. By reference to the first part of the solution we readily see that the probability that a third chord will also intersect AB is

$$P=\frac{16}{\pi^5}\int_0^{\frac{1}{2}\pi}\left\{2\int_0^\theta\int_{\theta-\phi}^{\theta+\phi}\sin^2\varphi\,d\varphi d\mu+2\int_\theta^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}\sin^2\varphi\,d\varphi d\mu\right\}^2\sin^2\theta\,d\theta$$
$$=\frac{2}{15}+\frac{1}{\pi^2}+\frac{49}{4\pi^4}.$$

Now P is evidently equal to the probability that the chords CD and EF will both intersect AB, and not each other, plus p_3; $\therefore \frac{1}{3}p_2+p_3=P$. (1)

The probability p, that AB and CD will intersect, is equal to the probability that they will intersect each other, and not EF, plus the probability that CD and EF will both intersect AB, and not each other, plus the prob. that AB and EF will both intersect CD, and not each other, plus p_3; hence

$$\tfrac{1}{3}p_1+\tfrac{2}{3}p_2+p_3=p. \qquad (2)$$

We also have
$$p_1+p_2+p_3=1-p_0, \qquad (3)$$

From (1), (2) and (3), knowing the values of P, p and p_0, we find

$$p_1=\frac{2}{5}+\frac{3}{\pi^2}-\frac{42}{\pi^4},\quad p_2=\frac{1}{5}+\frac{3}{2\pi^2}+\frac{21}{4\pi^4},\quad p_3=\frac{1}{15}+\frac{1}{2\pi^2}+\frac{21}{2\pi^4}.$$

PROBLEMS.

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359. *By George Lilley, A. M., Corning, Iowa.*—Find the greatest and least number of balls of equal diameter (radius r) that can be put in a given box, a feet long, b feet wide and c feet high.

360. *By Tho's Spencer, Meriden, Conn.*—Prove, of all spherical triangles of equal area, that of the least perimeter is equilateral.

361. *By John H. Christie, State College, Pa.*—A right cone, radius of base R and altitude a, is pierced by a cylinder whose radius is r, the axis of the cylinder intersecting the axis of the cone at right angles and at a point whose distance from the vertex of the cone is b. Required the solidity common to the cone and cylinder.

362. *By E. B. Opdycke, Pulaski, Ohio.*—A section of an embankment is a feet long; the top width of both ends is b feet; the width of the ends at bottom is c and d feet, respectively, and the corresponding depths of the ends are e and g feet. Develop a Rule, and give the contents.

363. *By Prof. W. W. Johnson, U. S. Naval Acad., Annap., Md.*—The tangent at one end of a chord of an ellipse is parallel to the line joining the other end with a fixed point within the ellipse. Show that the area of the locus of the middle point of the chord is one half the area of the ellipse.

364. *By W. E. Heal, Marion, Ind.*—Discuss the curve whose equation is $x = \log[y+\sqrt{(y^2-1)}]$, and find its area and length.

365. *Selected, By Prof. H. T. Eddy, Cincinnati, Ohio.*—Show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sqrt{(1-c)}\,.\,d\theta}{1-c\cos^n\theta} = \frac{\pi}{\sqrt{(2n)}}$$

when c is indefinitely nearly equal to unity, n being a positive quantity.

366. *By R. S. Woodward, Detroit, Mich.*—What is the probable error in a system of errors (y) given by the equation

$$\begin{aligned} y = {} & a_1\cos(\ rz + \beta_1) + \\ & a_2\cos(2rz + \beta_2) + \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & a_n\cos(nrz + \beta_n), \end{aligned}$$

wherein a_1, a_2, etc., β_1, β_2, etc., and r are constants, and all values of z are equally likely?

367. *By Prof. Simon Newcomb, Wash., D. C.*—Prove the equation

$$\begin{aligned} \log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) &= -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\text{etc.} \\ &\quad -2\eta\cos x-\tfrac{1}{2}\,2\eta^2\cos 2x-\tfrac{1}{3}\,2\eta^3\cos 3x-\text{etc.} \\ &= \sum_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\sum_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix. \end{aligned}$$

NOTE.—IN the demonstration of Maclaurin's Theorem, at p. 149 *et seq.*, for the covenience of the printer, the symbol (!) is substituted for the symbol (|____) used by the author; but, by an oversight, the parentheses that should include compound terms to which the symbol refers, was omitted. The omission, in this case, however, is unimportant, as the reader will readily perceive that, throughout the article, the symbol ! refers, in every case, to the whole of the compound divisor which precedes it.

ERRATA.

On page 127, line 6 from bottom, for u read α.
" " " " 5 " " for v read β.
" " " " 4 " " for w read γ.
" " 151, " 5 and 8, for $f^n(0)$ read $f^{n+1}.(0)$
" " " " 14, for L_n read L_{n-1}.
" " " last line, for Tx^{n+1} read L_{n+1}.
" " 152, line 17, for L_n read L_m.

// THE ANALYST.

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NOTE ON A SPECIAL SYMMETRICAL DETERMINANT.

BY THOMAS MUIR, M. A., F. R. S. E., BEECHCROFT, SCOTLAND.

1. IN the Cambridge and Dublin Mathematical Journal, Vol. I, p. 286 (1846), a correspondent, signing himself H (1), gives the identity

$$\begin{aligned}
&(a_1a_2-b_1b_2-c_1c_2)(b_1b_2-c_1c_2-a_1a_2)(c_1c_2-a_1a_2-b_1b_2)\\
&-(a_1a_2-b_1b_2-c_1c_2)(b_1c_2+b_2c_1)^2-(b_1b_2-c_1c_2-a_1a_2)(a_1c_2+a_2c_1)^2\\
&\qquad\qquad\qquad-(c_1c_2-a_1a_2-b_1b_2)(a_1b_2+a_2b_1)^2\\
&+2(b_1c_2+b_2c_1)(a_1c_2+a_2c_1)(a_1b_2+a_2b_1)\\
&\qquad=(a_1^2+b_1^2+c_1^2)(a_1a_2+b_1b_2+c_1c_2)(a_2^2+b_2^2+c_2^2).
\end{aligned}$$

No proof is added, and at first sight it might appear as if the verification of the identity would be a trifle laborious. The object of the present note is to give a proof interesting to some extent in itself and also as showing how a generalization of the identity may be effected.

2. The left hand member is expressible by a determinant, viz.,

$$\begin{vmatrix} a_1a_2-b_1b_2-c_1c_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & b_1b_2-c_1c_2-a_1a_2 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & c_1c_2-a_1a_2-b_1b_2 \end{vmatrix},$$

or, if we write S_3 for $a_1a_2+b_1b_2+c_1c_2$, by

$$\begin{vmatrix} 2a_1a_2-S_3 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & 2b_1b_2-S_3 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S_3 \end{vmatrix}.$$

Expanding this according to descending powers of S_3 we have

$$-S_3^3+S_3^2(2a_1a_2+2b_1b_2+2c_1c_2)$$

$$-S_3\left\{\begin{vmatrix} 2b_1b_2 & b_1c_2+b_2c_1 \\ b_1c_2+b_2c_1 & 2c_1c_2 \end{vmatrix}+\begin{vmatrix} 2a_1a_2 & a_1c_2+a_2c_1 \\ a_1c_2+a_2c_1 & 2c_1c_2 \end{vmatrix}\right.$$

$$+\begin{vmatrix} 2a_1a_2 & a_1b_2+a_2b_1 \\ a_1b_2+a_2b_1 & 2b_1b_2 \end{vmatrix}\Bigg\}$$

$$+\begin{vmatrix} 2a_1a_2 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 \\ a_1b_2+a_2b_1 & 2b_1b_2 & b_1c_2+b_2c_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2 \end{vmatrix}$$

where the term independent of S_3

$$=\begin{vmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} \times \begin{vmatrix} a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ c_2 & c_1 & 0 \end{vmatrix} = 0$$

and the coefficient of the first power of S_3

$$=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}^2$$

$$=\begin{vmatrix} a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2 \\ a_1a_2+b_1b_2+c_1c_2 & a_2^2+b_2^2+c_2^2 \end{vmatrix}$$

$$=(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-S_3^2.$$

The original determinant is thus found

$$=-S_3^3+S_3^2(2S_3)+S_3[(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-S_3^2]$$
$$=S_3(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)$$

as was to be proved.

3. A glance at the steps of this proof suffices to suggest a directio which the theorem may be extended. Writing S_4 for $a_1a_2+b_1b_2+$ $+d_1d_2$ we have

$$\begin{vmatrix} 2a_1a_2-S_4 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & a_1d_2+a_2d_1 \\ a_1b_2+a_2b_1 & 2b_1b_2-S_4 & b_1c_2+b_2c_1 & b_1d_2+b_2d_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S_4 & c_1d_2+c_2d_1 \\ a_1d_2+a_2d_1 & b_1d_2+b_2d_1 & c_1d_2+c_2d_1 & 2d_1d_2-S_4 \end{vmatrix}$$

$$=S_4^4-S_4^3(2a_1a_2+2b_1b_2+2c_1c_2+2d_1d_2)$$
$$+S_4^2(-|a_1b_2|^2-|a_1c_2|^2-|a_1d_2|^2-|b_1c_2|^2-|b_1d_2|^2-|c_1d_2|^2$$
$$-S_4(0+0+0+0)+0$$
$$=-S_4^4+S_4^2[(a_1a_2+b_1b_2+c_1c_2+d_1d_2)^2-(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+$$
$$=-(a_1a_2+b_1b_2+c_1c_2+d_1d_2)^2(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2).$$

The transition from these two cases to the general theorem dealing two sets of n letters can now be accomplished.

4. Putting $d_1=d_2=0$ in (2) and dividing both members by $-($ $+b_1b_2+c_1c_2)$ we obtain (1); and similarly any case may be derived that which follows it.

5. Putting $a_1 = a_2 = a$, $b_1 = b_2 = b$, $c_1 = c_2 = c$, $d_1 = d_2 = d$, in (2) we have

$$\begin{vmatrix} a^2-b^2-c^2-d^2 & 2ab & 2ac & 2ad \\ 2ab & b^2-c^2-d^2-a^2 & 2bc & 2bd \\ 2ac & 2bc & c^2-d^2-a^2-b^2 & 2cd \\ 2ad & 2bd & 2cd & d^2-a^2-b^2-c^2 \end{vmatrix} = -(a^2+b^2+c^2+d^2)^4$$

and from this, by making $d = 0$, there results

$$\begin{vmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{vmatrix} = (a^2+b^2+c^2)^3$$

and thence in the same way

$$\begin{vmatrix} a^2+b^2 & 2ab \\ 2ab & b^2-a^2 \end{vmatrix} = -(a^2+b^2)^2,$$

the identity well known in connection with Euc. I. 47, giving the sum of two squares as a square.

THE BITANGENTIAL.

BY WILLIAM E. HEAL, MARION, INDIANA.

THE curve which passes through the points of contact of bitangents of a given curve is called the *bitangential* of that curve.

Such a curve may be determined by the method of problem 331, ANALYST. It is, however, desirable to obtain a curve of lower order, and for this purpose Salmon has given two methods in Higher Plane Curves.

Let us put

$$A = \begin{vmatrix} \frac{d^2u}{dy^2} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dy\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad B = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dz\,dx} \\ \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz^2} \end{vmatrix}, \quad C = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dydx} \\ \frac{d^2u}{dxdy} & \frac{d^2u}{dy^2} \end{vmatrix},$$

$$D = \begin{vmatrix} \frac{d^2u}{dx\,dz} & \frac{d^2u}{dz\,dy} \\ \frac{d^2u}{dx^2} & \frac{d^2u}{dxdy} \end{vmatrix}, \quad E = \begin{vmatrix} \frac{d^2u}{dxdy} & \frac{d^2u}{dx\,dz} \\ \frac{d^2u}{dy^2} & \frac{d^2u}{dy\,dz} \end{vmatrix}, \quad F = \begin{vmatrix} \frac{d^2u}{dy\,dz} & \frac{d^2u}{dxdy} \\ \frac{d^2u}{dz^2} & \frac{d^2u}{dx\,dz} \end{vmatrix};$$

$$H = \begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2u}{dxdy} & \frac{d^2x}{dx\,dz} \\ \frac{d^2u}{dydx} & \frac{d^2u}{dy^2} & \frac{d^2u}{dy\,dz} \\ \frac{d^2u}{dz\,dx} & \frac{d^2u}{dz\,dy} & \frac{d^2u}{dz^2} \end{vmatrix};$$

$$\theta = A\frac{dH^2}{dx^2}+B\frac{dH^2}{dy^2}+C\frac{dH^2}{dz^2}+2D\frac{dH}{dy}\frac{dH}{dz}+2E\frac{dH}{dz}\frac{dH}{dx}+2F\frac{dH}{dx}\frac{dH}{dy},$$

$$\Phi = A\frac{d^2H}{dx^2}+B\frac{d^2H}{dy^2}+C\frac{d^2H}{dz^2}+2D\frac{d^2H}{dy\,dz}+2E\frac{d^2H}{dz\,dx}+2F\frac{d^2H}{dxdy},$$

$$\Psi = A\frac{d^2H}{dx^2}+B\frac{d^2\Phi}{dy^2}+C\frac{d^2\Phi}{dz^2}+2D\frac{d\Phi}{dy}\frac{d\Phi}{dz}+2E\frac{d\Phi}{dz}\frac{d\Phi}{dx}+2F\frac{d\Phi}{dx}\frac{d\Phi}{dy},$$

$$\chi = A\frac{dH}{dx}\frac{d\Phi}{dx}+B\frac{dH}{dy}\frac{d\Phi}{dy}+C\frac{dH}{dz}\frac{d\Phi}{dz}+D\left(\frac{dH}{dy}\frac{d\Phi}{dz}+\frac{dH}{dz}\frac{d\Phi}{dy}\right)$$
$$+E\left(\frac{dH}{dz}\frac{d\Phi}{dx}+\frac{dH}{dx}\frac{d\Phi}{dz}\right)+F\left(\frac{dH}{dx}\frac{d\Phi}{dy}+\frac{dH}{dy}\frac{d\Phi}{dx}\right).$$

Then the equation of the bitangential of a quartic is

$$\theta = 3H\Phi.$$

(Salmon's Higher Plane Curves, page 337.)

In attempting to find the bitangential of a quintic Salmon says (Higher Plane Curves, page 339), "In order to form the bitangential curve of a quintic, the quantity to be calculated is $(27q_2q_5-5q_3q_4)^2 = 5(4q_3^2-9q_2q_4)(5q_4^2-12q_3q_5)$, a quantity containing $\alpha\beta\gamma$ in the sixth order, and which it is necessary, by the help of the equation of the curve, to show is divisible by R^6.

"Now in virtue of a formula already obtained we have

$$4q_3^2 - 9q_2q_4 = R^2(4\theta - 9H\Phi).$$

It is also easy to show that $27q_2q_5 - 5q_3q_4$ and $5q_4^2 - 12q_3q_5$ are each divisible by R_6; but I have not been able to carry the reduction further."

In attemting to complete the solution of this problem I arrived at the following result:

The equation of the bitangential of a quintic is

$$529H^2\Psi - 270H\Phi\chi + 25\Phi^2\theta = 5(4\theta - 9H\Phi)(5\Phi^2 - 12\chi),$$

a curve of the forty-eighth order, as it should be.

I have not been able to *prove* this result, and submit it to the readers of the Analyst hoping that some one may be more successful than myself.

It will be observed that we have used trilinear coordinates and that the equation of the curve is

$$u = f(x, y, z) = 0.$$

ON THE RATIO OF THE AREA OF A GIVEN TRIANGLE TO THAT OF AN INSCRIBED TRIANGLE.

BY PROF. J. SCHEFFER, HARRISBURGH, PA.

LET us represent the sides BC, AC, AB, respectively by a, b, c; CD by $a\alpha$, AE by $b\beta$, BF by $c\gamma$; the area of the triangle ABC by Δ and that of the inscribed triangle DEF by Δ'.

We easily find triangle $CDE = \alpha(1-\beta).\Delta$; triangle $AEF = \beta(1-\gamma).\Delta$; triangle $BDF = \gamma(1-\alpha).\Delta$. Therefore

$$\frac{\Delta'}{\Delta} = 1-\alpha(1-\beta)-\beta(1-\gamma)-\gamma(1-\alpha)$$
$$= 1-(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma). \quad \text{(I)}$$

If the three transversals drawn from A, B, C, to the assumed points D, E, F, intersect in one point we have the relation

$\alpha\beta\gamma=(1-\alpha)(1-\beta)(1-\gamma)$, whence $1-(\alpha+\beta+\gamma)+(\alpha\beta+\alpha\gamma+\beta\gamma)=2\alpha\beta\gamma$.

In this case therefore

$$\frac{\Delta'}{\Delta} = 2\alpha\beta\gamma. \qquad \text{(II)}$$

1. *Let the three transversals be the medial lines*, then $\alpha = \beta = \gamma = \frac{1}{2}$, and according to (II)

$$\frac{\Delta'}{\Delta} = \tfrac{1}{4}.$$

2. *Let the transversals be the bisectors of the angles.* From $CD : BD = b : c$, we get

$$CD = \frac{ab}{b+c}, \therefore \alpha = \frac{b}{b+c}, \text{ and similarly } \beta = \frac{c}{a+c}, \gamma = \frac{a}{a+b};$$

$$\therefore \frac{\Delta'}{\Delta} = \frac{2abc}{(a+b)(a+c)(b+c)}, \text{ by (II).}$$

3. *Let the transversals be the altitudes*

$$CD = b\cos C, AE = c\cos A, BF = a\cos B.$$

$$\therefore \alpha = \frac{b}{a}\cos C, \beta = \frac{c}{b}\cos A, \gamma = \frac{a}{c}\cos B;$$

$$\therefore \frac{\Delta'}{\Delta} = 2\cos A\cos B\cos C = \frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b+c^2-a^2)}{4a^2b^2c^2}.$$

4. *Let the transversals intersect each other in one point and make equal angles with the radii.* (Vide Prob. 245, No. 1, Vol. VI.)

$$CD = \frac{ab^2}{b^2+c^2},\ AE = \frac{bc^2}{a^2+c^2},\ BF = \frac{a^2c}{a^2+b^2};$$

$$\therefore a = \frac{b^2}{b^2+c^2},\ \beta = \frac{c^2}{a^2+c^2},\ \gamma = \frac{a^2}{a^2+b^2};$$

$$\therefore \frac{\Delta'}{\Delta} = \frac{2a^2b^2c^2}{(a^2+b^2)(a^2+c^2)(b^2+c^2)}.$$

Compare this result with that in 2.

5. *Let the points D, E, F, be the feet of the perpendiculars let fall from the centre of the inscribed circle.*

Denote the radius of the inscribed circle by ρ and put $\frac{1}{2}(a+b+c) = s$; then $\alpha = (\rho \div a) \cot \frac{1}{2}C$, $\beta = (\rho \div b) \cot \frac{1}{2}A$, $\gamma = (\rho \div c) \cot \frac{1}{2}B$. Therefore

$$\frac{\Delta'}{\Delta} = \frac{2\rho^3}{abc} \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C = \frac{\rho^2 s}{abc} = \frac{2(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{(a+b-c)(a+c-b)(b+c-a)}{4abc}.$$

Compare this result with that in 3.

FIVE GEOMETRICAL PROPOSITIONS.

BY PROF. ELIAS SCHNEIDER, MILTON, PA.

I. LET A, B, C, D, &c., be the angular points of a regular polygon of n sides, and let AB, one of the equal sides, aqual unity; then will AB be contained once in AC, the chord which contains *two* of the equal sides, with a remainder which call x. Then is $\sqrt{(1-x)}$ = one side of a polygon of $2n$ sides inscribed in a circle whose radius is *one*.

II. AB will be contained twice in AD, the chord which contains *three* of the equal sides, with a remainder which call y. Then is $\sqrt{(1-y)}$ = one side of a polygon of n sides inscribed in a circle whose radius is *one*.

III. If the polygon be a Nonagon, AB will be contained twice in AE, the chord which contains *four* of the equal sides, with a remainder which call x. Then is $\sqrt{(1-x)}$ = one side of a polygon of 18 sides inscribed in a circle whose radius is *one*.

IV. If in Prop. II the polygon be also a Nonagon, then is

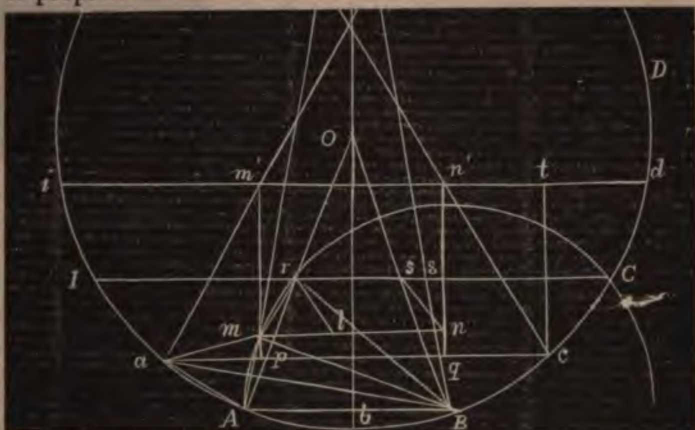
$$\sqrt{(1-x)} = x - y.$$

V. If the polygon be a Decagon, AB will be contained twice in AD, the chord which contains three of the equal sides, with a remainder which call z. Then is $\sqrt{(1-z)} = z$ = one side of a decagon inscribed in a circle whose radius is *one*.

[As the foregoing propositions are new, or at least original with Prof. Schneider (see ANALYST, Vol. I, p. 37), and have never been demonstrated geometrically, so far as we know; and as Prof. S. declined to submit his *proof*, we have attempted their demonstration, and submit the following sketch of our method and result.—Ed.]

It will be seen that the Figure is drawn for a Nonagon and that ac, IC, and id are chords containing 2, 3 and 4 of the equal sides, respectively.

The $\angle AOB = ABr = 2AFB = 2ABm$. Draw mn parallel, and mp and nq perpendicular to ac. Then, because aB is perpendicular to Am, we may easily prove that $am = aA$, and $ap = qc = \frac{1}{2}AB = \frac{1}{2}$; therefore $mn = pq = x$, of prop. I.



By Euclid, Prop. D, B'k VI, we have $Am \times Bn + AB \times mn = An \times Bm$; or, because $Bn = Am$, $Am^2 + x = 1$. Transpos'g and extracting root we have $Am = \surd(1-x)$, which proves Prop. I. And in like manner, substituting rs $(= y)$ for mn, we get $Ar = \surd(1-y)$, which proves Prop. II. The proof in both these cases is obviously independent of any particular value of n.

If the polygon is a Nonagon, ch will intersect id at an angle of 60° and therefore cdn' is an equilateral triangle and ct is equal and parallel to qn; therefore $td = n't = qc = \frac{1}{2}$; therefore $n'd = 1$, and $m'n' = mn = x$, and the demonstration of I applies also to III.

Join sn and draw rl parallel to sn, then, if the polygon is a Nonagon, the triangle lmr will be equilateral and $mr = \surd(1-x) = ml = mn - rs = x - y$, which proves Prop. IV.

If the polygon is a Decagon, we may easily prove that the triangle Ars is isosceles; therefore $Ar = rs$. But by Prop. II, $Ar = \surd(1-rs) = \surd(1-Ar)$. Or putting z for Ar, $z = \surd(1-z)$ = one side of a Decagon inscribed in a circle whose radius is *one*.

COR. If we put $R = AB$, the functions $\surd(1-x)$, $\surd(1-y)$ and $\surd(1-z)$, in the above cases, become $\surd(R^2 - Rx)$, $\surd(R^2 - Ry)$ and $\surd(R^2 - Rz)$.

NEW DEMONSTRATION OF PROP. K., EUCL., B. VI.

BY W. H. WEST, EX SUPREME JUDGE OF OHIO.

Theorem. In any plane triangle, the base is to the sum of the other two sides as the difference of those sides is to the difference of the segments of the base formed by a perpendicular from the angle opposite the base.

Demonstration.—Let ABC be the triangle, BC its base, AD the perpendicular upon it. Then will

$BC : AC+AB :: AC-AB : CD-BD$

With A as a center and radius AB describe an arc cutting BC in F and AC in E. Join E and F, and from E draw EG making $\angle GEC = \angle ABC$.

It is plain that $AFB+AFE=CEG+AEF$; hence $\angle FEG=\angle EFG$, and $EG=FG$.

From the similar triangles CEG and ABC,

$$CG+GE : CA+AB :: CE : BC.$$

But $CG+GE=CG+GF=CF=CD-BD$; and $CE=CA-AB$; hence the proportion

$$BC : AC+AB :: AC-AB : CD-BD.$$

If the given triangle be obtuse, as AFC, the second term of $CD + BD$ is to be regarded as negative, and we have

$$CF : CA+AF :: CA-AF : BC.$$

If ABC be a right angle, AF and AB become AD, CEG will be a right angle, and $CE+EG$ will $= CD$; then

$$CD : CA+AD :: CA-AD : CD,$$

or

$$AD^2+CD^2=AC^2.$$

[Produce CA to meet the circumference in H. Then, by Eucl. B. III, Prop. 36, Cor. 1, $CH\,(=CA+AB) \times CE = CB \times CF$, or

$$CB : CA+AB :: CA-AB : CD-BD.\text{—Ed.}]$$

Correction.—Mr. Adcock makes the following correction in his solution of Prob. 352:

On page 160, the last line should read

$$\int_0^{\frac{1}{2}\pi} \frac{4\varphi d\varphi}{2\pi} \div \int_0^{\frac{1}{2}\pi} d\varphi = \left[\frac{2\varphi^2}{2\pi\varphi}\right]_0^{\frac{1}{2}\pi} = \frac{1}{2}.$$

THE SECULAR DISPLACEMENT OF THE ORBIT OF A SATELLITE.

BY PROFESSOR ASAPH HALL.

(1). In his Mécanique Céleste, Tome II, p. 373, Laplace has considered the cause that holds the Rings of Saturn nearly in the plane of the equator of the planet, and he has explained the reason that led him to predict the rotation of Saturn and its Rings before Herschel had determined these rotations by observation. In this investigation Laplace employs the equations that determine the rotation of a solid body, the disturbing forces being those that arise from the action of the excess of matter around the equator of the planet and the attraction of a distant body. In the same chapter it is also shown why a satellite will be held nearly in the equator of its primary when the figure of the planet is elliptical. These results have been used by Professor J. C. Adams to explain why the orbits of the satellites of Mars are at the present time nearly coincident with its equator. M. F. Tisserand has discussed the same question by means of formulæ derived by him in a discussion of the secular perturbations of the orbit of Japetus, the outer satellite of Saturn. The investigations of these astronomers render it probable that the figure of Mars is slightly elliptical, and that the orbits of its satellites will always remain nearly coincident with the equator of the planet.

The question of the perturbations of satellites has been treated by Laplace in the fourth volume of his Mécanique Céleste. There Laplace determines the perturbations of the radius vector, and of the longitude and latitude of the satellite; and for astronomical purposes these coordinates seem to be as good as any that can be chosen. If we wish to know, however, the effect of the perturbation on any element of the orbit, it is more convenient to use the formulæ of Lagrange. In the unfinished essay on the Saturnian System left by Bessel he has given these formulæ, and also the various forms of the perturbative, or potential function that come into use in this complicated and interesting system. M. C. Souillart has lately published an elaborate memoir on the satellites of Jupiter in which he verifies and extends the investigations of Laplace referred to above.

The forces that disturb the motion of a satellite are, (1), the action of the protuberant matter around the equator of the primary planet; (2), the attraction of another satellite of this planet; (3), the attraction of a body outside this system, like the sun or another planet; and in the case of Saturn

the action of its Ring also produces a disturbance in the motion of its satellites. Since the perturbations are small they may be computed separately and then added together; I consider here only those which are produced by the protuberant matter around the equator of the planet, and by the action of the sun.

Denote by θ the longitude of the ascending node of the orbit of the satellite on the orbit of its primary planet, and by i the inclination of the satellite's orbit to the same plane. Let g be the angle between the line of apsides of the orbit of the satellite and its line of nodes; then if a be the semi-major axis of this orbit, e the eccentricity, and if n be the mean sidereal motion of the satellite, we shall have by the formulæ of Lagrange;

$$\frac{di}{dt}=\frac{an}{\sin i\surd(1-e^2)}\cdot\left(\frac{dR}{d\theta}\right)-\frac{an\cos i}{\sin i\surd(1-e^2)}\cdot\left(\frac{dR}{dg}\right)$$
$$\frac{d\theta}{dt}=-\frac{an}{\sin i\surd(1-e^2)}\cdot\left(\frac{dR}{di}\right). \tag{1}$$

R denotes the perturbative function used by Laplace.

We have now to expand this function for the two disturbing forces in elements of the orbit of the satellite, substitute the partial differential coefficients of R in equations (1), and then integrate them.

Assuming the satellite to be a material particle whose mass may be neglected, let x, y, z, be its rectangular coordinates referred to the centre of gravity of the planet, which is supposed to be fixed, and let

$$r=\surd(x^2+y^2+z^2).$$

Let X, Y, Z, be the coordinates of the centre of the sun, S its mass, and also let

$$D=\surd(X^2+Y^2+Z^2).$$

If m be the mass of the primary planet and $\frac{m}{r}+V$ the sum of the particles of the planet divided respectively by their distances from the satellite, then the value of the perturbative function will be,

$$R=\frac{S(Xx+Yy+Zz)}{D^3}-\frac{S}{[(X-x)^2+(Y-y)^2+(Z-z)^2]^{\frac{1}{2}}}-V.$$

The first two terms give the perturbations produced by the sun, and the last term those caused by the non spherical figure of the primary. We have now to develop this value of R and find those terms that produce the secular part of the perturbations. The denominator of the second term is

$$D.\left\{1+\frac{r^2}{D^2}-\frac{2(Xx+Yy+Zz)}{D^2}\right\}^{\frac{1}{2}},$$

and expanding the reciprocal of this we get for the second term of R,

$$-\frac{S}{D}+\frac{S.r^2}{2D^3}-\frac{S.(Xx+Yy+Zz)}{D^3}-\frac{3S.r^4}{8D^5}$$
$$-\frac{3S.(Xx+Yy+Zz)^2}{2D^5}+\frac{3S.r^2(Xx+Yy+Zz)}{2D^5}, \text{ \&c.}$$

The first term of this expansion may be omitted since it does not contain the elements of the orbit of the satellite and will disappear in the differentiation, and the third term cancels the first term in R. Again D is very great with respect to r, and hence the approximate value of the perturbative function that depends on the action of the sun becomes

$$R_1=\frac{S.r^2}{2D^3}-\frac{3S.(Xx+Yy+Zz)^2}{2D^5}. \tag{2}$$

If a great circle be drawn from the position of the satellite in its orbit to the sun, and we call the arc joining these bodies f, we shall have,

$$\cos f=\frac{Xx+Yy+Zz}{rD},$$

and therefore

$$\frac{r^2}{D^3}\cos f^2=\frac{(Xx+Yy+Zz)^2}{D^5}.$$

If we denote by u and U the angular distances of the satellite and the sun from the node, the spherical triangle between the node, the satellite and the sun gives

$$\cos f = \cos u \cos U + \sin u \sin U \cos i,$$

and hence

$$\cos f^2 = \cos u^2 \cos U^2 + \sin u^2 \sin U^2 \cos i^2$$
$$+ 2 \cos u \sin u \cos U \sin U \cos i.$$

Changing the powers into cosines of the multiple arcs we have

$$\cos f^2 = \tfrac{1}{4}.[2-\sin i^2 + 2\cos 2u \cos 2U + 2\sin 2u \sin 2U \cos i$$
$$+ (\cos 2u+\cos 2U-\cos 2u \cos 2U)\sin i^2].$$

As we wish to get only the secular part of this expression we have by omitting the periodical terms depending on the position of the sun,

$$\frac{r^2}{D^3}.\cos f^2 = \frac{1}{4D^3}.\left\{(2-\sin i^2).r^2+r^2\cos 2u.\sin i^2\right\}.$$

If now Φ be a function of the radius vector and true anomaly, r and v, in an ellipse, and if we denote by M and E the mean and excentric anomalies, then the non-periodical part of Φ is given by the definite integral,

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi.dM=\frac{1}{2\pi\sqrt{(1-e^2)}}.\int_0^{2\pi}\Phi.\frac{r^2}{a^2}.dv=\frac{1}{2\pi}\int_0^{2\pi}\Phi.\frac{r}{a}.dE.$$

Since from the equation of the ellipse we have

$$r=a(1-e\cos E),$$

the last integral gives for the non periodical part of r^2,

$$\frac{a^2}{2\pi}\int_0^{2\pi}(1-e\cos E)^3.dE = a^2.(1+\tfrac{3}{2}e^2).$$

Again since $u = v+g$ we have to find the non periodical parts of

$$\Phi = r^2\cos 2v: \qquad \text{and } \Phi = r^2\sin 2v.$$

The equation of the ellipse gives

$$\sin v = \frac{\sin E.\sqrt{(1-e^2)}}{1-e\cos E}: \qquad \cos v = \frac{\cos E - e}{1-e\cos E},$$

and hence

$$r^2\cos 2v = a^2.[(\cos E - e)^2 - (1-e^2).\sin E^2],$$
$$r^2\sin 2v = 2a^2.\sin E.(\cos E - e).\sqrt{(1-e^2)}.$$

Substituting these values of Φ and integrating we have

$$\Phi = \tfrac{5}{2}a^2e^2: \qquad \text{and } \Phi = 0.$$

If a_0 and e_0 be the semi-major axis and the excentricity of the orbit of the planet we shall have for the non periodical part of $\frac{1}{D^3}$,

$$\Phi = \frac{1}{2\pi a_0^3}\cdot\int_0^{2\pi}\frac{dE}{(1-e_0\cos E)^2} = \frac{1}{a_0^3(1-e_0^2)^{\frac{3}{2}}}.$$

From these results we find for the secular part of R_1, the value

$$R_1 = \frac{S.a^2}{a_0^3(1-e_0^2)^{\frac{3}{2}}}\cdot\left\{\left(-\frac{1}{4}+\frac{3}{8}\sin i^2\right).\left(1+\frac{3e^2}{2}\right)-\frac{15}{16}.e^2\sin i^2\cos 2g\right\}. \quad (3)$$

The term $\cos\frac{2u}{D^3}$ in $\cos f^2$ produces no constant terms of an order lower than e_0^4, and therefore it may be neglected.

(2). In the case of an ellipsoid of revolution Laplace gives a very simple form to the potential V of the disturbing force. If we denote by ρ the ellipticity of the spheroid, by φ the ratio of centrifugal force to gravity at the equator of the planet, by B the radius of this equator, and by μ the sine of the declination of the satellite with respect to the same equator, we shall have; Mec. Cel., Tome II, p. 103,

$$V = \frac{mB^2}{r^3}\cdot\left(\frac{\varphi}{2}-\rho\right)\cdot\left(\mu^2-\frac{1}{3}\right).$$

Taking the mass of the planet and its equatorial radius for the units of mass and distance, and putting $Q' = \frac{1}{2}\varphi-\rho$, we have

$$V = \frac{Q'}{r^3}\cdot(\mu^2-\tfrac{1}{3}).$$

Let ψ be the distance from the node of the orbit of the satellite to its intersection with the equator of the planet, and let γ be the inclination of its

orbit to this equator, then from the right angled spherical triangle we have

$$\mu = \sin\gamma \sin(u - \psi)$$

and

$$\mu^2 = \tfrac{1}{2}\sin\gamma^2 . [1 - \cos 2(u-\psi)].$$

The part of V which gives the secular terms is therefore

$$R_2 = -\frac{Q'}{2a^3(1-e^2)^{\frac{3}{2}}} . \sin\gamma^2. \tag{4}$$

Let A be the inclination of the equator of the planet to the orbit of the planet, and let θ_1 be the longitude of the node of the equator on this orbit; the spherical triangle formed by the three nodes gives,

$$\begin{aligned}
\cos\gamma &= \cos(\theta_1-\theta)\sin A \sin i + \cos A \cos i \\
\sin\gamma \sin\psi &= \sin(\theta_1-\theta)\sin A \\
\sin\gamma\cos\psi &= \cos(\theta_1-\theta)\sin A \cos i - \cos A \sin i.
\end{aligned}$$

These equations give the means of expressing $\sin\gamma^2$ in terms of θ and i, and hence R_2 becomes a function of the elements of the satellite's orbit. The values of A and θ_1, which determine the position of the equator of the planet on the plane of its orbit, are in fact subject to small variations similar to precession and nutation, but here these quantities are supposed to be constant.

The excentricities of the orbits of the satellites in our solar system are generally very small, and if we neglect the squares of these excentricities, and omit the constant part in R_1, equations (1), (3) and (4) give, if we put

$$Q = \frac{3S.n^2}{8a_0^3(1-e_0^2)^{\frac{3}{2}}},$$

$$\frac{di}{dt} = \frac{an}{\sin i} . \left(\frac{dR}{d\theta}\right)$$

$$\frac{d\theta}{dt} = -\frac{an}{\sin i} . \left(\frac{dR}{di}\right)$$

$$R = Q.\sin i^2 - \frac{Q'}{2a^3} . \sin\gamma^2; .$$

or we may write

$$R = \tfrac{1}{2}c . \sin i^2 + \tfrac{1}{2}c' . \sin\gamma^2.$$

Hence we find

$$\left(\frac{dR}{d\theta}\right) = -c'.\cos\gamma . \left(\frac{d.\cos\gamma}{d\theta}\right)$$

$$\left(\frac{dR}{di}\right) = c.\sin i \cos i - c'.\cos\gamma . \left(\frac{d.\cos\gamma}{di}\right).$$

The value of $\cos\gamma$ gives

$$\left(\frac{d.\cos\gamma}{d\theta}\right) = \sin(\theta_1-\theta)\sin A\sin i$$

$$\left(\frac{d.\cos\gamma}{di}\right) = \cos(\theta_1-\theta)\sin A\cos i - \cos A\sin i.$$

The other equations of the triangle enable us to change these values to

$$\left(\frac{d.\cos\gamma}{d\theta}\right) = \sin\gamma\sin\psi\sin i$$

$$\left(\frac{d.\cos\gamma}{di}\right) = \sin\gamma\cos\psi.$$

We have therefore finally

$$\frac{di}{dt} = -k'.\cos\gamma\sin\gamma\sin\psi$$

$$\frac{d\theta}{dt} = -k\cos i + k'.\frac{\cos\gamma\sin\gamma\cos\psi}{\sin i}, \tag{5}$$

where k and k' have the values,

$$k = \frac{3S.a^3}{4a_0^3(1-e_0^2)^{\frac{3}{2}}}.n: \qquad k' = \frac{(\rho-\frac{1}{2}\varphi)}{a^2}.n.$$

These are the formulæ given by Laplace, Mec. Cel., Tome IV, page 182. The quantity k can be computed accurately for all of the principal planets; but k' is not so well known since it depends on the ellipticity of the figure of the planet and on its law of density. If the planet has other satellites the secular perturbations of these should be comprised in the value of k', and in the case of Saturn k' would include also the secular action of the Ring.

The Italian astronomer Plana criticised equations (5), and considered them erroneous; but in the *Conn. des Tems* for 1829, p. 245, Laplace has verified his first proof of them, and has given also the simple derivation from the formulæ of Lagrange which has been followed above.

(3). In his investigation of the motion of the orbit of Japetus Laplace has introduced an auxiliary fixed plane passing through the line of nodes and placed between the equator of the planet and its orbit, the angle which this plane makes with the equator of the planet depending on the ratio of k to k'.

M. Tisserand has employed a property of the perturbative function which leads very simply to a knowledge of the curve described on the heavens by the pole of the orbit of the satellite. If ε be the mean longitude at the epoch, and π be the longitude of the inferior apsis, the complete differential of R is

$$dR = \frac{dR}{da}.da + \frac{dR}{d\varepsilon}.d\varepsilon + \frac{dR}{de}.de + \frac{dR}{d\pi}.d\pi + \frac{dR}{di}.di + \frac{dR}{d\theta}.d\theta,$$

but for the secular perturbations we have

$$da = 0, \qquad \text{and } \frac{dR}{d\epsilon} = 0,$$

and hence

$$dR = \frac{dR}{de}.de + \frac{dR}{d\pi}.d\pi + \frac{dR}{di}.di + \frac{dR}{d\theta}.d\theta.$$

Now we have by the theory of Lagrange,

$$\frac{dR}{de} = (e, \pi).\frac{d\pi}{dt} + (e, \theta).\frac{d\theta}{dt}$$

$$\frac{dR}{d\pi} = (\pi, e).\frac{de}{dt}$$

$$\frac{dR}{di} = (i, \theta).\frac{d\theta}{dt}$$

$$\frac{dR}{d\theta} = (\theta, i).\frac{di}{dt} + (\theta, e).\frac{de}{dt}.$$

If we substitute these values of the partial derivatives of R in the second value of dR and notice the well known relations

$$(e, \pi) = -(\pi, e), \ \&c.,$$

we shall have

$$dR = 0,$$

and hence

$$R = \text{constant.}$$

From the value of R that we have found we have,

$$k \sin i^2 + k' \sin \gamma^2 = \text{constant.} \tag{6}$$

If a satellite therefore be acted on by the two disturbing forces that we have considered the pole of its orbit will describe on the heavens a spherical ellipse. This equation shows also that if the disturbing force of the sun be zero, the orbit of the satellite will have a constant inclination to the equator of the planet, and if k' vanish, it will have a constant inclination to the orbit of the planet. The real position of the orbit of the satellite will depend on the ratio of the quantities k and k', and for this ratio we have

$$\frac{k'}{k} = \frac{4}{3}.\frac{a_0^3}{a^3 S}.\frac{1}{a^2}.\left(1-e_0^2\right)^{\frac{3}{2}}.\left(\rho-\frac{\varphi}{2}\right),$$

or, since the mass of the planet is the unit of mass, and

$$1 = a^3 n^2, \qquad S = a_0^3 n_0^2;$$

$$\frac{k'}{k} = \frac{4}{3}.\left(\frac{n}{n_0}\right)^2.\frac{1}{a^2}.\left(1-e_0^2\right)^{\frac{3}{2}}.\left(\rho-\frac{\varphi}{2}\right), \tag{7}$$

where the unit of distance is the equatorial radius of the planet. The uncertainty of this ratio lies in our ignorance of the factor $(\rho-\frac{1}{2}\varphi)$. For the

planets Mars, Jupiter and Saturn the quantity φ is known with a tolerable degree of accuracy, since it depends on the time of rotation of the planet about its axis; but the value of ρ must be inferred from the probable conformation of our Earth, or from some other analogy. In the case of the exterior planets of our system the ratio $k' \div k$ is probably large for all the satellites, with one exception, and for this reason these satellites are found nearly in the equators of the planets. This ratio, it will be seen, varies inversely as the fifth power of the mean distance of the satellite, and it is on account of this fact that Japetus, the outer satellite of the Saturnian system, presents an exception to the general rule, its orbit being inclined nearly fourteen degrees to the equator of Saturn, while the next interior satellites, Hyperion and Titan do not depart more than one or two degrees from the equator. In the case of Japetus Laplace found,

$$\frac{k'}{k} = 0.4219,$$

which is too large on account of the uncertain data used in the calculation. M. Tisserand finds from Bessel's value of the mass of the Ring and ellipticity of the planet,

$$\frac{k'}{k} = 0.2570;$$

and this ratio is still uncertain since the mass of the Ring is not well known, and the masses of all the satellites of Saturn are unknown. The better way probably would be to determine k' by observations made at two distant epochs which would enable us to fix the position of the node at those epochs and the values of the inclination, and by comparison of these values we should have k'. In order to distinguish in the case of Saturn the different parts of k' it will be necessary to determine accurately the motions of the interior satellites, since from these motions the mass of the Ring and the effect of the ellipticity of the planet can be found.

For the application of equations (5) to Japetus we need the values of the inclination and longitude of the node of the orbits of Saturn and Japetus and of the equator of Saturn on the ecliptic. These values are for 1880.0

	N.	J.
Saturn	112° 36′.5	2° 29′.6
Japetus	142 45 .0	18 31 .5
Equator of Saturn	167 55 .2	28 10 .3

The solutions of the three spherical triangles formed by the nodes of these great circles give

$$i = 16° \quad 25'.0$$
$$\gamma = 13 \quad 41.0$$
$$\psi = 53 \quad 39.0$$
$$A = 26 \quad 49.6.$$

The periodic times of Saturn and Japetus are

$$\tau_0 = 10759.2198 \text{ days}$$
$$\tau = 79.32936 \text{ " }.$$

The value of k may be written,

$$k = \frac{3}{4} \cdot \frac{\tau^2}{\tau_0^2} \cdot \frac{n}{(1-e_0^2)^{\frac{3}{2}}}.$$

For Saturn $e_0 = 0.0559956$; and if we take the Julian year of 365.25 days for the unit of time we have for Japetus $n = 5967071''.2$. The value of k is

$$k = 0.0000409651 . n.$$

Equations (5) give for the annual variations of i and θ,

$$\Delta i = -45''.250 . \frac{k'}{k}$$

$$\Delta \theta = -234''.476 + 117''.827 . \frac{k'}{k}.$$

With Tisserand's value of $k' \div k$ the annual motion of the node on the orbit of Saturn will be —204''.194.

For the outer satellite of Mars the values of the nodes and inclinations on the ecliptic for 1880.0 are:

	N.		J.	
Mars	48°	37'.9	1°	51'.0
Deimos	85	37.4	25	47.2
Equator of Mars	79	39.0	28	50.0

With regard to these quantities it must be noticed that the position of the equator of Mars is uncertain; but from the data given we find,

$$i = 24° \quad 20'.0$$
$$\gamma = 4 \quad 5.6$$
$$\psi = 312 \quad 38.5$$
$$A = 27 \quad 15.9$$

The periodic times of Mars and Deimos are

$$\tau_0 = 686.97965 \text{ days}$$
$$\tau = 1.2624350 \text{ "}$$

and in the orbit of Mars $e_0 = 0.09328975$; and for the satellite

$$\log n = 8.5739862.$$

The value of k is

$$k = 0.0000025661722 . n.$$

and equations (5) give for the annual motions of i and θ on the orbit of Mars:

$$\Delta i = + 50''.396 . \frac{k'}{k}$$

$$\Delta\theta = - 876''.735 + 112''.630 . \frac{k'}{k}.$$

The node would have therefore a retrograde motion by which it would complete a revolution on the heavens in 1478 years were it not for the influence of the second term. In the case of Mars the value of the ratio $k' \div k$ is unknown, but probably it is large enough to destroy the first term in the value of $\Delta\theta$.

(4). To find the equation of a spherical ellipse, let $2a$ be the part of a great circle forming the major axis of the ellipse, and let 2ϵ be the distance between the foci. Denote by ρ and ρ' the arcs drawn from a point on the ellipse to the foci, and we shall have the condition,

$$\rho + \rho' = 2a.$$

If we take the centre of the ellipse as the origin of the polar spherical coordinates S and V, S being the radius vector and V its angle with the major axis, we have,

$$\cos\rho = \cos S \cos\epsilon - \sin S \sin\epsilon \cos V$$

$$\cos\rho' = \cos S \cos\epsilon + \sin S \sin\epsilon \cos V.$$

Adding and subtracting these equations, and reducing, we find

$$\cos \tfrac{1}{2}(\rho'-\rho) = \frac{\cos S \cos\epsilon}{\cos a},$$

$$\sin \tfrac{1}{2}(\rho'-\rho) = \frac{-\sin S \sin\epsilon \cos V}{\sin a};$$

and squaring and adding these we have for the equation of the ellipse,

$$1 = \frac{\cos\epsilon^2}{\cos a^2} . \cos S^2 + \frac{\sin\epsilon^2}{\sin a^2} . \sin S^2 \cos V^2. \qquad (8)$$

The form of our perturbative function is the same as that of equation (8) since it may be written,

$$R = k \cos i^2 + k' \cos\gamma^2 = c.$$

The spherical ellipse will be symmetrical with respect to the great circle joining the pole of the orbit of the planet to the pole of its equator, and therefore the centre of the ellipse will be on this circle and between these poles. To find this centre M. Tisserand proceeds as follows; let η and η'

be the distances of the centre from the pole of the orbit of the planet and from the pole of the equator. Then

$$\eta + \eta' = A.$$

We shall have from the spherical triangles

$$\cos i = \cos \eta' \cos S - \sin \eta' \sin S \cos V$$

$$\cos \gamma = \cos \eta \cos S + \sin \eta \sin S \cos V.$$

The equation of the ellipse will be

$$k.(\cos \eta' \cos S - \sin \eta' \sin S \cos V)^2 + k'.(\cos \eta \cos S + \sin \eta \sin S \cos V)^2 = c.$$

By equation (8) the coefficient of $\sin S \cos S \cos V$ must be zero. Hence

$$-2k.\cos \eta' \sin \eta' + 2k'.\cos \eta \sin \eta = 0,$$

or

$$k.\sin 2(A - \eta) = k'.\sin 2\eta,$$

and

$$\text{tang}\, 2\eta = \frac{k.\sin 2A}{k' + k \cos 2A}.$$

The angle η is the same as the angle θ used by Laplace, Me'c. Ce'leste Tome IV, p. 178. For two satellites of the same planet the ratio $k' \div k$ will vary inversely as the fifth power of their mean distances. If we assume for Japetus this ratio to be

$$\frac{k'}{k} = 0.2570,$$

its value for Rhea, the fifth satellite of Saturn, will be

$$\frac{k'}{k} = 0.257 \times \left(\frac{64.54}{9.55}\right)^5$$

Hence $\eta = 22''.9$; or, for an observer on our Earth Rhea will move sensibly in the equator of Saturn. For Titan, the large satellite next beyond Rhea, we have

$$\frac{k'}{k} = 0.257 \times \left(\frac{64.54}{22.145}\right)^5.$$

Hence $\eta = 1520''.5$. This is smaller than the value given by observation, and indicates that the assumed ratio

$$\frac{k'}{k} = 0.2570$$

is too great.

CENTRIFUGAL TIDES.

BY PROF. ASAPH HALL.

THE earth and the moon move in ellipses around their common centre of gravity, the time of revolution being 27.32166 days. If we take the mass of the earth as the unit of mass, the mass of the moon will be $m = \frac{1}{81}$; and the centre of gravity will be 2912 miles from the earth's centre, and 6875 miles from the farther side of the earth. Let a be the distance of the moon from the centre of the earth, and take the radius of the earth for the unit of distance; then the difference of the attractions of the moon on the centre of the earth and on opposite points of its surface will be,

$$\frac{m}{(a-1)^2} - \frac{m}{a^2} = \frac{m(2a-1)}{a^2.(a-1)^2},$$

$$\frac{m}{a^2} - \frac{m}{(a+1)^2} = \frac{m(2a+1)}{a^2.(a+1)^2}.$$

Since $a = 60.257$, if we neglect the units we have for the approximate value of this difference $2m \div a^3$. Now the attraction of the moon on a point of the earth's surface ninety degrees from the line joining the centres tends to increase the attractive force of the earth on this point, and it is found that the preceding expression becomes $3m \div a^3$.

If we call m' the mass of the sun, and a' its distance from the earth, we have likewise $3m' \div a'^3$ for the difference of its attraction on the centre of the earth and on a point where the line joining the centres meets the earth's surface. Also

$$m' = 324439; \qquad a' = 23340.9$$

Turning these expressions of the disturbing force into numbers we find that the ratio of the moon's disturbing force to that of the sun's is 2.212; and that the combined disturbance of the sun and moon on a point of the earth's surface situated on a line joining their centres is only 0.0000002458 of the attraction of the earth on the same point.

The general expression for the centrifugal force is $v^2 \div r$; v being the velocity of the particle and r the radius of curvature of its orbit. If t expressed in seconds be the time of the revolution of the earth around the centre of gravity we have

$$t = 2360591.4$$

$$v = \frac{2\pi r}{t}.$$

The centrifugal force is therefore $r.\left(\frac{2\pi}{t}\right)^2$. If we express r in feet, $r=6875\times5280$; and if we notice that the earth's gravitating force measured in the same way is 32.2, we find for the centrifu'l force 0.000008024. The centrifugal tidal force is therefore nearly 33 times as great as the combined tidal force of the sun and moon, and should produce an enormous tide sweeping over our seaboard cities. But we know that no such tide exists. Will some one therefore point out the fallacy of the above reasoning. Such an investigation will not be useless, since the centrifugal theory of the tides is such an attractive one that it is frequently given in books on popular astronomy, and it is continually coming up in our popular scientific journals and at the meetings of our scientific associations.

[Because the centrifugal force of a particle moving in a circle, under the influence of an attracting body, is a function of the velocity of the particle and of the radius of curvature of its path; therefore the centrifugal force can only vary with the variation of one or both of these elements. But in the revolution of the earth about the common center of gravity of the earth and moon there is no rotation of the earth about that center; therefore in this revolution all points of the earth move in similar and equal curves, and the radius of curvature of these curves is not $r=6875$ miles but, approximately, $r=95{,}000{,}000$ miles; for the earth, in consequence, moves around the sun in a disturbed ellipse. And in this disturbed orbit, though the centrifugal force of the earth varies slightly at conjunction and opposition of the sun and moon in consequence of a variation in its radius of curvature and velocity at those periods, all its parts move with the same velocity and in parallel carves; therefore this disturbance can have no effect on the tides except through the variable attraction of the sun and moon.

The fallacy, manifested by the result in the foregoing argument, consists therefore in the application of the formula $r\left(\frac{2\pi}{t}\right)^2$, by which it is virtually assumed that the earth revolves around the common center of gravity of the earth and moon, as the moon does, making one rotation on its axis while it completes one revolution in that orbit.—Ed.]

CORRESPONDENCE.

EDITOR ANALYST:

I notice you inquire in "*Science*", as to the evidence of the conversion of gravitation into heat and light. Are not the nearest approaches to a demonstration of such convertibility, the following?

1. Herschel's confirmation of the hypothesis of Grove and Fresnel, by computing the relation of æthereal elasticity to density from the formula $v \propto \sqrt{(e \div d)}$.

2. The demonstration by Peirce and Helmholtz, that all the energies of solar heat can be explained by gradual contraction of solar mass.

3. My own application of the formula of projectile velocity, $v = gt$, to determining the velocity of projection of the solar particles from the c. g of the system, and my demonstration that $v =$ velocity of light if $t =$ time of solar rotation, and $g =$ superficial gravitating reaction against the action of æthereal undulation at the c. g. of the system.

4. My demonstration of interstellar parabolic action, in fixing planetary positions in accordance with the laws of harmonic vibration, the parabolic elements being solar mass and velocity of light.

I enclose my latest approximation to the harmonic abscissas [see Proceedings of the Amer. Philos. Soc., June 1881, pp. 446–448], and subjoin some newly-discovered evidences of the influence of parametral *vis viva*.

Taking $\xi = \frac{1}{8}$, $\eta = .9758534$, $\eta^2 = 2p\xi = \frac{1}{8}(5.950342)$,

$$2p = 5.950342.$$

In the primative cometoid nebula, let us investigate nucleal tendency in the axis of abscissas, in the following order; Neptune (N), Jupiter (J), Mars (M), Sun (S), Mercury (Me), Venus (V), Earth (E), Saturn (S), Uranus (U). Let secular perihelion, mean perihelion, mean, mean aphelion, secular aphelion, be designated by subscripts, 1, 2, 3, 4, 5, respectively.

Then if $x = E_3$, $\quad 2px = 5.950$

$$J_1 + E_5 = 5.954$$

$$J_3 + V_4 = 5.953$$

$$J_5 + Me_4 = 5.975$$

$$N_1 - U_1 = 11.910 \qquad 4p = 11.9$$

$$U_5 - Sa_1 = 11.945$$

$$U_1 = 17.688 \qquad 6p = 17.85$$

$$N_2 = 29.732 \qquad 10p = 29.75$$

$$M_2 + E_4 = 2.437 \qquad \sqrt{(2p)} = 2.439$$

$$J_1 = 4.886 \qquad 2\sqrt{(2p)} = 4.879.$$

If $x = M_3$, $\quad 2px = 9.066 \qquad Sa_2 = 9.078.$

If $x = M_5$, $\quad 2px = 10.332 \qquad Sa_5 = 10.343.$

If $x = J_2$, $\quad 2px = 29.622 \qquad N_1 = 29.732.$

If $x = Sa_4$, $\quad 2px = 59.504 \qquad 2N_2 = 59.465.$

If $x = Me_1$, $\quad 2px = 1.770 \qquad M_5 = 1.736.$

PLINY E. CHASE.

Haverford College, Sept. 27, 1881.

EDITOR ANALYST:

What you represent as my correction [see p. 176] to 352 is not accepted. The only correction I can admit to my original solution is that of introducing the probability of selecting a second chord equal to the first, which is $2d\varphi \div \pi$, as given in your first letter of objection to that solution. You gave no reason that $2d\varphi \div \pi$ is this probability. The reason however is given at length in my revised solution. And the reason for the contingent probability $2\varphi \div \pi$, is given at length in my original solution.

My objection to your construction on page 160 is that the chords should be drawn from every point on the circumference of the circle AC to every other point on that circle at the distance $2AC \sin\varphi$ from the first one, instead of tangent at every point of circumference of the circle CD. The whole number in the one case being proportional to $2\pi AC$, and in the other $2\pi CD$.

R. J. ADCOCK.

Roseville, Ill., Sept. 24, 1881.

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BY PROF. L. G. BARBOUR, RICHMOND, KY.

To find the sûm of an infinit number of terms of the series

$$\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\frac{1}{13.15}+\frac{1}{17.19} \text{ \&c.}$$

First calculate directly the sum of these five terms $= .380229950$. Next to find the sum of $\frac{1}{21.23}$ &c. Let

$$S=\frac{1}{21.23}+\frac{1}{25.27}+\frac{1}{29.31} \text{ \&c.}$$

By the formula

$$S=\frac{1}{p}\Big(\frac{q}{n}-\frac{q}{n+p}\Big),$$

we have

$$S=\tfrac{1}{2}\Big\{\frac{1}{21}-\frac{1}{23}+\frac{1}{25}-\frac{1}{27}+\frac{1}{29}-\frac{1}{31} \text{ \&c.}\Big\}.$$

Now since $\frac{1}{23}-\frac{1}{25}$ is *nearly* midway between $\frac{1}{21}-\frac{1}{23}$ and $\frac{1}{25}-\frac{1}{27}$, assume $\frac{1}{23}-\frac{1}{25}=\frac{1}{2}(\frac{1}{21}-\frac{1}{23}+\frac{1}{25}-\frac{1}{27})$. So also assume $\frac{1}{27}-\frac{1}{29}=\frac{1}{2}(\frac{1}{25}-\frac{1}{27}+\frac{1}{29}-\frac{1}{31})$, and so on.

$$\therefore 2S=\frac{1}{21}-\frac{1}{13}+\frac{1}{25}-\frac{1}{27}+\frac{1}{29}-\frac{1}{31}.$$

Then the sum of the new series $\frac{1}{23}-\frac{1}{25}+\frac{1}{27}-\frac{1}{29}$ &c. $=\frac{1}{2}(\frac{1}{21}-\frac{1}{23})+\frac{1}{25}-\frac{1}{27}+\frac{1}{29}-\frac{1}{31}$ &c. $=2S-\frac{1}{2}(\frac{1}{21}-\frac{1}{23})$. Adding the two series we have

$4S - \frac{1}{2}(\frac{1}{21} - \frac{1}{23}) = (\frac{1}{21} - \frac{1}{23} + \frac{1}{23} - \frac{1}{25} + \frac{1}{25}$ &c.); $\therefore\ 4S = \frac{1}{2}(\frac{1}{21} - \frac{1}{23}) + \frac{1}{21}$ $= \frac{3}{2}\cdot\frac{1}{21} - \frac{1}{2}\cdot\frac{1}{23}$. $S = \frac{3}{8}\cdot\frac{1}{21} - \frac{1}{8}\cdot\frac{1}{23} = .012422360$

Add sum of first five terms	$= .380229950$
	$.392652310$
True result	$= .392699082$
Error	$.000047372$

A closer approximation could be obtained by going farther down the line and adding directly the first six, seven or more terms. Thus take six terms, the last one being $\frac{1}{21\cdot 23}$; add $\frac{3}{8}$ of $\frac{1}{25}$ and subtract $\frac{1}{8}$ of $\frac{1}{27}$ and we get the sum $= .392670714$; the error $= .000028368$.

But there is a more rapid method of approximation. The sum of the first six terms is .382300343. Then $\frac{1}{25\cdot 27} + \frac{1}{29\cdot 31}$ &c. $= \frac{1}{2}(\frac{1}{25} - \frac{1}{27} + \frac{1}{29} - \frac{1}{31}$ &c.) $= \frac{1}{2}\cdot\frac{1}{25} - \frac{1}{2}(\frac{1}{27} - \frac{1}{29} + \frac{1}{31}$ &c.). Add $\frac{1}{2}\cdot\frac{1}{25}$ to the sum of the first six terms and we get .402300344. We have now to subtract $\frac{1}{2}(\frac{1}{27} - \frac{1}{29} + \frac{1}{31} - \frac{1}{33}$ &c) $= \frac{3}{8}\cdot\frac{1}{27} - \frac{1}{8}\cdot\frac{1}{29}$.

$$\begin{array}{rl}
 & .402300343 \\
- & .013888888 \\ \hline
 & .388411455 \\
+ & .004310345 \\ \hline
 & .392721800, \text{ a result too large by } .000022718 \\
 & .392670714 = \text{former result} \\
 & .392696257 = \text{average} \\
 & .392699082 = \text{true result} \\ \hline
 & .000002825 = \text{error.}
\end{array}$$

The reader may be curious to know how the true result referred to is obtained. Develope $\tan^{-1}x$ by Maclaurin's theorem. It is equal to $x - \frac{1}{3}x^2 + \frac{1}{5}x^3 -$ &c. Let the arc $= 45° = \frac{1}{4}\pi$; $\therefore x = 1$;

$$\therefore \text{arc } 45° = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \text{ \&c.} = 2\left\{\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} \text{ \&c.}\right\}.$$

Hence divide 3.1415962535 by 8 and we find

$$\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} \text{ \&c.} = .392699082 = \text{arc of } 22\tfrac{1}{2}°.$$

But this method by the Calculus is not general. The Algebraic method just given is not beyond the reach of the student in Algebra, and admits moreover of quite a wide extension.

Thus to find the sum of

$$\frac{1}{1.3} + \frac{1}{7.9} + \frac{1}{13.15} + \frac{1}{19.21} \text{ \&c.}$$

The sum of the first four terms = .356840768, and by a process analogous to the foregoing we find $S = .007160494,$

and the required sum is $.364001262.$

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in No. 5 have been received as follows:

From Prof. W. P. Casey, 359, 360, 361; George Eastwood, 361, 363; H. Heaton, 365, 367, A. Hall, Jr., 367, W. E. Heal, 360, 362, 367; Prof. P. H. Philbrick, 362; Prof. E. B. Seitz, 360, 363, 367; Thomas Spencer, 367; Prof. J. Scheffer, 360, 364, 367; R. S. Woodward, 360, 362, 364, 367.

359. "Find the greatest and least number of balls of equal diameter (radius r) that can be put in a given box, a feet long, b feet wide and c feet high."

ANSWER BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Balls whose diameters are equal to the greatest common divisor of a, b and c will give the least number, and those whose diameters are equal to the least c. d. of a, b and c will give the greatest number. The various forms of this question lead to a long discussion.

360. "Prove, of all spherical triangles of equal area, that of the least perimeter is equilateral".

SOLUTION BY PROF. J. SCHEFFER, HARRISBURGH, PA.

Since the area is to be constant, the spherical excess will also be constant, and consequently the three angles; hence, putting

$$A + B + C = 2S, \tag{1}$$

S is constant. Denoting $a+b+c$ by $2s$, we have the formula

$$\tan^2\tfrac{1}{2}s = \tan\tfrac{1}{2}(S-\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-C+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-B+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-A+\tfrac{1}{2}\pi)$$

which may, without difficulty, be derived from L'Huilier's formula.

Consequently

$$M = \tan\tfrac{1}{2}(S-C+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-B+\tfrac{1}{2}\pi)\tan\tfrac{1}{2}(S-A+\tfrac{1}{2}\pi) \tag{2}$$

must be a minimum.

Considering A and B the independent variables, we obtain

$$\frac{dM}{dA} = -\tfrac{1}{2}\frac{\tan\frac{1}{2}(S-C+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-A+\frac{1}{2}\pi)} - \tfrac{1}{2}\frac{\tan\frac{1}{2}(S-A+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-C+\frac{1}{2}\pi)}\frac{dC}{dA},$$

$$\frac{dM}{dB} = -\tfrac{1}{2}\frac{\tan\frac{1}{2}(S-C+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-B+\frac{1}{2}\pi)} - \tfrac{1}{2}\frac{\tan\frac{1}{2}(S-B+\frac{1}{2}\pi)}{\cos^2\frac{1}{2}(S-C+\frac{1}{2}\pi)}\frac{dC}{dB}.$$

From (1) $$\frac{dC}{dA} = -1,\quad \frac{dC}{dB} = -1.$$

Substituting, putting the diff. coef. $= 0$, and clearing of fractions, we obtain the two equations:

$$\sin(S-C+\tfrac{1}{2}\pi) = \sin(S-A+\tfrac{1}{2}\pi),$$
$$\sin(S-C+\tfrac{1}{2}\pi) = \sin(S-B+\tfrac{1}{2}\pi).$$

Wherefore $A = B = C$, and the triangle is equiangular and consequently equilateral.

SOLUTION BY W. E. HEAL, MARION, IND.

The area of a spherical triangle T, whose sides are a, b, c, is

$$\tan\tfrac{1}{4}A(T) = \sqrt{[\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)]}.$$

If the perimeter $P(T)$, $= 2s$, is constant this area will be the greatest when the three factors, $\tan\frac{1}{2}(s-a)$, $\tan\frac{1}{2}(s-b)$, $\tan\frac{1}{2}(s-c)$, are equal; and if we suppose that each side of the triangle is less than π it follows that

$$a = b = c.$$

Now suppose t_1 is an equilateral triangle having the same area as T, that is $A(T) = A(t_1)$. Also let t_2 be an equilateral triangle having the same perimeter as T, that is $P(T) = P(t_2)$. Then by what precedes we must have

$$A(T) < A(t_2),$$
$$\therefore A(t_1) < A(t_2),$$
$$\therefore P(t_1) < P(t_2),$$
$$\therefore P(t_1) < P(T).\quad \text{Q. E. D.}$$

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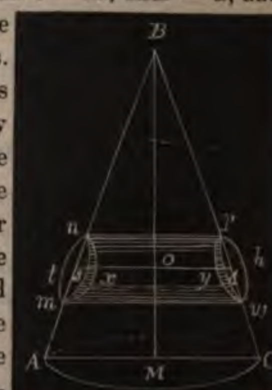
361. "A right cone, radius of base R and altitude a, is pierced by a cylinder whose radius is r, the axis of the cylinder intersecting the axis of the cone at right angles and at a point whose distance from the vertex of the cone is b. Required the solidity common to the cone and cylinder."

SOLUTION BY PROF. CASEY.

Let ABC be the cone, M the center of its base, $MC = R$, $MB = a$, and let $npwm$ represent the solid common to the cone and cylinder; nxm and pyw are equal semi-ellipses.



As $BO = b$, and the radius of the cylinder is given $= r$, $\therefore np$ and mw are known lines, and by well known properties in descriptive geometry the ellipses which form the ends are given. Then if we pass a plane through the element AB perpendicular to the plane ABC and conceive the lateral surface of the cylinder to intersect in the ellipse $nsmt$, and do the same at the other end, forming the ellipse $phwd$, we will have a solid whose ends are plane surfaces bounded by ellipses; and as all the dimen-

sions of this solid are known, its volume is easily found; and as all the dimensions of the ungula *nxmn* are known, its volume may be found; and as there are four of such ungula, subtracting their sum from the volume of *nsmtphwd* we get the required volume.

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362. "A section of an embankment is a feet long; the top width of both ends is b feet; the width of the ends at bottom is c and d feet, respectively, and the corresponding depths of the ends are e and g feet. Develop a Rule, and give the contents."

SOLUTION BY PROF. P. H. PHILBRICK, STATE UNIV., IOWA CITY.

The bottom width at a distance x from first end $= c + x(d-c)\div a$, and the depth at the same point $= e+x(g-e)\div a$. Therefore

$$d\,\text{Vol.} = \tfrac{1}{2}\left[b+c+\frac{x}{a}\left(d-c\right)\right]\left[e+\frac{x}{a}\left(g-e\right)\right]dx$$

$$= \frac{1}{2a^2}\left\{\begin{array}{l} a^2be + a^2ce + aedx + cex^2 \\ \qquad\qquad\quad + abgx + dgx^2 \\ \qquad\qquad\quad + acgx - cgx^2 \\ \qquad\qquad\quad - abex - dex^2 \\ \qquad\qquad\quad - 2acex \end{array}\right\} dx;$$

$$\therefore \text{Vol.} = \frac{1}{2a^2}\left[\begin{array}{l}(a^2be+a^2ce)a+(aed+abg+acg-abe-2ace)\tfrac{1}{2}a^2 \\ +(dg+ce-cg-de)\tfrac{1}{3}a^3\end{array}\right]$$

$$= \tfrac{1}{12}a(3be+3bg+2ce+2dg+ed+cg)$$

$$= \tfrac{1}{12}a[(b+c)e+(b+d)g+(2b+c+d)(e+g)].$$

But $(b+c)e =$ twice the area of the first end $= 2A_1$ say, and

$(b+d)g =$ " " second " $= 2A_2$ " .

Again, the width at the middle section $=[\tfrac{1}{2}(c+d)+b]\div 2=\tfrac{1}{4}(2b+c+d)$; and the depth $= \tfrac{1}{2}(e+g)$ and therefore $(2b+c+d)(e+g) =$ eight times the area of the middle section $= 8M$ say.

$$\therefore\ V = \tfrac{1}{12}a(2A_1+8M+2A_2) = \tfrac{1}{6}a(A_1+4M+A_2),$$

which is known as the Prismoidal Formula.

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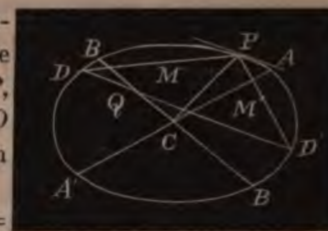
363. "The tangent at one end of a chord of an ellipse is parallel to the line joining the other end with a fixed point within the ellipse. Show that the area of the locus of the middle point of the chord is one half the area of the ellipse."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Let C be the center of the ellipse, Q the fixed point, P any point of the ellipse, DD' the chord through Q parallel to the tangent at P, and M, M' the middle points of the chords PD and PD'. Draw the diameter BB' through Q, and the diameter AA' conjugate to BB'.

Let $CA = a$, $CB = b$, $CQ = c$, $\angle ACB = \psi$, and let $(a \cos\varphi, b \sin\varphi)$ be the co-ordinates of P, referred to AA' and BB', (x, y) those of D or D', and (x', y') those of M or M'.

Then the equation of the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2, \tag{1}$$

and the eq. of DD' is

$$ay \sin\varphi + bx \cos\varphi = ac \sin\varphi. \tag{2}$$

From (1) and (2) we readily find

$$y = c\sin^2\varphi \pm \cos\varphi\sqrt{(b^2 - c^2\sin^2\varphi)},$$
$$bx = ac\sin\varphi\cos \mp a\sin\varphi\sqrt{(b^2 - b^2\sin^2\varphi},$$

the upper sign being taken for D, and the lower for D'.

Hence we have

$$2bx' = b(a\cos\varphi + x) = ab\cos\varphi + ac\sin\varphi\cos\varphi \mp a\sin\varphi\sqrt{(b^2 - c^2\sin^2\varphi)},$$
$$2y' = b\sin\varphi + y = b\sin\varphi + c\sin^2\varphi \pm \cos\varphi\sqrt{(b^2 - c^2\sin^2\varphi)}, \text{ and}$$
$$2dy' = b\cos\varphi\, d\varphi + 2c\sin\varphi\cos\varphi\, d\varphi \mp \sin\varphi\sqrt{(b^2 - c^2\sin^2\varphi)}d\varphi$$
$$\mp c^2\sin\varphi\cos d\varphi \div \sqrt{(b^2 - c^2\sin^2\varphi)}.$$

Therefore the area of the locus of M or the locus of M' is

$$\int x'\sin\psi\, dy' = \int_0^{2\pi}\left\{ \tfrac{1}{4}ab\cos^2\varphi + \frac{3ac^2}{4b}\sin^2\varphi\cos^2\varphi + \frac{a}{4b}\sin^2\varphi(b^2 - c^2\sin^2\varphi) \right\}$$
$$\times \sin\psi\, d\varphi,$$

omitting all terms containing odd powers of $\sin\varphi$ or $\cos\varphi$,

$$= \int_0^{2\pi}\left(\tfrac{1}{4}ab + \frac{ac^2}{8b}\cos 2\varphi + \frac{ac^2}{8b}\cos 4\varphi\right)\sin\psi\, d\varphi$$
$$= \tfrac{1}{2}\pi ab\sin\psi = \text{half the area of the ellipse.}$$

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364. "Discuss the curve whose equation is $x = \log[y + \sqrt{(y^2 - 1)}]$, and find its area and length."

SOLUTION BY R. S. WOODWARD, DETROIT, MICH.

Since $x = \log[y + \sqrt{(y^2 - 1)}]$, $y + \sqrt{(y^2 - 1)} = e^{+x}$, and

$$y^2 - 1 = e^{+2x} - 2e^{+x}y + y^2, \text{ whence}$$

$$y = \tfrac{1}{2}(e^{+x} + e^{-x}).$$

This is the equation of a catenary whose directrix is the axis of x. The **area lying** between the curve and the axis of x for the limits of $+x$ and $-x$ is

$$2\int_0^x y\,dx = \int_0^x e^{+x}\,dx + e^{-x}\,dx = e^{+x} - e^{-x}.$$

Therefore the area within the curve between the limits $+x$ and $-x$ is

$$x(e^{+x} + e^{-x}) - (e^{+x} - e^{-x})$$
$$= e^{+x}(x-1) + e^{-x}(x+1).$$

The length of the curve between $x = 0$ and $x = \pm x$ is

$$\int_{x=0}^{x=\pm x}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \tfrac{1}{2}(e^{\pm x} - e^{\mp x}).$$

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365. "Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}\,.\,d\theta}{1-c\cos^n\theta} = \frac{\pi}{\sqrt{(2n)}}$$

when c is indefinitely nearly equal to unity, n being a positive quantity."

SOLUTION BY H. HEATON, LEWIS, IOWA.

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}.d\theta}{1-c\cos\theta}$$

$$= \frac{1}{n}\int_0^{\frac{1}{2}\pi}\left(\frac{\sqrt{(1-c)}.d\theta}{1-c^{\frac{1}{n}}\cos\theta} + \frac{\sqrt{(1-c)}[n-1+(n-2)c^{\frac{1}{n}}\cos\theta\ldots+c^{\frac{n-2}{n}}\cos^{(n-2)}\theta]d\theta}{1+c^{\frac{1}{n}}\cos\theta\ldots c^{\frac{n-1}{n}}\cos^{(n-1)}\theta}\right)$$

But the second member on the right-hand of this equa'n$=0$ when $c=1$; because it does not take the form $\frac{0}{0}\,d\theta$ for any value of θ. Hence

$$\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}.d\theta}{1-c\cos^n\theta} = \frac{1}{n}\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(1-c)}d\theta}{1-c^{1\div n}\cos\theta}$$

$$= \frac{2\sqrt{(1-c)}}{n\sqrt{(1-c^{1\div n})}}\tan^{-1}\left(\frac{\sqrt{(1-c^{1\div n})}\tan\frac{1}{2}\theta}{\sqrt{(1-c)}}\right)$$

$$= \frac{2\sqrt{(1+c^{1\div n}+c^{2\div n}\ldots+c^{(n-1)\div n})}}{n\sqrt{(1+c^{1\div n})}}\tan^{-1}\left(\frac{\sqrt{(1+c^{1\div n})}\mathrm{tn}\,\frac{1}{2}\theta}{\sqrt{(1-c)}}\right)$$

$$= \frac{\pi}{\sqrt{(2n)}} \text{ when } c = 1.$$

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367. "Prove the equation

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\text{ etc.}$$
$$-2\eta\cos x-\tfrac{1}{2}\,2\eta^2\cos 2x-\tfrac{1}{3}\,2\eta^3\cos 3x-\text{etc.}$$
$$= \sum_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i} - \sum_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

SOLUTION BY THOMAS SPENCER, SOUTH MERIDEN, CONN.

From Trigonometry we have the expansion

$$\frac{\eta \sin x}{1-2\eta\cos x+\eta^2} = \eta \sin x + \eta^2 \sin 2x + \eta^3 \sin 3x + \&c.$$

Multiply both sides of this equation by $2dx$, and integrate, we have

$$\log(1-2\eta\cos x+\eta^2) = -2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.$$

Also we know that

$$\log(1+\eta^2) = \eta^2 - \tfrac{1}{2}\eta^4 + \tfrac{1}{3}\eta^6 - \&c.$$

Therefore we have

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = \log(1-2\eta\cos x+\eta^2) - \log(1+\eta^2)$$
$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \tfrac{1}{3}\eta^6 + \&c.$$
$$-2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.$$
$$= \sum_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i} - \sum_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

SOLUTION BY H. HEATON.

Because $2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$; therefore

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = \log\left[1+\eta^2-\eta(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})\right] - \log(1+\eta^2)$$
$$= -\log(1+\eta^2) + \log(1-\eta e^{x\sqrt{-1}}) + \log(1-\eta e^{-x\sqrt{-1}})$$
$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \tfrac{1}{3}\eta^6 + \&c.$$
$$-2\eta\cos x - \tfrac{1}{2}\,2\eta^2\cos 2x - \tfrac{1}{3}\,2\eta^3\cos 3x - \&c.$$

PROBLEMS.

368. *By Prof. J. Scheffer.*—In a quadrilateral $ABCD$, the diagonal AC makes with the sides the four angles $CAB = \alpha$, $ACB = \beta$, $ACD = \gamma$, $CAD = \delta$. Find the angles which the other diagonal BD makes with the sides.

369. *By R. J. Adcock.*—Show that the radius of curvature of an ellipse equals the cube of the radius vector divided by the rectangle of the semi axes; the radius vector being through the centre at right angles to the radius of curvature.

370. *By Prof. Edmonds.*—Divide a right angle into three parts α, β, γ, such that $(\cos\alpha) \div m = (\cos\beta) \div n = (\cos\gamma) \div p$.

371. *By Prof. E. B. Seitz.*—ACB is the quadrant of a circle, O the center of its inscribed circle; $O_1, O_2, O_3, \ldots O_n$ are the centers of a series of circles, each of which touches the preceding, the arc AB and the radius AC, the circle O_1 touches the circle O; and OH, O_1H_1, $O_2H_2, \ldots O_nH_n$ are the perpendiculars on AC, or the radii of the inscribed circles. If $AC = r$, $O_nH_n = x_n$, and $CH_n : O_nH_n = u_n$, prove that

$$u_n = \tfrac{1}{2}(\sqrt{2}+1)^{2n+1} - \tfrac{1}{2}(\sqrt{2}-1)^{2n+1},$$

$$x_n = \frac{2r}{2 + (\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}}.$$

372. *By William Hoover, A. M.*—A hemisphere, radius r, is resting with its convex surface on two planes, one perfectly smooth and inclined to the horizen at an angle α, the other being inclined at an angle β; if m be the coefficient of friction between the latter and the hemisphere, what is the position for rest?

373. *Selected by Prof. Eddy.*—Two particles of masses m and m' respectively, are connected by a string passing through a small fixed ring and are held so that the string is horizontal; their distances from the ring being a and a', they are let go. If ρ and ρ' be the initial radii of curvature of their paths, prove that

$$\frac{m}{\rho} = \frac{m'}{\rho'}, \text{ and } \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{a} + \frac{1}{a'}.$$

374. *By R. S. Woodward.*—Prove 1st, that the probable error of any tabular value in a table of logarithms, trigonometric functions etc., is 0.25 of a unit of the last decimal place, supposing this place correct to the nearest unit; 2nd, that the average of the squares of probable errors of interpolated values depending on first differences only is $\frac{2}{3}(0.25)^2$.

Announcement of Vol. IX.—As this No. completes the 8th annual volume of the Analyst, we are pleased to say to our readers that we have no thought of abandoning the publication, so long as we continue to receive the support and encouragement of the many able mathematicians who give character to our publication by their contributions to its pages.

The publication of the Analyst was commenced with no exalted expectations of success, as the history of like publications in this country attests the difficulty of sustaining a periodical devoted exclusively to severe and exact scientific research.

It is therefore with some degree of gratification that we are able to state that the publication has been thus far conducted without pecuniary loss.

Though our subscription list is not large yet an examination of the published volumes will show that we number among our contributors many of the ablest mathematicians and astronomers in America, besides several eminent Europeans. And it is with considerable satisfaction that we have been able to present to our readers eight annual volumes made wholly of original contributions, with a few translations. For this we take no praise to ourself but freely acknowledge our indebtedness to our learned and liberal subscribers.

We shall commence the ninth volume of the ANALYST, therefore, with confidence that the friends who have thus far stood by us will continue their patronage and support, and that many new names will be added to our list of subscribers and contributors to Vol. IX.

J. E. HENDRICKS.

PUBLICATIONS RECEIVED.

A Treatise on the Calculus of Variations. By LEWIS BUFFETT CARLL, A. M. 8vo. 568 pp. New York: John Wiley & Sons. 1881.

In this volume the author has presented, in concise and elegant style, all the important applications that have been made of this abstruse branch of analysis, and has illustrated the several principles which embody the science by the discussion of 71 special problems.

The publishers have manifested their appreciation of the permanent value of this book by the superior paper and faultless typography in which it is brought out.

An Elementary Treatise on the Integral Calculus, Founded on the Method of Rates, or Fluxions. By WILLIAM WOOLSEY JOHNSON, Professor of Mathematics at the United States Naval Academy, Annapolis, Maryland. 228 pp. 8vo. New York: John Wiley & Sons. 1881.

The author of this book is so well known to the readers of the ANALYST that any commendation by us would be superfluous. We need only say that the elements of the Integral Calc. are here presented with the clearness and elegance that are characteristic of the author.

On Binomial Congruences: comprising an Extension of Fermat's and Wilson's Theorems, and a Theorem of which both are Special Cases. By O. H. MITCHELL, Fellow of the Johns Hopkins University. Reprinted from the American Journal of Math., Vol. III.

On the Ratio between Sector and Triangle in the Orbit of a Celestial Body. By ORMOND STONE. Reprinted from the American Journal of Mathematics, Vol. III.

ERRATA.

On page 147, line 14, for "and radius" read and squared radius.
" " 174, " 6 from bottom, for "Prop. II." read Props. I. and II.
" " 175, *dele* the *s* contiguous to the line $n'q$ in the Fig.
" " 190, line 8, for "t = time of" read t = half time of.
" " ", " 19, " "primative" read primitive.
" " ", " 5 from bottom, for "$N_1 = 29.732$", read $N_1 = 29.598$.

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ALPHABETICAL INDEX.

// THE ANALYST.

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A GENERAL ALGEBRAIC METHOD FOR THE SOLUTION OF EQUATIONS.

BY T. S. E. DIXON, ESQ., CHICAGO, ILLINOIS.

In Algebra, as now taught, equations of the second, third and fourth degrees, are resolved, each by a specific method or artifice specially adapted to the degree of the equation. It may not be an inappropriate addition to supplement these diverse methods by an uniform general method, applicable independently of the degree of the proposed equation.

Space will permit only of a condensed statement of the principles and outlines of the proposed method.

I.

Each root of an equation of the nth degree, wanting its second term, is equal to the sum of the roots of $n-1$ binomial equations of the form $y^n = a^n$, $y^n = b^n$, $y^n = c^n$, &c., one root being taken from each binomial equation in the formation of each root of the original equation.

II.

The order in which the n roots of each of the $n-1$ binomial equations are arranged to form by their sum the n roots of the original equation is symmetrical, and is given in the following table, in which ε represents any one of the imaginary nth roots of unity:

$$\underline{x = \sqrt[n]{a^n} + \sqrt[n]{b^n} + \sqrt[n]{c^n} + \sqrt[n]{d^n} + \ldots + \text{to } n-1 \text{ terms.}}$$

$$\begin{aligned}
x &= a(\varepsilon^n) + b(\varepsilon^n)^2 + c(\varepsilon^n)^3 + d(\varepsilon^n)^4 + \ldots + m(\varepsilon^n)^{n-1},\\
x &= a(\varepsilon\,) + b(\varepsilon^2) + c(\varepsilon^3) + d(\varepsilon^4) + \ldots + m\,(\varepsilon^{n-1}),\\
x &= a(\varepsilon^2) + b(\varepsilon^2)^2 + c(\varepsilon^2)^3 + d(\varepsilon^2)^4 + \ldots + m(\varepsilon^2)^{n-1},\\
x &= a(\varepsilon^3) + b(\varepsilon^3)^2 + c(\varepsilon^3)^3 + d(\varepsilon^3)^4 + \ldots + m(\varepsilon^3)^{n-1},\\
x &= \quad + \quad + \quad + \quad + \quad + \quad ,\\
x &= a(\varepsilon^{n-1}) + b(\varepsilon^{n-1})^2 + c(\varepsilon^{n-1})^3 + d(\varepsilon^{n-1})^4 + \ldots + m(\varepsilon^{n-1})^{n-1}.
\end{aligned}$$

It will be noticed that in the first perpendicular column a is combined with *all* the nth roots of unity, b in the second column with the squares of these roots, c in the third column with their cubes &c. But as obviously $\varepsilon^n = 1$, $\varepsilon^{n+1} = \varepsilon$, $\varepsilon^{n+2} = \varepsilon^2$, &c., each letter in the table is combined in each instance with an nth root of unity, and this combination is therefore always one of the nth roots of the binomial equation in that letter.

III.

If n values of x be taken from the above table and combined to form an equation of the nth degree, the imaginary quantities ε, ε^2, ε^3, &c., will wholly *disappear* and the coefficients and the absolute term of the equation will all be rational functions of the $n—1$ quantities a, b, c, &c.

If numerical values be assigned to the letters a, b, c, &c., the equation produced, of whatever degree, will verify all the values of x, and its coefficients and the absolute term will all be real numbers.

IV.

When n is a prime number, the coefficients and the absolute term of the equation produced are symmetrical rational functions of the $n—1$ quantities a, b, c, &c.; symmetrical in the important sense that each coefficient and the absolute term is exactly the same function of one letter that it is of another, and the same interchange between the letters is permissible as when n letters are used to represent the n values of the unknown quantity.

When n is a composite number, while the coefficients are no longer symmetrical functions of a, b, c, &c., the relations established between these quantities afford facilities for the reduction of the equation.

V.

The resolution of a given equation is effected by giving expression to its coefficients and the absolute term as functions of the n-1 letters a, b, c, &c., and then, by eliminating from the supplemental equations thus formed all the letters but one, producing an auxiliary equation of the $(n—1)$th degree, whose resolution affords the elements forming by their sum the values of the unknown quantity in the given equation.

Applying this method to the general equations of the several degrees in their order, we have—

VI.

Quadratic Equations.

The artifice usually employed is termed "completing the square" by an addition to each member of the equation. While this is effective and the

most available for practical use, the general method is, under its terms equally applicable.

Given the general quadratic equation

$$x^2 + 2px = q;$$

removing the second term we have the equation $y^2 = p^2 + q$. As $n = 2$ and $n-1 = 1$, only one letter, a, can be employed, and this letter is to be combined with the two square roots of unity to form the two values of y.

$$y = a(\ \ 1) = \ \ \sqrt{(p^2 + q)} \qquad x = -p + \sqrt{(p^2 + q)},$$
$$y = a(-1) = -\sqrt{(p^2 + q)} \qquad x = -p - \sqrt{(p^2 + q)}.$$

VII.

Cubic Equations.

Here the general method corresponds closely with the particular method usually employed.

Given the general cubic equation

$$x^3 + 3px = 2q;$$

we form from the above table the following:

$$\underline{x = \sqrt[3]{a^3} + \sqrt[3]{b^3}.}$$
$$x = a \ \ + b$$
$$x = a\varepsilon \ \ + b\varepsilon^2$$
$$x = a\varepsilon^2 + b\varepsilon.$$

From these three values of x we construct the general equation

$$x^3 - 3abx = a^3 + b^3,$$

whose coefficient and absolute term are symmetrical functions of a and b. The supplemental equations become $ab = -p$ and $a^3 + b^3 = 2q$.

Eliminating b the final equation is a quadratic. But as $\sqrt[3]{a^3}$ and $\sqrt[3]{b^3}$ have each three values, these six unknown quantities, expressed in the table, must all enter into the final equation of which they will be the roots; this equation will, therefore, be not only a quadratic, but it must also be an equation of the 6th degree, to wit:

$$a^6 - 2qa^3 = p^3,$$

whence

$$a^3 = q + \sqrt{(q^2 + p^3)}, \text{ and } b^3 = q - \sqrt{(q^2 + p^3)},$$

and

$$x = \sqrt[3]{[q + \sqrt{(q^2 + p^3)}]} + \sqrt[3]{[q - \sqrt{(q^2 + p^3)}]}.$$

It may be observed that the interchange of a and b with each other in the equation $x^3 - 3abx = a^3 + b^3$ does not change the value of the roots, its only effect being to change the order of the radicals in the above expression of the value of x.

The essential reason for the existence of the "irreducible case" is also here made manifest.

The values of x expressed in the foregoing table are one real and two imaginary; an attempt therefore to combine them to form a cubic equation whose roots are all real must necessarily result in the imposition of impossible conditions in the final formula.

VIII.

Biquadratic Equations.

Instead of resorting to the variety of expedients heretofore employed, apply the above method directly to the general equation

$$x^4 - 2px^2 - 8qx = r.$$

The four 4th roots of unity are $1, -1, \sqrt{-1}$ and $-\sqrt{-1}$. Applying these to the general table we have:—

$$\begin{aligned} x &= a + b + c \\ x &= -a + b - c \\ x &= a\sqrt{-1} - b - c\sqrt{-1} \\ x &= -a\sqrt{-1} - b + c\sqrt{-1}. \end{aligned}$$

Here x in general $= \sqrt[4]{a^4} + \sqrt{b^2} + \sqrt[4]{c^4}$, but as the square roots of b^2 are 4th roots of b^4 the conditions of sections I and II are still observed.

Constructing an equation from the four values of x in the table we have:

$$x^4 - 2(b^2 + 2ac)x^2 - 4b(a^2 + c^2)x = a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c;$$

giving the supplemental equations

$$\begin{aligned} b^2 + 2ac &= p, \\ b(a^2 + c^2) &= 2q, \\ a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c &= r. \end{aligned}$$

As 4 is not a prime number, there is here an absence of symmetry; the relations established, however, in the supplemental equations, enable us to ascertain with great facility the values of a, b and c.

For brevity take first the simplest general form

$$x^4 - 8qx = r,$$

and hence $p = 0, -2ac = b^2$ and $2q \div (a^2 + c^2) = b$. Substituting these values there is produced the auxiliary cubic equation

$$b^6 + \tfrac{1}{4}rb^2 = q^2,$$

whence the value of b is readily obtained and the values of a and c also determined, when the four values of x may be expressed as indicated in the table. When p does not equal 0 the auxiliary cubic equation is found, but with a second term in b^4.

As in the forgoing table the values of x are two real and two imaginary, an "irreducible case" must arise whenever the roots are all real or all imaginary, and this essential reason is here first disclosed, without reference to the character of the auxiliary cubic equation.

IX.

Equations of the 5th Degree.

In entering upon this almost *terra incognita*, we have in the proposed general method a simple and direct line of attack upon the difficulties of the case.

Representing by ε any one of the imaginary 5th roots of unity we have from the general table:—

$$\underline{x = \sqrt[5]{a} + \sqrt[5]{b} + \sqrt[5]{c} + \sqrt[5]{d}.}$$

$$\begin{aligned}
x &= a + b + c + d, \\
x &= a\varepsilon + b\varepsilon^2 + c\varepsilon^3 + d\varepsilon^4, \\
x &= a\varepsilon^2 + b\varepsilon^4 + c\varepsilon + d\varepsilon^3, \\
x &= a\varepsilon^3 + b\varepsilon + c\varepsilon^4 + d\varepsilon^2, \\
x &= a\varepsilon^4 + b\varepsilon^3 + c\varepsilon^2 + d\varepsilon.
\end{aligned}$$

We then construct, directly, an equation of the 5th degree, having the above five values of x for its roots.

Again the imaginary quantities wholly disappear, and we have the eq'n

$$x^5 - 5(ad+bc)x^3 - 5(a^2c+ab^2+bd^2+c^2d)x^2 + 5(a^2d^2 - a^3b - abcd - ac^3 - b^3d - cd^3 + b^2c^2)x = a^5+b^5+c^5+d^5 - 5(a^3cd - a^2b^2d - a^2bc^2 + ab^3c + abd^3 - ac^2d^2 - b^2cd^2 + bc^3d).$$

Given the general equation

$$x^5 + 5px^3 + 5qx^2 + 5rx = s,$$

we have the supplemental equations

$$\begin{aligned}
&ad + bc = -p, \\
&a^2c + ab^2 + bd^2 + c^2d = -q, \\
&a^2d^2 - a^3b - abcd - ac^3 - b^3d - cd^3 + b^2c^2 = r, \text{ and} \\
&a^5 + b^5 + c^5 + d^5 - 5(a^2cd - a^2b^2d - a^2bc^2 + ab^3c + abd^3 \\
&\quad - ac^2d^2 - b^2cd^2 + bc^3d) = s.
\end{aligned}$$

Since the above supplemental equations are each exactly the same function of one letter as of another, and as both in them and in the table one letter may be interchanged with another without disturbing any relation, the final equation obtained by elimination will be of one identical form, whichever three of the four unknown quantities are eliminated.

This final equation will, therefore, contain within it, as its roots, all the values of each of the four letters a, b, c and d, which values are expressed in the above table, and it can contain no other values of the unknown quantity.

Were it not for the permutations which must take place, this final auxiliary equation would be an equation of the fourth degree, of the form

$$a^{20} - Aa^{15} + Ba^{10} - Ca^{5} + D = 0.$$

This is readily made apparent. By reference to the above table it will be observed that twenty distinct quantities a, $a\epsilon^2$, $b\epsilon^4$, &c, all unknown, enter into the composition of the coefficients of the given general equation. If any one of the five values of a in the table be assigned to a in the supplemental equations, this value of a, together with the corresponding values of b, c and d in the same horizontal column, or value of x, will satisfy all the equations, and the same is true of the five values of b, c and d respectively. The table also embraces all the possible values of a, b, c, and d, as it is impossible to assign any other value and verify the equations, all of which is in strict analogy with the same relations in the table of the cubic eq'n.

Consequently the final equation, in which there is centered in one letter, by elimination, all the values of the four letters, must have for its roots, did permutation not take place, precisely these twenty quantities contained in the table, in strict analogy to the final auxillary equation containing the six quantities of the cubic table.

But the equation having for its roots these twenty quantities (as is made manifest by its formation directly from these roots), can only be of the form:

$$y^{20} - (a^5 + b^5 + c^5 + d^5)y^{15} + (a^5b^5 + a^5c^5 + a^5d^5 + b^5c^5 + b^5d^5 + c^5d^5)y^{10}$$
$$- (a^5b^5c^5 + a^5b^5d^5 + a^5c^5d^5 + b^5c^5d^5)y^5 + a^5b^5c^5d^5 = 0,$$

which is the form of the auxiliary equation above indicated.

But permutations take place. The four perpendicular rows in the table in a, b, c and d respectively are capable of $1 . 2 . 3 . 4 = 24$ permutations with each other. Consequently the final equation in a will not be of the 20th, but of the 120th degree. But by these permutations no unknown quantities additional to the twenty in the table are introduced into the final equation, and each of these twenty quantities enters into that equation precisely the same number of times. The entire twenty quantities are therefore included six times in the formation of the equation of the 120th degree. It is a necessary conclusion that the first member of this equation, the second member being zero, is a polynomial which is a perfect sixth power, and that the sixth root of this polynomial is the first member of the required equation in a^{20}, whose roots are the twenty quantities expressed in the table. As this equation is a biquadratic, its solution is practicable, and its roots may be combined, as in the table, to form the five values of x in the given general equation of the fifth degree.

It will be observed that in the above table the values of x are one real and four imaginary, consequently when all the roots are real, or when only two are imaginary, an "irreducible case" will again be presented.

X

It is a remarkable fact, in this connection, that, by reason of the radicals therein, the algebraic expression for the value of a in the foregoing equation

$$a^{20} - Aa^{15} + Ba^{10} - Ca^{5} + D = 0,$$

is capable of expressing algebraically 120 values of a, while it only expresses 20 actually different values.

For convenience transform the equation into one wanting its second term:

$$a^{20} + 2pa^{10} + 8qa^{5} = r.$$

Let

$$a^{5} = \sqrt{n} + \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)};$$

make the substitution and we have the cubic equation

$$n^{3} + pn^{2} + \frac{p^{2}+r}{4}n = q^{2},$$

and the three values of n may be each expressed in the regular cubic formula.

The four values of a^5 in the above equation are expressed as follows:

$$a^{5} = \sqrt{n} + \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)},$$

$$a^{5} = \sqrt{n} - \sqrt{-\left(p+n+\frac{2q}{\sqrt{n}}\right)},$$

$$a^{5} = -\sqrt{n} + \sqrt{-\left(p+n-\frac{2q}{\sqrt{n}}\right)},$$

$$a^{5} = -\sqrt{n} - \sqrt{-\left(p+n-\frac{2q}{\sqrt{n}}\right)}.$$

But here are four different algebraic expressions for the value of a^5 containing the symbol n, as yet unexpressed in its cubic formula. As n, however, is capable of expression in six different algebraic forms, these values of n successively substituted in each of the above four expressions will give twenty four different algebraic expressions for the value of a^5.

(To obtain the six algebraic forms expressing the value of n, for convenience, transform the cubic equation into the well known general form

$$x^3 + 3px = 2y;$$

let

$$x = \sqrt[3]{y} - \frac{p}{\sqrt[3]{y}}$$

and make the substitution, resulting in the quadratic

$$y^2 - 2qy = p^3,$$

in which y has the two values $q + \sqrt{(q^2 + p^3)}$ and $q - \sqrt{(q^2 + p^3)}$.

But the three values of x are expressed as follows, representing by ϵ one of the imaginary cubic roots of unity:

$$x = \sqrt[3]{y} - \frac{p}{\sqrt[3]{y}},$$

$$x = \epsilon\sqrt[3]{y} - \frac{p}{\epsilon\sqrt[3]{y}},$$

$$x = \epsilon^2\sqrt[3]{y} - \frac{p}{\epsilon^2\sqrt[3]{y}}.$$

Substituting for y in the above formulæ each of its two values we have the six different algebraic expressions for the value of x in the cubic equation.)

As a is the 5th root of each of the 24 algebraic expressions for the value of a^5, and as there are five 5th roots in each case, expressed by combination with the one real and four imaginary 5th roots of unity, there are therefore 120 different algebraic expressions for the value of a in the foregoing equation.

These 120 varied expressions are the necessary result of the permutations which have taken place, but upon analysis they are found to represent only twenty different quantities, and these are the quantities found in the table.

[We feel much interest in the foregoing paper and invite a critical examination of it by our readers.

We have never devoted much attention to the efforts that have been made to solve the general equation of the fifth degree, and hence do not regard ourself as a competent critic; and as the extended efforts of Hirsch, Schulenburg and others, have resulted in failures, and especially as so exalted a mathematician as ABEL has pronounced the solution impossible, it would seem presumptious to assert that the problem has been solved in the foregoing paper. Nevertheless, as we have not been able to detect any error in the foregoing very elegant discussion of the question, we earnestly hope that, if it shall not be found in every respect complete, it may lead to the desired solution, and we commend it to the careful consideration and critical scrutiny of our readers.—Ed.]

ZERO AND INFINITY.

ILLUSTRATIONS BY PROF. C. H. JUDSON, GREENVILLE, S. C.

An objection has been raised against the conclusions reached in my article on Zero and Infinity, in the July No. of the Analyst, that if these interpretations be admitted, then we must reject many very beautiful and interesting generalizations of Analytical Geometry. Thus,— "Every strai't line meets a curve of the second degree in two distinct, coincident, or imaginary points." (See Salmon's Conic Sections, 5th edition, Art. 135.)

Let us take the equations

$$y = nx + a, \tag{1}$$

$$y^2 = 4ax. \tag{2}$$

Eliminating x we find for the ordinate of the point of intersection

$$y = \frac{2a}{n}\Big(1 \pm \surd[1-n]\Big). \tag{3}$$

If $n > 1$, the line meets the curve in two imaginary points: if $n = 1$, in two coincident points at P, and is a tangent: if $n < 1$, in two distinct p'ts at P' and P'': if $n = 0$ the line becomes parallel to the axis, and $y = 2a.\frac{0}{0}$, which by (1) is $y = a$; or $y = 4a \div 0$, and the second point of intersection is said to be "at infinity". But how can there be a real point of intersection at infinity when DC remains constant and equal to a, while CP'' increases without limit? Can we believe that the two p'ts D and P'', though infinitely distant from each other, are yet coincident?

It would seem that the only intelligible explanation is, that if n is an infinitesimal, the line AP'' is not quite paral'l to OC; and as the ordinate to this right line increases proportionally to its abscissa, while that to the curve increases proportionally to the square root of its abscissa, therefore the ordinate to the right line will eventually become the greater, and there will be a point of intersection indefinitely remote from the origin; but if $n = 0$ then $4a \div 0$ is a symbol of impossibility and there can be no second point of intersection.

If n is negative there will be two real points of intersection, whose abscissas are, one positive and one negative, each approaching zero as n increases. This indicates two coincident points at O when $n = -\infty$.

Again, let us consider an hyperbola

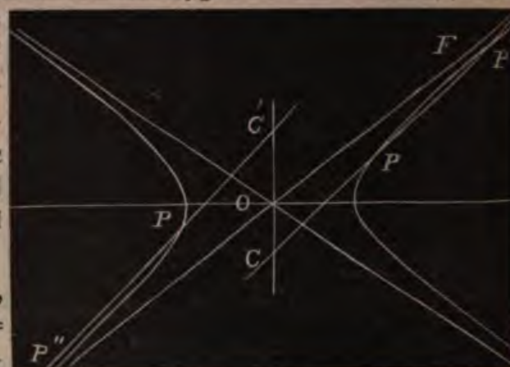
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and a right line

$$y = nx + c.$$

If c is negative and $n < (b \div a)$ we have two distinct points of intersection at P and P'.

If n is very nearly equal to $b \div a$ P' will be indefinitely remote from the origin. If now $c, = OC$, decreases indefinitely the line PP' will approach indefinitely to coincidence with the asymptote OT, and PP' may be said to meet the curve in two coincident points at positive infinity. If c is positive the point of intersection will be on the left hand branch of the curve at minus infinity.



According to the older view, a line through the origin ($y = nx$) meets the curve in two imaginary points if $n > (b \div a)$. If $n < (b \div a)$ it meets the curve in two distinct points, one in the positive direction as at P', the other in the negative, as P''. Now when $n = (b \div a)$ the coordinates of P' become + infinity while those of P'' become — infinity, and since $a \div (+0) = a \div (-0)$, plus infinity and minus infinity have been supposed to be two coincident points. This is manifestly absurd. The true statement is: (1), An asymptote to an hyperbola does not meet the curve. (2), Any other line through the centre meets the curve in two imaginary, or in two distinct points (the coordinates of one being +, those of the other —). (3), A line passing very near the centre and very nearly parallel to the asymptote meets the curve in two p'ts very remote and very nearly coincident.

Once more—"A right line meets a curve of the third degree in three p'ts, one of which must be real; and an asymptote to a curve of the third degree must meet the curve in one real point besides the two coincident points at infinity". (See Salmon's Higher Plane Curves, Chapt. III; also, Williamson's Diff. Calc., Art. 197.)

Let us take the curve $x^3 + y^3 = a^3$.

The equation of its asymptote is

$$y + x = 0.$$

Now what are the three *real* points of intersection of the asymptote with the curve? If we substitute the value of $y = -x$ in the first equation, we have

$$x^3 - x^3 = a^3, \text{ or } 0x^3 = a^3, \text{ or } x^3 = \frac{a^3}{0}.$$

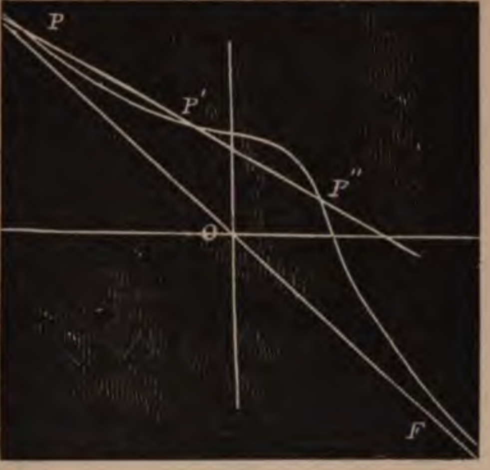
What are the three roots of this equation? It may be written

$$0x^3 + 0x^2 + 0x = a^3,$$

which is said th have "*three roots, each equal to infinity*", thus giving three points of intersection. This is to the writer *profoundly obscure*.

The line $y = nx + c$, if $c < a$, meets the curve in 3 distinct p'ts, P', P'', P'''.

If this line is very nearly parallel to the asymptote P will be very remote from the origin.



If now c decreases indefinitely, P' will approach P while P'' will recede indefinitely toward T, and the asymptote is the limiting position of $P\,P'\,P''$. Hence the line $y = nx + c$ may be said to meet the curve in two coincident points infinitely remote on one side of the origin, and at a distinct point, also infinitely remote, on the other side, if c is an infinitesimal ($c = \circ$) and $n = \pm \circ$.

Hence we adhere to the interpretations

$$\frac{a}{0} = \text{impossibility; and } \frac{a}{\circ} = \infty.$$

NOTE.—The value of Ψ, at p. 172, should be

$$\Psi = A\frac{d\Phi^2}{dx^2} + B\frac{d\Phi^2}{dy^2} + C\frac{d\Phi^2}{dz^2} + \&c.$$

The error was made by me, in copying

W. E. HEAL

A DEMONSTRATION OF MACLAURIN'S THEOREM.

BY J. S. HAYES.

[Continued from page 154, Vol. VIII.]

WE now proceed to prove that $f(x)$ is *equal* to $A+Bx+Cx^2+$&c. *ad infinitum* when all the terms of the series after Sx^n have the same sign, on the condition that between the values $x=0$ and $x=x'$ $f(x)$ is neither imaginary nor infinite. This is done by proving that the limit of L_n, when n is infinite, is zero. L_n is of such a nature that it and its first $n-1$ differential coefficients vanish with x. It therefore must be of the form $(\varphi x)^n$ where φx is a function that vanishes with x. In fact it is proved by the ordinary methods of demonstrating Maclaurin's Theorem that L_n being the remainder after n terms, is equal to $\frac{x^n}{n!}f(\theta x)$ (Tod. Dif. Cal. p. 74) $=\frac{\{x\sqrt[n]{[f(\theta x)]}\}^n}{n!}$ $=(\varphi x)^n$. Now φx is a continuous function of x and is equal to zero when $x=0$. Therefore as x increases it gradually increases numerically. Of course at first it is less than 1 and greater than -1. While this is the case, it may be made as small as we please by sufficiently increasing n. Therefore, its limit is zero. If φx becomes greater than 1 or less than -1, $(\varphi x)^n$ may be made numerically as large as we please by sufficiently increasing n, If $\varphi x=\pm 1$, $(\varphi x)^n=\pm 1$. Therefore $(\varphi x)^n$ is either 0, infinity or $\pm$ 1. when n is indefinitely increased. But in the present case L_n is numerically less than $L=f(x)$. Therefore it cannot be infinite. Suppose that it is $=\pm 1$ when $x=x'$. Now let x change from x_1 to x_2, numerically less than x_1; φx also changes. Let the change be so small that $f(x_1)\sim f(x_2)$ $<\frac{1}{2}$ and $A+Bx_1+\ldots+Sx_1^n\sim(A+Bx_2+\ldots+Sx_2^n)<\frac{1}{2}$. Then φx becomes greater or less than ± 1. It cannot become numerically greater, else $(\varphi x_2)^n=\infty$ when n is infinite. If it becomes numerically less, $(\varphi x_2)^n$ $=0$, when n is infinite. Then $f(x_1)=A+Bx_1+\ldots\pm 1$ and $f(x_2)=A$ $+Bx_2+\ldots+0$, x_1 being numerically greater than x_2 and all the signs after Sx^n being the same. Then $f(x_1)\sim f(x_2)>\frac{1}{2}$. But this is contrary to the hypothesis. Therefore φx cannot be $\pm$ 1, and L_n must be zero when n is infinite.

The value of the preceding demonstration (see No. 5, Vol. VIII) lies in its results. We are able, by the conditions established, to determine with reference to the convergence of a series from the series itself without reference to a remainder. When the signs change, we can also determine from the series itself within what degree of nearness the series at any required point (if convergent) approximates to the true value of the function. Some examples will be given in illustration.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}}{n} x^n + \&c.$$

By condition 2°, this is true for all values of x less than unity.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \&c.$$

By condition 1°, this is true for all values of x less than unity. If $x=1$, $\log(1-x) = -\infty$. If $x > 1$, $\log(1-x)$ is imaginary.

$$\sqrt{(l^2+x)^2(l^2-x) - 4lx(l^2-x)^{\frac{3}{2}} + 8l^4x} = l^3 + \tfrac{5}{2}lx - \frac{5x^2}{8l} + \frac{15x^3}{48l^3} - \frac{123x^4}{384l^5} + \ldots$$

By condition 2°, this series is convergent to the fourth term for all values of x less than l^2. Also it approaches to within $15x^3 \div 48l^3$ of the true value of the function.

Lagrange's and Laplace's Theorems are so dependent on Maclurin's that the conditions just established of the convergence of the latter apply equally to the former. This needs no proof.

EXAMPLES.—Given $y = z + xe^y$; expand y in powers of x.

The result by Lagrange's Theorem is

$$y = z + xe^z + \frac{x^2}{2} 2e^{2z} + \frac{x^3}{3!} 3^2 e^{3z} + \ldots + \frac{x^n}{n!} n^{n-1} e^{nz} + \ldots$$

By condition 1°, this furnishes a true value of y for all positive values of x, unless, in the change from $x = 0$ to the value of x under consideration, y becomes infinite. But, if x and z are both equal to unity, the second member is an increasing series, and its sum, *ad infinitum*, is infinite. Therefore we may conclude that y *does* become infinite (z being $>$ or $=1$) for some value of x between zero and infinity. In this problem $\frac{dy}{dx} = e^y + xe^y \frac{dy}{dx}$, whence $\frac{dy}{dx} = \frac{e^y}{1-xe^y} = \frac{e^y}{1-xe^{z+}}$, which becomes infinite before $xe^z = 1$. This corroborates the preceding conclusion.

In Todhunter's Dif. Cal., p. 120, the following is established by the use of Lagrange's Theorem:

$$\log \frac{1-\sqrt{(1-4t)}}{2} = t + \tfrac{3}{2}t^2 + \frac{4.5}{2.3}t^3 + \frac{5.6.7}{2.3.4} 4t, \&c.$$

By condition 1°, this is true for all values of $4t$ less than unity. If $4t > 1$, the function is imaginary.

In the ordinary demonstration of Maclaurin's Theorem there is a condition that $f(x)$ and all its differential coefficients shall be continuous. In the demonstration here given, it is only necessary that $f(x)$ shall be continuous. This demostration therefore covers a wider field.

For example, by Maclaurin's Theorem

$$(a-x)^{\frac{1}{3}} = a^{\frac{1}{3}} - \frac{x}{3\,a^{\frac{2}{3}}} - \frac{1.2}{3.3}\frac{x^2}{2!a^{\frac{5}{3}}} - \frac{1.2.5}{3.3.3}\frac{x^3}{3!a^{\frac{8}{3}}} - \dots$$

Here $f(x)$ is continuous for all values of x, but its differential coefficients are not continuous between $x=0$ and $x>a$. The ordinary demonstrations, therefore, fail for all values of x greater than a, while this establishes the validity of the expansion for *all* values of x.

There is another advantage in the demonstration here given. All other methods of expansion are placed on a solid basis. In some works on the Differential Calculus it is simply *assumed* that

$$f(x) = A + Bx + Cx^2 + \dots \text{ to infinity,}$$

A, B, C, &c. being unknown constants. The same assumption is made in demonstrations of the Binomial Theorem. But this assumption, it is well known in the best mathematical circles, is unwarrantable; that is, although what is assumed may be true, we have no right to assume it. But in the demonstration here given it is *proved* that $f(x) = A+Bx+Cx^2+\dots$ to infinity, under the conditions specified. An example will be taken from Tod. Diff. Cal., p. 91.

"Expand $\tan^{-1}x$ in powers of x.

Assume $\tan^{-1}x = A_0 + A_1x + A_2x^2 + \dots A_nx^n + \dots +$ (1)

Defferentiate both sides with respect to x, then

$$\frac{1}{1+x^2} = A_1 + 2A_2x + \dots + nA_nx^{n-1} + \dots \qquad (2)$$

But $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots \qquad (3)$$

by simple division or by the binomial theorem.

Equating coefficients of like powers of x in (2) and (3) we have

$$A_1 = 1,\ A_2 = 0,\ A_3 = -\tfrac{1}{3},\ A_4 = 0, \dots$$

and putting $x=0$ in (1) we get $A_0 = 0$; therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\text{"}$$

The fault in this solution lies in the assumption in (1). Todhunter says concerning it and others of the same kind: "We do not lay much stress upon them as exact investigations, but they may serve as exercises in differentiation." But, by the demonstration here given, the assumption in (1) is *proved* under conditions. Therefore, with it as a basis, the investigation is exact, the expression being valid by condition 2° for all values of x less than unity.

NOTE ON DIRECTION.

BY T. M. BLAKSLEE, PH. D., PROF. OF MATH., DES MOINES UNIVERSITY.

1. THE *path* of a moving thing is the sum total of all its positions.

2. The moving thing is called the *generatrix.*

3. The first and last positions of the generatrix are called *termini* of the path.

4. AXIOM. Whatever holds true as we approach indefinitely near a limit may be conceived of as true at the limit.

5. For brevity any *position* of the generatrix is called a generatrix.

6. An element of space is called a *point of space*; its limit, or mere position, a Geometrical point.

(*a*). The path of a point is called a *line.*
(*b*). " " " " line " " a *surface.*
(*c*). " " " " surface is " a *solid.*

7. The property of the motion of a generatrix which determines, and is is itself determined by, its next consecutive position, is called the direction of the path at that generatrix.

8. A *Straight path* is one which has constantly the same direction, i. e., one in which any other position of the generatrix can be taken instead of the next consecutive, in determining its direction; .·. a St. P. is determined by any two of its genertarices. (Abbreviations, St. P., straight path, G., generatrix.)

9. A curved path is one that changes its direction at every generatrix.

10. (*a*), Any path determined by two of its generatrices is a St. P.

(*b*). The path of least extent is determined by its termini, .·. it is a St. P. Special case. The shortest distance between two points is measured on a straight line.

11. Two St. Ps. can intersect in but one G., for if they did in two they would coincide throughout.

12. If one of two intersecting St. Ps. and their common G. be given, the other is determined, (*a*), by a G. in the St. P. to be determined; (*b*), by the *opening* between the paths. For we may suppose the two paths to have coincided at first, and then one of them to have been turned about their common G. (as a pivot or axis, as the case may be) until it contains the second G., and since the second G. determines the second path, it determines the opening, and conversely, the opening determines the path.

13. The opening between any path and a fixed path can be taken as the direction of that path.

14. Paths which have the same direction with respect to a *fixed* or initial path are called parallel paths.

15. Parallel paths can never intersect, for if they did they must form equal openings with a third path through the common G., and hence must coincide.

16. An opening can be measured by revolving a unit opening about an axis; $\therefore$ it is proper to speak of the sum or difference of openings.

17. If the *initial* path pass through the intersection of two paths, the opening, called the *angle*, between them equals the difference of their directions. The two paths are called termini of the angle.

18. If two St. Ps. intersect, any two angles on opposite sides of one path and on the same side of the other are called adjacent angles, on opposite sides of both, opposite or vertical angles.

19. Equal adjacent angles are called *right angles*.

A right angle will be indicated by r.

20. If (1) and (2) are adjacent angles, then $(1) + (2) = 2r$.

Proof. Revolve the common terminus of the ang's through an angle A,

$$\begin{aligned} &(1) \text{ becomes } (1) \pm A = (1)', \\ &(2) \quad `` \quad\;\; (2) \mp A = (2)'; \\ &\therefore (1) + (2) = (1)' + (2)' = \text{constant.} \end{aligned}$$

If $(1)' = (2)'$, by def. $(1)' + (2)' = 2r$; $\therefore (1) + (2) = 2r$; Q. E. D.

Cor. The sum of the angles about a G., on one side of a St. P., equals $2r$; on both sides, their sum equals $4r$.

21. The angle between two paths b and a, is indicated thus $\frac{b}{a}$; the angle being positive when generated by motion from right to left.*

22. To show that this definition of an angle (17) is general, let a, b and c be three paths intersecting in three Gs.; then is

$$\frac{a}{b} = \frac{a}{c} - \frac{b}{c}. \quad \text{Now } \frac{b}{c} = -\frac{c}{b}; \therefore \frac{a}{b} = \frac{a}{c} + \frac{c}{b}.$$

Proof. Pass a St. P. β, through the intersection of a and c parallel to b.

By def. of parallel paths $\frac{a}{\beta} = \frac{a}{b}$ and $\frac{c}{\beta} = \frac{c}{b}$; but since a, β and c pass through a common G.,

$$\frac{a}{\beta} = \frac{a}{c} + \frac{c}{\beta}; \therefore \frac{a}{b} = \frac{a}{c} + \frac{c}{b} = \frac{a}{c} - \frac{b}{c}; \text{ Q. E. D.}$$

*If a and b be the directions of the corresponding paths, by (17)

$$\frac{b}{a} = b - a \text{ and } \frac{a}{b} = a - b = -(b - a) = -\frac{b}{a}.$$

ON THE COMPUTATION OF THE ECCENTRIC ANOMALY FROM THE MEAN ANOMALY OF A PLANET.

BY DR. J. MORRISON, NAUTICAL ALMANAC OFFICE, WASH., D. C.

THE relation between the eccentric and mean anomalies is given by the equation

$$m = E - e \sin E, \tag{1}$$

in which m is the mean anomaly, E the eccentric anomaly and e the eccentricity of the orbit.

In computing a series of values of E, it is not necessary to go beyond 180°, for any mean anomaly 360°—m corresponds to the eccentric anomaly 360°—E. Since E is greater than m, let us put

$$E = m + x,$$

where x is generally a small angle depending of course on the value of e.

In the case of the orbit of Mercury, whose eccentricity is .2056, the maximum value of x is less than 12°. Substituting in (1) we have

$$m = m + x - e \sin(m + x)$$

or

$$\begin{aligned} x &= e \sin(m + x) \\ &= e \sin m \cos x + e \cos m \sin x \\ &= e \sin m \left(1 - \frac{x^2}{1.2} + \ldots\right) + e \cos m \left(x - \frac{x^3}{1.2.3} + \ldots\right) \end{aligned}$$

Neglecting, for the first approximation, the third and all higher powers of x, we have

$$x^2 + \frac{2}{e}\left(\frac{1 - e\cos m}{\sin m}\right) x - 2 = 0, \tag{2}$$

whence

$$x = \sqrt{2} . \tan \tfrac{1}{2}\theta, \tag{3}$$

where

$$\tan\theta = e\sqrt{2} . \frac{\sin m}{1 - e \cos m}, \tag{4}$$

in which $\sqrt{2}$ is to be taken with the positive sign and θ always *less* than 90°.

The computation of the second member of the last equation will be much facilitated by the use of Zech's addition and subtraction Tables. Hence we have $E = m + x$ which will never differ more than $\pm 15''$ from the correct value of E, even when the eccentricity is as large as that of the orbit of Mercury. Let us represent this first approximate value of E, viz., $m + x$, by E_1, that is, let

$$E_1 = m + x. \tag{5}$$

Substituting E_1 for E in (1) we have

$$m_1 = E_1 - e \sin E_1, \qquad (6)$$

which subtracted from (1) gives

$$\begin{aligned} m - m_1 &= E - E_1 - e(\sin E - \sin E_1) \\ &= E - E_1 - 2e \cos \tfrac{1}{2}(E + E_1) \sin \tfrac{1}{2}(E - E_1) \\ &= E - E_1 - e(E - E_1) \cos \tfrac{1}{2}(E + E_1) \\ &= (E - E_1)(1 - e \cos E_1) \text{ very nearly;} \end{aligned}$$

whence we have

$$E - E_1 = \frac{m - m_1}{1 - e \cos E_1}, \qquad (7)$$

and $E = E_1 + \dfrac{m - m_1}{1 - e \cos E_1}$, for the second approximaon. Therefore we have finally

$$E = m + x + \frac{m - m_1}{1 - e \cos E_1}. \qquad (8)$$

If we require a third approximation, we may repeat the last operation with the corrected value of E as given by (8), but it will seldom or never be necessary as the last equation will generally give E within $0''.01$.

The second approximation given by (7) is substantially the same as that given by Gauss in his Theoria Motus, Art. 11, but the method here employed for obtaining it is much easier and more direct.

The second correction (7) is easily computed by the aid of Zech's Tables before referred to.

We will now test our formulæ by the following example:

Given $m = 143°$ and $e = .2056$, find E.

$$\begin{aligned} \log \sqrt{2} &= 0.1505150 \\ \log e &= \bar{1}.3130231 & &= \bar{1}.3130231 \\ \sin m &= 9.7794630 & \cos m &= \underline{9.9023486\,n} \\ \text{Co. log}(1 - e\cos m) &= \underline{9.9340098} \text{ (Zech)} & & \ 9.2153717\,n \\ \tan \theta &= 9.1770109 & \theta &= 8°\ 32'\ 54''.8 \\ \tan \tfrac{1}{2}\theta &= 8.8735468 \\ \sqrt{2} &= 0.1505150 \\ \text{cosec } 1'' &= \underline{5.3144251} \\ \log x'' &= 4.3384869 & x &= 6°\ 3'21''.525 \\ & & E_1 &= 149°\ 3'21''.525 \\ & & e \sin E_1 &= 6°\ 3'26''.23 \\ & & m_1 &= 142°59'55''.295 \\ & & m - m_1 &= 4''.705. \end{aligned}$$

$$\log(m - m_1) = 0.6725596$$
$$\log(1 - e\cos E_1) = \underline{0.0705317} \text{ (Zech).}$$
$$\log 3''.9997 = 0.6020279.$$
$$\therefore E = m + x + 3''.9997$$
$$= 143^\circ + 6^\circ\ 3'21''.525 + 3''.9997$$
$$= 149^\circ\ 3'\ 25''.5247.$$

Check.
$$e\sin E = 6^\circ\ 3'\ 25''.52$$
$$\therefore m = E - e\sin E$$
$$= 143^\circ\ 0'\ 0''.0047.$$

If the second correction be taken at 4″, which it is very nearly, E will be found to be 149°3′25″.52 exactly.

In computing a series of values of E, the labor may be lessened a little by preparing the constant logarithms, viz., $\log e\sqrt{2}$, $\log \sqrt{2} \operatorname{cosec} 1''$ and $\log e \operatorname{cosec} 1''$.

PLANE TRIGONOMETRY BY QUATERNIONS.

BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

A QUATERNION may be expressed under a variety of forms. Thus if a and b are the tensors respectively of the unit vectors α and β we have (see ANALYST, Vol. VII, pp. 124 and 127)

$$q = \frac{b}{a} i^{t} \tag{1}$$
$$= Tq.Uq \tag{2}$$
$$= Sq + Vq \tag{3}$$
$$= Tq\,(SUq + VUq). \tag{4}$$

Each of these forms has special advantages for the solution of certain problems, but in this article we will make use of the third one.

Two quaternions are equal when the elements of one equal respectively those of the other. Thus, if

$$q = q',$$

we have

$$Sq + Vq = Sq' + Vq', \tag{5}$$

and hence from the definition,

$$Sq = Sq', \text{ and } Vq = Vq'. \tag{6}$$

The principle in algebra corresponding to this is that where an equation is composed partly of real and partly of imaginary terms. Such an equation

is equivalent to two others—thus we have

$$ax + \sqrt{(-b)}\,.\,y - cy = \sqrt{(-d)}\,.\,x + e,$$

then

$$ax - cy = e,$$

$$\sqrt{(-b)}\,.\,y = \sqrt{(-d)}\,.\,x.$$

An application will now be made of equations (5) and (6) to the solution of a plane triangle.

Let A, B, C be the angles of the triangle, α, β, γ, the respective vectors opposite, and a, b, c, the corresponding tensors. Let i be an axis perpendicular to the plane of the triangle and positive in front of the plane. Let vector $\beta = CA$, $\gamma = AB$, $\alpha = CB$, then will $-\beta = AC$, $-\gamma = BA$, and $-\alpha = BC$. Left handed rotation being positive, the angle between β and γ will be $\pi - A$, between β and α it will be C, and between α and γ, B, or between γ and α it will be $-B$ or $360^\circ - B$.

The plane triangle gives the vector equation

$$\alpha = \beta + \gamma. \tag{7}$$

This equation may be operated upon in any manner that will produce scalar results.

Multiplying both members by γ gives

$$\alpha\gamma = \beta\gamma + \gamma^2.$$

Substituting the value of the product of the vectors (see 3d of eq's A, p. 124, Vol. VII) and assuming that $\gamma^2 = -c^2$ (Art. 22, p. 70, Vol. VII), we have

$$-ac\cos B + ac\sin B.i = -bc\cos(180^\circ - A) + bc\sin(180^\circ - A) - c^2.$$

Hence by equations (6) taking the scalar and vector parts, we have

$$-ac\cos B = +bc\cos A - c^2,$$

and

$$ac\sin Bi = bc\sin Ai.$$

From these we readily deduce

$$a\cos B + b\cos A = c,$$

$$a\sin B = b\sin A,$$

both of which are well known trigonometrical results.

Again, dividing both members of (7) by β gives

$$\frac{\alpha}{\beta} = 1 + \frac{\gamma}{\beta}.$$

Taking the scalar and vector parts, as indicated by equations (6) gives

$$S\frac{\alpha}{\beta} = 1 + S\frac{\gamma}{\beta},$$

$$V\frac{\alpha}{\beta} = V\frac{\gamma}{\beta};$$

and substituting the values of the scalar and vector parts (see eq'ns (36) and (43), p. 126 Vol. VII) gives

$$\frac{a}{b}\cos C = 1 + \frac{c}{b}\cos(180^\circ - A),$$

$$\frac{a}{b}\sin C.i = \frac{c}{b}\sin(180^\circ - A).i.$$

These reduced give

$$a\cos C + c\cos A = b,$$
$$a\sin C = c\sin A;$$

which are the same relations as those found above.

Again, squaring (7) gives

$$\alpha^2 = \beta^2 + \beta\gamma + \gamma\beta + \gamma^2,$$

and taking the scalar parts we have

$$-a^2 = -b^2 + S\beta\gamma + S\gamma\beta - c^2.$$

But
$$S\beta\gamma = -bc\cos(180^\circ - A),$$
$$S\gamma\beta = -cb\cos(180^\circ - A),$$

which substituted gives

$$a^2 = b^2 + c^2 - 2bc\cos A, \text{ a well known result.}$$

Taking the vectors gives

$$0 = V\beta\gamma + V\gamma\beta,$$

or
$$0 = bc\sin(180^\circ - B).i + cb\sin(180^\circ - B)(-i),$$

which is simply a true equation and make known no new property.

Again, from (7) we have by transformation, $\alpha - \beta = \gamma$, and $\alpha - \gamma = \beta$. Multiplying the former by the latter, member by member, gives

$$\alpha^2 - \alpha\beta - \gamma\alpha + \gamma\beta = \beta\gamma;$$

and taking the vectors gives

$$b(a\sin C + c\sin A) = c(b\sin A + a\sin B),$$

each member of which expresses twice the area of the triangle.

As a further exercise, transpose $\frac{1}{2}\beta$ and square, thus $\alpha - \frac{1}{2}\beta = \frac{1}{2}\beta + \gamma$.

Squaring, $\alpha^2 - \frac{1}{2}\alpha\beta - \frac{1}{2}\beta\alpha + \frac{1}{4}\beta^2 = \frac{1}{4}\beta^2 + \frac{1}{2}\beta\gamma + \frac{1}{2}\gamma\beta + \gamma^2$,

and taking the scalar parts gives

$$-a^2 + ab\cos C = bc\cos A - c^2,$$

or
$$a^2 = c^2 + ab\cos C - bc\cos A,$$

a form which is not common, but which may readily be reduced to

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

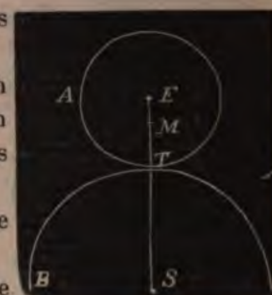
as given above, so that no essentially new relation is thus determined. The essential relations are determined by simply multiplying or dividing equation (7) by one of the vectors of the equation, or by squaring it.

REPESENTATION OF THE MOON'S PATH ALONG THE LINE OF THE EARTH'S ORBIT.

BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLAND.

LET E represent the earth, S the sun, and ES the distance between them. On ES lay off ET so that ET : TS :: 1 : 13. Then is

$$ET : ES :: 1 : 14. \qquad (1)$$



With ET and TS as radii describe the circles TA and TB tangent to each other at T.

If we assume the moon's distance from the earth to be 240,000 miles and the earth's distance from the sun say 90,000,000 miles, these distances are as 1 to 375.

Let EM represent the moon's distance from the earth, then EM : ES :: 1 : 375,
and ET : ES :: 1 : 14, by (1); hence
EM : ET :: 14 : 375 :: 1 : 27, nearly.

Since circumferences are to each other as their radii, the circle AT will make 13 revolutions in rolling around the circle BT, and the point M will make 13 revolutions around the centre E during the same time.

If two such wheels are made and pointed pencils are fixd at E and M, by rolling AT toward the left around BT the pencil at M will describe the moon's path along the line of the earth's orbit as traced by the pencil at E.

A simpler plan is to cut from pasteboard any circle AT, and from centre E take EM $= \frac{1}{27}$ of the radius, insert pencils at E and M, and roll the circle along the inner edge of a black-board, and the relative path of the moon along the line of the earth's orbit will be traced.

PROB. 357 (p. 135, Vol. VIII).—"An elastic string without weight and of given length, has one end fixed in a perfectly smooth horizontal plane, and the other to a point in the surface of a sphere, the string being unwound. The sphere is projected on the plane from the fixed point with a linear velocity v and an angular velocity ω, winding the string on the circumference of a great circle; required the elongation of the string when fully stretched, and the subsequent motion of the sphere."

SOLUTION BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

Let r = the radius of the sphere, a = the original length of the string, ω = the initial angular velocity of the body, v = the initial velocity of the centre of the body, and t_1 = the time of winding the slack. Then

$$vt_1 + r\omega t_1 = a;$$

$$\therefore t_1 = \frac{a}{v + r\omega},$$

and the initial stretched part will be

$$vt_1 = \frac{va}{v + r\omega} = l \text{ (say)}.$$

Immediately following this time the string will be stretched, and the tension at first diminishes both the linear and angular velocities. Take the origin at the remote end of l for the variable motion. Let m = the mass of the body, s = the space passed over by the centre during time t, θ = the angular distance passed by the initial radius in the same time, k = the radius of gyration of the body, e = the coefficient of elasticity of the string, A = cross section of string, and λ = the elongation produced by the tension T of the string. Then Mariotte's law gives

$$T = \frac{eA\lambda}{l - r\theta}. \tag{1}$$

Assume that l is so long compared with $r\theta$ that the latter may be neglected, and let $B = eA \div l$, then

$$T = B\lambda.$$

The conditions of the problem give

$$d\lambda = ds + rd\theta; \tag{2}$$

$$\therefore \quad d^2\lambda = d^2s + rd^2\theta.$$

Also, for motion of the centre,

$$m\frac{d^2s}{dt^2} = -T = -B\lambda, \tag{3}$$

and for the rotary motion,

$$mk^2\frac{d^2\theta}{dt^2} = -Tr = -Br\lambda, \tag{4}$$

which two equations in the preceding give

$$\frac{d^2\lambda}{dt^2} = -\frac{B}{mk^2}\left(k^2 + r^2\right)\lambda = -D^2\lambda.$$

Integrating, observing that for $\lambda = 0$, $t = 0$, and $d\lambda \div dt = v + \omega$, we have

$$\lambda = \frac{v+\omega}{D}\sin Dt. \tag{5}$$

The elongation λ, will be a max. for sin $Dt = 1$, or $t = \pi \div 2D$, for which

$$\lambda = \frac{v+\omega}{D} = \frac{k\sqrt{(ml)}}{\sqrt{[eA(k^2+r^2)]}}(v+\omega).$$

The *time* of producing the maximum stretch of the string is independent of the initial motions. When the string returns to its original length, λ will again be zero, and sin $Dt = 0$, or $Dt = \pi$; $\therefore t = \pi \div D$.

All the circumstances of the variable motion may be determined by integrating equations (3) and (4). Integrating, after substituting from equation (5), observing that for $t = 0$, $ds \div dt = v$, $s = 0$, $d\theta \div dt = \omega$ and $\theta = 0$, we have, if we put F for $eA(v+\omega) \div mlD^2$,

$$\frac{ds}{dt} = F[\cos Dt - 1] + v, \tag{6}$$

$$s = \frac{F}{D}\Big[\sin Dt - Dt\Big] + vt, \tag{7}$$

$$\frac{d\theta}{dt} = F\frac{r}{k^2}\Big[\cos Dt - 1\Big] + \omega, \tag{8}$$

$$\theta = \frac{F}{D}\frac{r}{k^2}\Big[\sin Dt - Dt\Big] + \omega t. \tag{9}$$

For the maximum of (5), $d\lambda \div dt = 0$, which in (2) gives

$$\frac{ds}{dt} = -r\frac{d\theta}{dt},$$

which combined with (6) and (9) gives $\cos Dt - 1 = -1$; $\therefore Dt = \frac{1}{2}\pi$ as before found, and serves as a check upon the work. The relation $ds = -rd\theta$ shows that the direction of one of the motions changes signs. At the point where the linear motion is reversed $ds \div dt = 0$, and for this we have

$$t_2 = \frac{1}{D}\cos^{-1}\Big(1 - \frac{v}{F}\Big);$$

and if the direction of the rotation is reversed, $d\theta \div dt = 0$, and (8) gives

$$t_3 = \frac{1}{D}\cos^{-1}\Big(1 - \frac{\omega k^2}{Fr}\Big);$$

from which it appears that if $v < (k^2\omega \div r)$ the motion of the centre will be reversed, but otherwise the angular motion will be reversed. The value of t_2 in the former case will be less than $\pi \div 2D$. Both motions will change at the instant of greatest elongation if $rv = k^2\omega$.

If the values of t_2 and t_3 are both less than $\pi \div D$, one motion will change signs before the instant of greatest elongation and the other after; otherwise only one will change signs. To find the total variable movement, make $Dt = \pi$, and (7) and (9) give

$$s = \left(v - F\right)\frac{\pi}{D},$$

$$\theta = \left(\omega - F\frac{r}{k^2}\right)\frac{\pi}{D}.$$

If (6) reduces to zero when $Dt = \pi$, the body would be at rest at the moment the string regains its original length and $F = \frac{1}{2}v$, but it would still have an angular velocity of $\omega + (rv \div k^2)$ as shown by (8). Similarly, if the rotary motion is destroyed at that instant, the linear velocity will be $v + (k^2\omega \div r)$, and will continue uniform. It may be shown that the kinetic energy of the moving body at the end of the variable motion is the same as at the beginning.

I have not attempted to solve the general case represented by equation (1). It is evidently very intricate.

Note on Prof. Casey's Treatment of Prob. 361, by Prof. E. W. Hyde. — Referring to Prof. Casey's figure on page 194, Vol. VIII, the curves *nxm* and *pyw* are *not* semi-ellipses. They are tortuous curves whose projections on the plane ABC are arcs of an *hyperbola*, those on a plane perpendicular to BM equal curves, forming together a curve of the 4th order, and those on a plane perpendicular to these two planes, the circle which is the right section of the cylinder.

The differential expression for the volume is easily written but can be integrated only by expansion into series. Thus the equation of the cone is $x^2 + y^2 = (R^2 \div a^2)(z - a)^2$, and that of the cylinder, $y^2 + z^2 = r^2$; whence

$$V = 8R\int_0^r\int_0^{\sqrt{(r^2-z^2)}} dz\,dy \sqrt{\left[\left(\frac{z-a}{a}\right)^2 - \left(\frac{y}{R}\right)^2\right]} = 4R\int_0^r dz\left\{\sqrt{(r^2-z^2)}\times\right.$$

$$\left.\sqrt{\left[\left(\frac{z-a}{a}\right)^2 - \frac{r^2-z^2}{R^2}\right]} + \left(\frac{z-r}{a}\right)^2 R\sin^{-1}\frac{a}{R}\frac{(r^2-z^2)^{\frac{1}{2}}}{z-a}\right\}$$

[Mr. Eastwood puts for the equation of the cone and cylinder, respectively, $y^2 + x^2 = z^2\tan^2\beta$, and $(z - c)^2 + x^2 = r^2$, and gets

$$V = 4\int\sqrt{(r^2-x^2)}\left\{[\sqrt{(r^2-x^2)}+c]^2\tan^2\beta - x^2\right\}^{\frac{1}{2}}dx.$$

Mr. Heaton puts $x =$ the distance of a horizontal plane through the cylinder from the axis of the cylinder, and finds, for the area of such plane,

$$A = 2\sqrt{\{(r^2-x^2)[(R^2 \div a^2)(b+x)^2 - r^2 + x^2]\}} + 2(R^2 \div a^2)(b+x)^2 \times$$

$$\sin^{-1}\sqrt{\{(r^2-x^2) \div [(R^2 \div a^2)(b+x)^2]\}};\ \text{and therefore}$$

$$V = 2\int_{-r}^{+r}\left\{\left[(r^2-x^2)\left(\frac{R^2}{a^2}(b+x)^2 - r^2 + x^2\right)\right]^{\frac{1}{2}} + \frac{R^2}{a^2}(b+x)^2\times\right.$$

$$\left.\sin^{-1}\left(\frac{a\sqrt{(r^2-x^2)}}{R(b+x)}\right)\right\}dx.$$

Note on the Solution of Prob. 363, by Prof. W. W. Johnson.—The following proof of Prob. 363 shows that the theorem is true for all convex ovals.

Let O be the fixed point and AB the chord, the tangent at B being parallel to OA; and complete the parallelogram $AOBT$. As B travels about the oval the tangent BT is at every instant rotating about B; Hence, since BT is equal and parallel to OA it generates an area equal to that generated by OA, that is, an area equal to the given oval. Thus the area of the locus of T is double that of the oval. Now the middle point of AB is also the middle point of OT, hence its locus is similar to that of T, and its area is one fourth the area of the locus of T or one half the area of the given oval.

Another Solution of Prob. 365, by Prof. Asaph Hall.—It is plain that the indetermination will occur only for very small values of θ.

Put therefore $\cos\theta = 1 - \frac{1}{2}\theta^2$, and, neglecting higher powers of θ, we shall have,

$$\int_0^\theta \frac{\sqrt{(1-c)}.d\theta}{1-c+\frac{1}{2}nc.\theta^2} = \sqrt{\frac{2}{nc}}\cdot\tan^{-1}\frac{\theta.\sqrt{(nc)}}{\sqrt{[2(1-c)]}};$$
$$= \sqrt{\frac{2}{n}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{(2n)}},$$

when $c = 1$. This is the value required, since the upper limit may be changed from θ to $\frac{1}{2}\pi$.

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. VIII.

Solutions of problems in No. 6, Vol. VIII, have been rec'd as follows: From R. J. Adcock, 374; Prof. W. P. Casey, 368, 370, 371; George E. Curtis, 370; Dr. H. Eggers, 368, 370; Prof. A. B. Evans, 371; Prof. E. J. Edmunds, 368, 369, 370; George Eastwood, 372; Prof. E. W. Hyde, 369; W. E. Heal, 369; William Hoover, 372; Prof. J. Scheffer, 368, 369, 370, 372; Prof. E. B. Seitz, 369, 370, 371; Thos. Spencer, 369; R. S. Woodward, 374.

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368. *By Prof. J. Scheffer.*—"In a quadrilateral $ABCD$, the diagonal AC makes with the sides the four angles $CAB = \alpha$, $ACB = \beta$, $ACD = \gamma$, $CAD = \delta$. Find the angles which the other diagonal BD makes with the sides."

SOLUTION BY PROF. J. SCHEFFER, HARRISBURG, PA.

Denoting $\angle BDC$ by θ, and $\angle BDA$ by φ, we have

$$BC : CD = \sin\theta : \sin(\beta+\gamma+\theta),$$
$$CD : AC = \sin\delta : \sin(\delta+\gamma).$$

Multiplying:

$$BC : AC = \sin\delta\sin\theta : \sin(\delta+\gamma)\sin(\beta+\gamma+\theta);$$

but $$BC : AC = \sin\alpha : \sin(\alpha+\beta);\text{ therefore}$$

$$\sin\alpha : \sin(\alpha+\beta) = \sin\delta\sin\theta : \sin(\delta+\gamma)\sin(\beta+\gamma+\theta),$$

whence $\sin\alpha\sin(\delta+\gamma)\sin(\beta+\gamma+\theta) = \sin(\alpha+\beta)\sin\delta\sin\theta$, or

$$\sin\alpha\sin(\delta+\gamma)[\sin(\beta+\gamma)\cos\theta+\cos(\beta+\gamma)\sin\theta]$$
$$= \sin(\alpha+\beta)\sin\delta\sin\theta.$$

Dividing by $\sin\theta$, we obtain

$$\cot\theta = \frac{\sin\delta\sin(\alpha+\beta)}{\sin\alpha\sin(\beta+\gamma)\sin(\delta+\gamma)} - \cot(\beta+\gamma).$$

Similarly

$$\cot\varphi = \frac{\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\alpha+\delta)\sin(\delta+\gamma)} - \cot(\alpha+\delta).$$

Substituting the values of θ and φ, as here found, in the equations

$$\angle ABD = \pi - (\alpha+\delta+\varphi),$$
$$\angle CBD = \pi - (\beta+\gamma+\theta),$$

these angles also become known.

[Dr. Eggers' solution of this problem is similar to the foregoing. Prof. Casey assumes that the sides of the quadrilateral are given, and hence concludes that the solution involves only the application of a well known case in trigonometry. Prof. Edmunds obtains, from the figure, four equations involving the four unknown angles with known quantities, by the reduction of which, he assumes, the solution may be effected. His method, however, will not succeed because his equations are not independent.—Ed.]

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369. *By R. J. Adcock.*—"Show that the radius of curvature of an ellipse equals the cube of the radius vector divided by the rectangle of the semi axes; the radius vector being through the centre at right angles to the radius of curvature."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MISSOURI.

Let R be the radius of curvature at the point $(a\cos\varphi, b\sin\varphi)$, and r the radius vector from the center perpendicular to R. Then by the usual formula we have $R = \sqrt{(a^2\sin^2\varphi+b^2\cos^2\varphi)^3}\div ab$ (1). Since r is parallel to the tangent at $(a\cos\varphi, b\sin\varphi)$, and the radius vector of this point is $\sqrt{(a^2\cos^2\varphi+b^2\sin^2\varphi)}$, we have $r=\sqrt{(a^2\sin^2\varphi+b^2\cos^2\varphi)}$, (2). From (1) & (2) $R=r^3\div ab$.

370. *By Prof. Edmunds.*—"Divide a right angle into three parts α, β, γ, such that $(\cos\alpha)\div m = (\cos\beta)\div n = (\cos\gamma)\div p$."

SOLUTION BY DR. H. EGGERS, MILWAUKEE, WISCONSIN.

Construct a triangle with m, n, p as sides; then the complements α, β, γ of its three angles are the angles required.

For if a, b, c be the angles of the triangle, we have

$$m : n : p = \sin a : \sin b : \sin c,$$
$$= \cos(\tfrac{1}{2}\pi - a) : \cos(\tfrac{1}{2}\pi - b) : \cos(\tfrac{1}{2}\pi - c).$$

Now $\alpha + \beta + \gamma = (\tfrac{1}{2}\pi - a) + (\tfrac{1}{2}\pi - b) + (\tfrac{1}{2}\pi - c)$
$$= \tfrac{3}{2}\pi - (a + b + c)$$
$$= \tfrac{3}{2}\pi - \pi = \tfrac{1}{2}\pi.$$

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371. *By Prof. E. B. Seitz.*—"ACB is the quadrant of a circle, O the center of its inscribed circle; O_1, O_2, O_3, ... O_n are the centers of a series of circles, each of which touches the preceding, the arc AB and the radius AC, the circle O_1 touches the circle O; and OH, O_1H_1, O_2H_2, ... O_nH_n are the perpendiculars on AC, or the radii of the inscribed circles. If $AC = r$, $O_nH_n = x_n$, and $CH_n : O_nH_n = u_n$, prove that

$$u_n = \tfrac{1}{2}(\sqrt{2} + 1)^{2n+1} - \tfrac{1}{2}(\sqrt{2} - 1)^{2n+1},$$
$$x_n = \frac{2r}{2 + (\sqrt{2} + 1)^{2n+1} + (\sqrt{2} - 1)^{2n+1}}."$$

SOLUTION BY PROF A. B. EVENS, LOCKPORT, N. Y.

Let $CH_n = y_n$; then from the geometry of the figure

$$4x_n y_{n-1} = (y_n - y_{n-1})^2, \quad (1)$$
$$x_n = \frac{1}{2r}(r^2 - y_n^2). \quad (2)$$

Similarly
$$x_{n-1} = \frac{1}{2r}(r^2 - y_{n-1}^2). \quad (3)$$
$$\therefore 4x_n x_{n-1} = \frac{1}{r^2}(r^2 - y_n^2)(r^2 - y_{n-1}^2). \quad (4)$$

From (1) and (4), by elimination and evolution,

$$r^2 - y_n y_{n-1} = (y_n - y_{n-1})r\sqrt{2}. \quad (5)$$

From (5), by solving for y_n and then for y_{n-1}, we find

$$y_n = r(r + y_{n-1}\sqrt{2}) \div (r\sqrt{2} + y_{n-1}), \quad (6)$$

and
$$y_{n-1} = r(y_n\sqrt{2} - r) \div (r\sqrt{2} - y_n); \quad (7)$$

$\therefore$
$$y_{n-2} = r(y_{n-1}\sqrt{2} - r) \div (r\sqrt{2} - y_{n-1}). \quad (8)$$

By eliminating y_n between (2) and (6) we find

$$x_n = \tfrac{1}{2}r(r^2 - y_{n-1}^2) \div (r\sqrt{2} + y_{n-1})^2, \tag{9}$$

and by writing n—2 for n in (2) and then eliminating y_{n-2} by aid of (8),

$$x_{n-2} = \tfrac{1}{2}r(r^2 - y_{n-1}^2) \div (r\sqrt{2} - y_{n-1})^2. \tag{10}$$

By aid of (6), (8), (9), and (10) we may write

$$u_n = \frac{y_n}{x_n} = \frac{2}{r^2 - y_{n-1}^2}\left\{3ry_{n-1} + (r^2 + y_{n-1}^2)\sqrt{2}\right\}$$

and $$u_{n-2} = \frac{y_{n-2}}{x_{n-2}} = \frac{2}{r^2 - y_{n-1}^2}\left\{3ry_{n-1} - (r^2 + y_{n-1}^2)\sqrt{2}\right\};$$

whence $$u_n + u_{n-2} = \frac{12ry_{n-1}}{r^2 - y_{n-1}^2} = 6u_{n-1}.$$

$$\therefore u_n - 6u_{n-1} + u_{n-2} = 0. \tag{11}$$

The solution of (11) by Finite Differences gives

$$u_n = C_1(r_1)^n + C_2(r_2)^n; \tag{12}$$

where $r_1 = (\sqrt{2} + 1)^2$ and $r_2 = (\sqrt{2} - 1)^2$ are the roots of the equation $a^2 - 6a + 1 = 0$, and C_1 and C_2 are constants of integration.

To determine these constants, observe that when $n = 0$ and $n = 1$, $u = 1$ and $u_1 = 7$, and therefore $1 = C_1 + C_2$ and $7 = C_1(\sqrt{2}+1)^2 + C_2(\sqrt{2}-1)^2$; whence $C_1 = \tfrac{1}{2}(\sqrt{2} + 1)$ and $C_2 = \tfrac{1}{2}(\sqrt{2} - 1)$. These values of C_1 and C_2 reduce (12) to $u_n = \tfrac{1}{2}(\sqrt{2}+1)^{2n+1} - \tfrac{1}{2}(\sqrt{2}-1)^{2n+1}$. (13)

Since $y_n \div x_n = \sqrt{(r^2 - 2rx_n)} \div x_n = u_n$, we readily find from (13),

$$x_n = \frac{2r}{2 + (\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}}.$$

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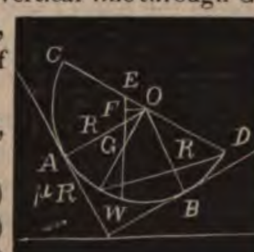
372. *By William Hoover, A. M.*—"A hemisphere, radius r, is resting with its convex surface on two planes, one perfectly smooth and inclined to the horizon at an angle α, the other being inclined at an angle β; if m be the coefficient of friction between the latter and the hemisphere, what is the position for rest?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let θ represent the angle the base makes with the horizon when the hemisphere is at rest; G, its centre of grav. and EGW a vertical line through G.



We have for data, $\theta = EOF = EGO$; R, μR, and R' for forces, and α and β for inclinations of planes; also $OG = \tfrac{3}{8}r$.

Take vertical and horizontal components of R, μR, R'; and moments about centre O. Then

$$R\cos\alpha + \mu R\sin\alpha + R'\cos\beta = W, \tag{1}$$

$$R\sin\alpha + \mu R\cos\alpha - R'\sin\beta = 0, \tag{2}$$

$$r.\mu R = FO.W = \tfrac{3}{8}r.W\sin\theta. \tag{3}$$

Multiply (1) by $\sin\beta$, and (2) by $\cos\beta$, and add: then

$$R\sin(\alpha+\beta)+\mu R\cos(\alpha-\beta) = W\sin\beta.$$

$$\therefore R = \frac{W\sin\beta}{\sin(\alpha+\beta)+\mu\cos(\alpha-\beta)}. \qquad (4)$$

Substituting this value of R in (3), we have

$$\sin\theta = \frac{8\mu\sin\beta}{3\sin(\alpha+\beta)+3\mu\cos(\alpha-\beta)}.$$

[Solved in a similar manner by the proposer, and by Prof. Scheffer.]

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373. No solution received.

374. *By R. S. Woodward.*—"Prove 1st, that the probable error of any tabular value in a table of logarithms, trigonometric functions, etc., is 0.25 of a unit of the last decimal place, supposing this place correct to the nearest unit; 2nd, that the average of the squares of probable errors of interpolated values depending on first differences only is $\frac{2}{3}(0.25)^2$."

SOLUTION BY THE PROPOSER.

The actual errors of tabular values are confined within the limit $+0.5$ and -0.5 of a unit of the last place. All errors between these limits are equally probable. Hence the probable error of any tabular value is one-half the maximum error, or ± 0.25.

Let v and v' be two consecutive tabular values, and x an interpolated value $\frac{1}{10}t$ from v. Then

$$x = v + \tfrac{1}{10}t(v'-v) = v(1-\tfrac{1}{10}t) + v'\tfrac{1}{10}t.$$

The square of the probable error of x is

$$(\text{p. e. } x)^2 = (0.25)^2\left\{\left(1-\frac{t}{10}\right)^2 + \frac{t^2}{100}\right\}$$

$$= (0.25)^2\left\{1 - 2\frac{t}{10} + 2\frac{t^2}{100}\right\}. \qquad (1)$$

The average of the squares of probable errors given by this formula between the limits $t=0$ and $t=10$ is

$$(0.25)^2\int_0^{10}\left(1-2\frac{t}{10}+2\frac{t^2}{100}\right)dt \div \int_0^{10}dt = \tfrac{2}{3}(0.25)^2.$$

From (1) it appears that the probable error of an interpolated value is always less than that of a tabular value, and that the probable error is least for the interpolated values midway between the two tabulated values.

[R. J. Adcock submits the following remarks on the solution of 374:]

"At p. 189, Vol. VII, is found the probable error $x = \surd[S(d_1^2) \div n] \times \tan\frac{1}{2}\tan^{-1} cl = .707\surd(S(d_1^2) \div n$, where $S(d_1^2)$ is the sum of the squares of the errors, and , their number.

In the first case of 374, all possible errors, without regard to sign, are included between 0 and 0.5 of the last decimal place, their number is infinite, there is no greater density or accumulation of errors of one value between these limits than of another; therefore

$$\frac{S(d_1^2)}{n} = \int_0^{0.5} y^2 dy \div y = \tfrac{1}{3}y^2 + C = \tfrac{1}{12}.$$

Hence the probable error $x = \frac{1}{12}\surd 6 = 0.204$, instead of 0.25.

R. J. ADCOCK."

PROBLEMS.

375. *By W. B. Bates.*—*A* and *B* enter into partnership and gain \$200. Now six times *A*'s accumulated stock (capital and profit) equals five times *B*'s original stock, and six times *B*'s profit exceeds *A*'s original stock by \$200. Required the original stock of each.

376. *By Dr. H. Eggers.* — Divide a right angle into three parts, such that the tangents of the several angles are proportional to three given numbers.

377. *By W. E. Heal.*—If the equations,

$$x^2 + a\,x + b \;= 0$$
$$x^2 + a_1x + b_1 = 0,$$

have a common root, find the remaining roots.

378. *By Isaac H. Turrell.* – *O* is the center of a circle circumscribing a triangle, and *a*, *b*, *c*, are the middle points of the sides opposite the angles *A*, *B*, *C*, respectively. If a circle be drawn through *A* to touch *Ob*, *Oc*, and another through *B* to touch *Oa*, *Oc*, prove that their common tangent passes through *C*.

379. *By Paul Peltier, A. M., Waterloo, Ill.*—If any number of circles touch one another in one point, all their polars which correspond to a common pole, pass through a single point.

380. *By Lieut. Chas. A. Stone, U. S. Naval Acad., Ann., Md.* — Find the equation of the curve in which the tangent of the angle which the tangent line makes with the axis of *X*, increases proportionally to the length of the curve.

381. *By Prof. W. P. Casey.* — To find a point in a given line so that the rectangle contained by two lines drawn to it from two given points may be given or a minimum (without the aid of the Cassinian Ovals).

382. *By Thomas Spencer.* — Prove in general that the chord drawn though a given point so as to cut off the minimum area from a given curve is bisected at that point.

383.—*By Prof. Edmunds* —Solve and discuss:

$$\begin{cases} x^2 + y^2 = a^2, \\ \log x + \log y = n. \end{cases}$$

384. *By Prof. Asaph Hall.*—"Show that

$$\int_0^a dx \int_0^x \varphi(x, y) . dy = \int_0^a dy \int_y^a \varphi(x, y) dx.$$

(Dirichlet's theorem.)"

385. *Selected by Prof. H. T. Eddy.*—Show that

$$\int_{-\infty}^{+\infty} \varepsilon^{(-\ {}^2\cos 2\theta + \frac{a^2}{2x^2}\sin 2\theta)} \substack{\cos \\ \sin} \left[x^2 \sin 2\theta + \frac{a^2}{2x^2} \cos 2\theta \right] dx$$

$$= \pi^{\frac{1}{2}} \varepsilon^{-a} \substack{\cos \\ \sin} \left[\theta + a \right].$$

386. *By George Eastwood.*—Integrate the equation

$$\frac{d^2\psi}{dt^2} \cdot \frac{d^2\phi}{dx^2} - \left(\frac{d\psi}{dt} \cdot \frac{d\phi}{dx} \right)^2 = 0.$$

PUBLICATIONS RECEIVED.

On Gauss's Method of Computing Secular Perturbations with an Application to the action of Venus on Mercury, by GEORGE W. HILL, Assistant American Ephemeris. 4to. 1881.

The Stophoids, by WILLIAM WOOLSEY JOHNSON. Reprinted from the American Journal of Mathematics, Vol. III.

Solution of a Geometrical Problem, by PROF. E. B. SEITZ. Reprinted from the Mathematical Visitor, Vol. II, No. 1.

ERRATA.

On page 5, line 7, for a, b, c and d under the radicals, read a^5, b^5, c^5, d^5.
" " 8, " 6 from bottom, for presumptious, read presumptuous.
" " 12, lines 6 and 22, for x', read x_1.
" " 16, " 16, for 2γ, read $2r$.
" " 20, line 18, for by, read into.
" " 21, " 11, from bottom, for twice, read four times.

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LAW OF ERROR IN THE POSITION OF A POINT IN SPACE.

BY E. L. DE FOREST, WATERTOWN, CONN.

THE law of facility of error in the observed position of a point varies according as the point is supposed to lie on a given straight line, or in a given plane, or in unrestricted space of three dimensions. Most quantities which are the subjects of observation are capable of having their magnitude represented by the length of a straight line, and their errors are represented by the errors in the position of the point which forms one extremity of the line, the other extremity being fixed. Taking this line as an axis of X, the probability y that the point of error will fall within any arbitrary but very small interval dx on this line, is a function of the form

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \tag{1}$$

where x is the distance from the most probable position of the point to the middle of the interval dx, and h is a constant. The most probable position of the point, and the value of h, are to be determined by known methods. If an observed point may be anywhere on a given plane, as in the case of shot-marks on a target, the probability that the point will fall within any very small area $dxdy$ whose middle point has the coordinates x and y, will be

$$z = \frac{h_1 h_2 dx\, dy}{\pi} e^{-(h_1^2x^2+h_2^2y^2)}. \tag{2}$$

The positions of the axes of X and Y, and the values of h_1 and h_2, are to be found as in my recent article, ANALYST, Jan. to May '81. In this and other articles (Sept. '81, p. 145), it was shown that the functions (1) and (2) are the limiting forms of the series of coefficients in the expansion, to an infinitely high power, of a polynomial of one and two variables respectively. On similar principles, the law of errors in space of three

dimensions may be obtained by the expansion of a polynomial of three variables, whose coefficients are all positive and their sum equal to unity. It is the object of the present paper to show this, and to demonstrate the limiting form of the expansion.

Taking at first the true position of the observed point as an origin, and drawing through it any three rectangular axes, we will refer to these axes any other positions which the point may take through errors of observation. The coordinates x, y, z of any such point of error will be regarded as multiples of the arbitrary intervals or units of measure Δx, Δy and Δz respectively, which may be taken as small as need be. If for example a single observation places the point at $x = 4\Delta x$, $y = 2\Delta y$, $z = 3\Delta z$, we will write $L_{4,2,3}$ for the probability that this particular error will occur, and in general, we write $L_{a,b,c}$ for the probability of occurrence of the error $x = a\Delta x$, $y = b\Delta y$, $z = c\Delta z$, where a, b, c are any whole numbers, positive or negative. Now take the probabilities L as coefficients, and their sub-indices as exponents of ξ, η, ζ, in the polynomial

$$u = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L_{a,b,c}\, \xi^a \eta^b \zeta^c). \tag{3}$$

Under this notation, the terms of the polynomial are supposed to be arranged in the form of a rectangular block, so that exponents and sub-indices vary from term to term by differences equal to unity. Each term is of the form included within the brackets, and the signs of summation denote that the whole polynomial is the sum of all the terms, and is formed by assigning to one of the exponents and sub-indices, a for instance, all the integer values from $-m$ to m, while b and c remain constant, then in this sum assigning integer values to another exponent, b for instance, from $-m$ to m, while c remains constant, and finally in this sum assigning integer values to c from $-m$ to m. In other words, the block-formed polynomial is the sum of all the different terms which can be formed by giving to a, b and c, independently of each other, all the integer values from $-m$ to m. We might have adopted a notation analogous to that which I employed for polynomials of two variables, in ANALYST, Jan. '81, p. 4, but it would be quite cumbrous, if the eight corner terms of the block were written in full.

The number m is supposed to be taken so large that no point of error will occur whose coordinates exceed the limits $x = \pm m\Delta x$, $y = \pm m\Delta y$, $z = \pm m\Delta z$. Any entire polynomial of three variables can be put in this block form, by adding or intercalating terms with zero coefficients if req'd. The whole number of terms in the block is $(2m+1)^3$. From the manner in which the coefficients in the successive powers of a polynomial are formed, it is evident that when n observations are made, the probability that the sum of the errors in the x direction will be $s\Delta x$, and that at the same time

the sums of those in the y and z directions will be $t\Delta y$ and $v\Delta z$ respectively, is the coefficient of $\xi^s\eta^t\zeta^v$ in the expansion of the polynomial (3) to the nth power.

If L is the probability that a particular error or event will occur, in one observation, then in n observations the most probable number of times for it to occur is nL. (ANALYST, May '79, p. 66.) This is strictly true when nL is a whole number, and approximately true in other cases. Hence, the most probable total effect, in the x, y and z directions, of the possible error $4\Delta x$, $2\Delta y$, $3\Delta z$ for example, will be

$$nL_{4,2,3}(4\Delta x), \qquad nL_{4,2,3}(2\Delta y), \qquad nL_{4,2,3}(3\Delta z),$$

and in like manner for the other errors. The most probable algebraic sum of the n errors which occur will be, in the x direction,

$$n\Delta x[\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}(aL_{a,b,c})], \tag{4}$$

where the coefficient a takes the same values as the sub-index a does. This is the approximate value of the exponent of ξ in that term of the expansion of (3) whose coefficient is a maximum. The most probable value of the arithmetical mean of the n errors which occur, in the x direction, is found by dividing (4) by n. The quotient thus obtained is evidently the statical moment, with respect to the YZ plane, of the system ot coefficients L regarded as the masses of material points. It also represents the lever arm of the system with respect to that plane, since $\Sigma L = 1$. In like manner it will be found that the most probable arith. means of the errors in the Y and Z directions, respectively, are the lever arms of the system with respect to the XZ and XY planes. Thus the most probable mean position of the n points of error is the centre of gravity of the system of masses L. From what was shown in my article of May '80, p. 81, respecting the position of the centre of gravity of the coefficients in an expanded polynomial, joined with the fact that the coefficients in the expansion are not altered by subtracting a constant quantity from each exponent in the first power, so as to bring the place of $\xi^0\eta^0\zeta^0$ (or $x^0y^0z^0$ in the notation there used) into the middle of the block instead of at one corner, it follows that the lever arms of the system of coefficients in the expansion of (3) to the nth power are n times what they are in the first power, the coordinate planes remaining the same. Hence, according to (4) &c, the maximum coefficient in the expansion coincides approximately with the centre of gravity of the whole system of coefficients in the expansion. It will otherwise appear hereafter, that this is true at the limit, when n becomes infinite.

Let us write the expansion of (3) to the nth power as follows, retaining the same block form.

$$U = \Sigma_{c=-mn}^{c=mn}\Sigma_{b=-mn}^{b=mn}\Sigma_{a=-mn}^{a=mn}(l_{a,b,c}\xi^a\eta^b\zeta^c), \tag{5}$$

where a, b, c take only integer values as before. The whole number of terms in the expansion is $(2mn+1)^3$. We shall now employ the method of indeterminate coefficients, to find a relation between the $(2m+1)^3$ coefficients L in the first power, and any similar block of an equal number of coefficients l in the expansion. Since the L and l are independent of ξ, η and ζ, we may take the relation

$$u^n = U, \tag{6}$$

and differentiate it with respect to the variables, obtaining

$$nu^{n-1}\left(\frac{du}{d\xi}\right)=\frac{dU}{d\xi},\qquad nu^{n-1}\left(\frac{du}{d\eta}\right)=\frac{dU}{d\eta},\qquad nu^{n-1}\left(\frac{du}{d\zeta}\right)=\frac{dU}{d\zeta},$$

which being divided by (6), gives

$$nU\left(\frac{du}{d\xi}\right)=u\left(\frac{dU}{d\xi}\right),\quad nU\left(\frac{du}{d\eta}\right)=u\left(\frac{dU}{d\eta}\right),\quad nU\left(\frac{du}{d\zeta}\right)=u\left(\frac{dU}{d\zeta}\right). \tag{7}$$

In the first of these three equations (7) substitute the expressions for u and U as in (3) and (5), and also expressions for $\frac{du}{d\xi}$ and $\frac{dU}{d\xi}$ as obtained from the same by differentiation. This gives an equation, each member of which contains the product of two block-formed polynomials. Any such product will itself be a block-formed polynomial, and any term in it will have for its coefficient the sum of products of L and l, such that this sum can be expressed in the block notation. Forming in this way the coefficient of $\xi^{i-1}\eta^j\zeta^k$ in the product in each member of the equ'n referred to, and placing them equal to each other by the principle of indeterminate coefficients, we get

$$\begin{aligned} &n[\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}(-aL_{-a,-b,-c}\,l_{(i+a),(j+b),(k+c)})] \\ &=\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}[(i+a)L_{-a,-b-c}\,l_{(i+a),(j+b),(k+c)}]. \end{aligned} \tag{8}$$

In the second member of this equation, let the part which does not have the coefficient i be transferred to the first member, then change the signs of both members and divide by $n+1$. Take V as an auxiliary letter and write

$$V=\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}(L_{-a,-b,-c}\,l_{(i+a),(j+b),(k+c)}). \tag{9}$$

Then (8) becomes

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(aL_{-a,-b,-c}\,l_{(i+a),(j+b),(k+c)}\right)=\frac{-i}{n+1}V. \tag{10}$$

In like manner, starting from the two last equations in (7), and equating to each other the coefficients of $\xi^i\eta^{j-1}\zeta^k$ in the one case and those of $\xi^i\eta^j\zeta^{k-1}$ in the other, we get after reduction

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(bL_{-a,-b,-c}\,l_{(i+a),(j+b),(k+c)}\right)=\frac{-j}{n+1}V, \tag{11}$$

$$\sum_{c=-m}^{c=m}\sum_{b=-m}^{b=m}\sum_{a=-m}^{a=m}\left(cL_{-a,-b,-c}\,l_{(i+a),(j+b),(k+c)}\right)=\frac{-k}{n+1}V. \tag{12}$$

It will be seen that the terms in (10), (11) and (12) all contain products of L and l, so taken that any L occupying a given position in its block, is the multiplier of that l which occupies just the opposite position in the equal block of terms in the expansion. These three equations express the relation which exists between the coefficients L in the given polynomial, and any similar block of coefficients l in the expansion to the nth power. The middle coefficient in this block is $l_{i,j,k}$, and its coordinates are $i\Delta x$, $j\Delta y$ and $k\Delta z$, the expansion being referred to the same axes as the first power. The coefficients l are also regarded as the masses of material points. When the exponent n is made very large, or infinite, the masses l attain their limiting values, and we denote by w the limiting value of $l_{i,j,k}$. Supposing the p'ts to be set so close together as to become consecutive, Δx, Δy, Δz are represented by dx, dy and dz, the coordinates of the point whose mass is w are

$$x = idx, \qquad y = jdy, \qquad z = kdz, \qquad (13)$$

and we wish to express w as a function of x, y and z. Let n be made an infinity of the second order. The system of material p'ts will then extend throughout infinite space, their number $(2mn+1)^3$ being the whole number of consecutive points in the infinite block, while $(2m+1)^3$ is the number included within the small block under consideration which has w at its centre, so that the portion of space occupied by this last block is infinitesimal in its length, breadth and depth. The masses of the consecutive points within its limits may therefore be regarded as varying from each other by constant differences, under the assumption, of course, that w is a continuous function of x, y, z.

Let d_xw, d_yw and d_zw denote the differentials of w taken in the three coordinate directions. The values of all the $(2m+1)^3$ masses l in the block will be formed from w by successive additions and subtractions of its differentials, and will be the terms in this block-formed total,

$$\Sigma_{c=-m}^{c=m}\Sigma_{b=-m}^{b=m}\Sigma_{a=-m}^{a=m}(w+ad_xw+bd_yw+cd_zw). \qquad (14)$$

Substitute them for the correspond'g terms l in (10), (11) and (12). Collect separately the coefficients of w, d_xw, d_yw and d_zw in the result, remembering that $\Sigma L = 1$. Employ auxiliary letters, such that α_1, α_2, α_3 denote the three sums of all the products formed by multiplying each L by its first, second, and third sub-index respectively; also let β_1, β_2, β_3 denote the three sums of the products of each L into the square of its first, second, and third sub-index respectively; and let γ_1, γ_2, γ_3 denote the three sums of the products formed by multiplying each L by the product of its first and second, first and third, and second and third sub-indices respectively. It will be found that (10), (11) and (12) are then reducible to

$$\left.\begin{aligned}
-a_1 w+\beta_1 d_x w+\gamma_1 d_y w+\gamma_2 d_z w &= \frac{-i}{n+1}(w-a_1 d_x w-a_2 d_y w-a_3 d_z w),\\
-a_2 w+\gamma_1 d_x w+\beta_2 d_y w+\gamma_3 d_z w &= \frac{-j}{n+1}(w-a_1 d_x w-a_2 d_y w-a_3 d_z w),\\
-a_3 w+\gamma_2 d_x w+\gamma_3 d_y w+\beta_3 d_z w &= \frac{-k}{n+1}(w-a_1 d_x w-a_2 d_y w-a_3 d_z w).
\end{aligned}\right\} \quad (15)$$

These are differential equations in which a_1, β_1 &c. are constants. From the manner in which the latter are formed, it is evident that if the coefficients L of the given polynomial are regarded as the masses of material points, the intervals between them being $\varDelta x$, $\varDelta y$, $\varDelta z$, represented by dx, dy, dz at the limit, then $a_1 dx$, $a_2 dy$, $a_3 dz$ are the statical moments of the system with respect to the coordinate planes YZ, XZ and XY. Also $\beta_1 (dx)^2$, $\beta_2 (dy)^2$ and $\beta_3 (dz)^2$ are the three sums of the products formed by multiplying each L into the square of its distance from the three planes respectively.

Likewise $\gamma_1 dxdy$, $\gamma_2 dxdz$ and $\gamma_3 dydz$ are the three sums of the products formed by multiplying each L into the product of its distances from the planes YZ and XZ, YZ and XY, and XZ and XY respectively. It is evident that if the coefficients L in the given polynomial (3) are such that their centre of gravity falls at the origin, that is, at the place of $L_{0,0,0}$, the statical moment of the system with respect to any plane through this centre will be null, and we shall have $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. Furthermore, if the coefficients and assumed coordinate axes are such that these axes are the free axes of the system of masses L, that is, the principal axes through the centre of gravity, we shall have also $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$. This follows from the known properties of the principal axes in bodies of three dimensions. (See for instance Chapter II. of Vol. II. of Poisson's *Traité de Mécanique.*)

Of course, if the possible points of error and their probabilities are arbitrarily distributed, in other words, if any arbitrary set of values is assigned to the coefficients L in (3), their centre of gravity will not in general coincide with the origin or place of $L_{0,0,0}$ and the axes taken will not in general be free axes. But the position of the free axes can always be found by means of the formulas demonstrated in Poisson's treatise, and being at right angles to each other, they can be taken as coordinate axes. By reasoning precisely analogous to that employed in my article on errors in two dimensions (ANALYST, March '81, pp. 44 to 47), it will appear that when the coefficients L are referred to any new rectangular axes, and the coefficients l in the expansion are also referred to the same axes, the coefficients in the expansion are unchanged and retain the same positions relatively to each other which they had under the old system, differing only in position relatively to the axes.

For the sake of simplicity we will assume the same arbitrary interval or unit of measure in all directions, so that $\Delta x = \Delta y = \Delta z$. A given system of coefficients located in space will be here said to be *referred to* any given rectangular axes, when the variables ξ, η, ζ attached to each coefficient receive exponents equal to the three coordinates of the coefficient, expressed in units of measure. And coefficients whose attached variables have any given exponents are said to be referred to given axes, when they are so placed in space that their coordinates, expressed in the units of measure, are equal to the corresponding exponents.

Suppose that the directions of the axes in (3) are changed while the origin remains the same. Let x_1, y_1, z_1 and x_2, y_2, z_2 be the coordinates, in units of measure, of two given points of error referred to the old axes, and let x'_1, y'_1, z'_1 and x'_2, y'_2, z'_2 be those of the same points referred to the new axes. Multiplying together the two corresponding terms of the polynomial, omitting their coefficients L since the products of these have the same value in both cases, we find in the two systems

$$\left.\begin{aligned} \xi^{x_1}\eta^{y_1}\zeta^{z_1} \times \xi^{x_2}\eta^{y_2}\zeta^{z_2} &= \xi^{x_1+x_2}\eta^{y_1+y_2}\zeta^{z_1+z_2}, \\ \xi^{x'_1}\eta^{y'_1}\zeta^{z'_1} \times \xi^{x'_2}\eta^{y'_2}\zeta^{z'_2} &= \xi^{x'_1+x'_2}\eta^{y'_1+y'_2}\zeta^{z'_1+z'_2}. \end{aligned}\right\} \qquad (16)$$

The exponents of the second members are the coordinates of the place of the coefficient of the product of the two terms, in the two systems. The known relations between the two systems of coordinates give

$$\left.\begin{aligned} x_1 &= a\,x'_1 + b\,y'_1 + c\,z'_1, & x_2 &= a\,x'_2 + b\,y'_2 + c\,z'_2, \\ y_1 &= a'x'_1 + b'y'_1 + c'z'_1, & y_2 &= a'x'_2 + b'y'_2 + c'z'_2, \\ z_1 &= a''x'_1 + b''y'_1 + c''z'_1, & z_2 &= a''x'_2 + b''y'_2 + c''z'_2, \end{aligned}\right\} \quad (17)$$

where a, b, c &c. are the direction-cosines of the new axes, that is to say, a, b, c are the cosines of the angles which the new X, Y and Z axes make with the old X axis; a', b', c' are those of the angles which they make with the Y axis; and so on. By addition we get

$$\left.\begin{aligned} x_1+x_2 &= a\,(x'_1+x'_2)+b\,(y'_1+y'_2)+c\,(z'_1+z'_2), \\ y_1+y_2 &= a'(x'_1+x'_2)+b'(y'_1+y'_2)+c'(z'_1+z'_2), \\ z_1+z_2 &= a''(x'_1+x'_2)+b''(y'_1+y'_2)+c''(z'_1+z'_2). \end{aligned}\right\} \qquad (18)$$

These relations, being of the same kind as those in (17), show that just as x_1, y_1, z_1 and x'_1, y'_1, z'_1 are coordinates of the same point in the two systems, so the point whose coordinates in the old system are x_1+x_2, y_1+y_2, z_1+z_2, is the same point whose coordinates in the new system are $x'_1+x'_2$, $y'_1+y'_2$, $z'_1+z'_2$. In other words, the product keeps the same absolute position under either system. And as this is true for all the partial products which make up the total, it follows that whatever new directions we give to the rectangular axes, the origin being unchanged, the absolute places of the coefficients in the squared polynomial will be the same.

In like manner it may be shown that they will continue the same for higher powers, and up to the limit. If we also transfer the origin to some new point, and draw axes through it parallel to the new directions as above, and denote by a_1, b_1, c_1 the coordinates of the old origin referred to the new, then in the first power of the polynomial all the exponents of ξ, η, ζ will be increased by the constant quantities a_1, b_1, c_1 respectively, in the second power they will all be increased by $2a_1$, $2b_1$, $2c_1$ respectively, and in the nth power, by na_1, nb_1, nc_1 respectively. Thus the positions of the coefficients in the expansion, relatively to each other, will be unaffected by a change in the origin, or in the directions of the axes, or in both together.

Supposing now that all the exponents and corresponding sub-indices in the polynomial (3) have been altered so as to refer to the new axes, it is evident that they will in general be fractional numbers, and i, j, and k likewise, whereas we regarded them as integers, in our demonstration of the results (10), (11), (12). But we may imagine that the given block of coefficients L in (3) is enclosed within a cubical block whose centre is at the desired new origin and whose sides are parallel to the new axes, and that this new block is subdivided into very small cubes by equidistant planes parallel to its sides and taken very near together.

The points at which these planes intersect are supposed to be each occupied by a coefficient L, only all these coefficients are zero except at the points which come nearest to the coefficients in the original block; and to these points the original coefficients are supposed to be transferred. It is evident that by diminishing the intervals between the planes, these positions of the coefficients L in the new block can be made to approach to their true positions in the old block, within any assignable limits. Hence the relations (10), (11), (12) will hold good, it being understood that m is now the number of new intervals from the centre to the surface of the new block, in any one of the three coordinate directions, and that i, j, k denote the number of such intervals from the centre of any equal block of coefficients in the expansion, to the respective coordinate planes. But we may restore, if we please, the larger unit of measure $\Delta x = \Delta y = \Delta z$, by dividing (10), (11) and (12) throughout by the ratio which the small interval or unit bears to the larger one. The effect of this will be, that the coefficients from m to $-m$, as well as i, j, k, will now, in general, be fractional numbers. The same is true of the coefficients from m to $-m$ in (14). Here $d_x w$, $d_y w$ and $d_z w$ are the increments of w corresponding to those increments of x, y and z which are equal to $\Delta x = \Delta y = \Delta z$, and which are represented by $dx = dy = dz$ at the limit.

[To be continued.]

REVIEW OF "THEORY OF THE MOON'S MOTION DEDUC'D FROM THE LAW OF UNIVERSAL GRAVITATION By John N. Stockwell, Ph. D."

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MR. Stockwell has, in the last fifteen or twenty years, presented to the public several memoirs on the lunar theory, of which the one, whose title is given above, is the latest and most extensive. In all of these, at least in all that have come to our notice, and we believe we have seen all, the *raison d'être* of the memoir appears to be the author's desire to controvert certain of the results arrived at by his predecessors in this subject; results, too, which have long been regarded by astronomers as definitely settled and acquired to science. But he has been singularly unfortunate in these criticisms, for there is no case, in which he has called in question the legitimacy of some procedure followed by previous investigators, but that he is himself in error, or, at least, that his objections are without foundation. This is the more to be regretted, as when the author does not come in conflict with the generally received theory, and his results agree with what was known before, he is often right in the methods employed; though it must be mentioned that the latter are sometimes unnecessarily prolix.

To follow Mr. Stockwell in his many aberrations, would make too long a story for this place, we shall be obliged to confine our attention to the above-named volume. And, even with this restriction, the space at our disposal will not allow us to write out, in full, a logical refutation of the mistakes contained in this. Nor is it necessary. For what could we do, but repeat, in a slightly varied form, perhaps, what has already been said sufficiently well by the mathematicians whom Mr. Stockwell so often controverts?

On p. XXII of his Introduction, we find the author possessed by the notion that each coordinate of the moon can be divided absolutely, and in only one way, into two parts; of the first of which we can say, this is elliptic and produced by the action of the earth, and of the second, this is perturbational and is due to the action of the sun. Now the fact is, the curve described by the moon is described under the simultaneous action of both bodies, and what belongs to each cannot be separated in the way mentioned. Our analytical expressions for the coordinates may be in such a form that it is not easy to see what they would become on making the solar force zero. For this, Mr. Stockwell appears to think that it is only necessary, in the expresssions of Plana, Pontécoulant and Delaunay, to put $m=0$. And he

observes that, when this is done, there still remain some terms which he is pleased to attribute to the solar action only. Now the fact is, the putting $m=0$ implies that we have either an infinitely short month or an infinitely long year; that is, the semi-axis major of the moon's orbit is infinitely small or the semi-axis major of the sun's orbit is infinitely great. Hence, when we put $m = 0$, to be consistent, we are obliged also to put $\frac{a}{a'} = 0$: also we notice that, under these conditions, the perigee and node become stationary in the heavens. The terms which are now left in the expressions of the coordinates have only, in their arguments, multiples of the mean anomaly and the mean argument of the latitude, of the moon; and, hence, return to the same values after each revolution of the moon, since the perigee and node are fixed. Thus the curve described under these conditions is a closed one; and, when we investigate its nature, we find it to be an ellipse, and the laws of the motion of the moon in it we find to be those of elliptic motion, the central force being equivalent to that of the earth.

Mr. Stockwell's notion, that what are called the perturbational parts of the coördinates, must necessarily vanish when the disturbing force vanishes, is entirely unfounded. All that is necessary, under such a condition, is, that the resulting expressions for the coordinates should satisfy the differential equations of elliptic motion with the earth as central body. The terms which appear to have puzzled Mr. Stockwell so much, are nothing but the expression of the passage from one ellipse to another with slightly different elements. Thus falls to the ground Mr. Stockwell's startling statement "that there must be something seriously wrong in the published theories, notwithstanding their intricacy and refinement".

Mr. Stockwell, having noticed that many of the terms, which refuse to vanish when m is put equal to zero, have arguments whose motion differs from the mean motion of the moon only by quantities of the order of the disturbing force, proceeds to inquire how Laplace, Plana and Pontécoulant got such terms into their expressions. He gives his equation [3]

$$r = \frac{m^2}{m^2} \cos (it - \epsilon),$$

which seems to trouble him a great deal. So great is his dread of introducing any term, of the zero order with respect to the disturbing force, into the expressions for the coordinates, that he will not allow us to cancel the factor m^2 common to the numerator and denominator of the fraction. How this cancelling can vitiate the computation of the coefficient, the only thing we care about, one does not see. The only reason he can give for this, is, that the fraction takes the indeterminate form $\frac{0}{0}$ when $m = 0$. Now what,

ever weight such an argument might have in the case where m actually vanishes, it has no application in a lunar theory where m has some definite finite value, as is the case in the motion of the moon about the earth. But suppose that m is really zero, why need we be troubled about the result obtained by cancelling the factors? Has Mr. Stockwell never read the chapter, in our treatises on the dff. calc., on vanishing fractions? Has he, for instance, never seen the proof that $\frac{\sin x}{x} = 1$, when $x = 0$? And, moreover, why does he neglect to notice that the term, in this case, having a period exactly equal to the period of revolution of the moon, merely expresses the change of radius vector caused by passing from one ellipse to another?

Hence we cannot accept, as true, the strange assertion that "some of the most remarkable cases of perturbation, which have hitherto been supposed to affect the moon's motion, have no existence in nature".

Mr. Stockwell next turns his attention to Plana's determination of one of the terms under discussion, where the latter uses the method of variation of the elements. He follows Plana until he arrives at his equation [11], which he says agrees with what Plana has. But then follows this remarkable statement, "But this conclusion is not satisfactory". Why a conclusion, legitimately deduced from correct principles, should be thrown aside at a mere *arbitrium*, certainly surpasses our powers of explanation. Mr. Stockwell proceeds, "I shall now show that, if we neglect the square of the disturbing force, we may suppose the elements to be constant in the differential equations, and that then we shall have $\delta e = 0$, $\delta \omega = 0$". It appears from this that Mr. Stockwell does not admit there may be terms in the lunar coordinates, to the values of which we cannot obtain the lowest degree of approximation, without consenting to consider the square of the disturbing force as well as the first power of it. There are such terms, and the term under consideration is one of them. Mr. Stockwell, however, proceeds in his own way and arrives at his equation [20]. We give it here:—

$$\delta e = -\frac{21}{8}\frac{m^2}{\alpha} e\gamma^2 \cos(\beta_0 + \alpha v) + \frac{21}{8}\frac{m^2}{\alpha} e\gamma^2 \cos \beta_0.$$

Now, what is strange, Mr. Stockwell does not perceive that this equation is virtually identical with his equation [10]. The last term is constant and coalesces with the elliptic value of e, and nothing variable is left but the first term, which differs only in notation from the second member of [10]. Mr. Stockwell continues, "In the case of constant elements we shall have $\alpha = 0$, and then the value of δe will become $\delta e = 0$, [21]". How [21] results from [20] by putting $\alpha = 0$, we must leave analysts more competent than ourselves to discover. Certainly the diff. of two infinities is not always zero.

But why insist on putting $a = 0$, when we know, both from observation and our analysis, that is not actually the value of this quantity: the perigee and node are actually in motion, whatever we may say to the contrary? To forbid the perigee and node to move, is virtually to say there shall be no disturbing force, and then we are driven back on elliptic values, as Mr. Stockwell's equation [22] shows. Who does not see that, to get even the first approximation to this inequality, we must admit the movement of the perigee and node?

Our author's dislike to quantities of the order of the square of the disturbing force is remarkable; on p. XXXIV, we find him pronouncing a correction of Laplace erroneous for no other reason than that he thinks it is, or ought to be, of the order of the square of the disturbing force.

From his closing remark on this inequality, we see that Mr. Stockwell regards the coefficient of it, which is used in the present Tables, as being no less than 84″.8 in error. But how could such a large error have escaped detection before now? We must remember that the effect of this inequality goes through all its phases in about three years, and that there is no other inequality having nearly the same period and thus capable of concealing the former. Tobias Mayer, a hundred and twenty years ago, was able to discover the small term in the moon's longitude, whose argument is the longitude of the node, simply by comparison of his rude theory with observation, and without any assistance from theory, although its coefficient does not exceed 7″. And, recently, Sir. G. B. Airy and Prof. S. Newcomb have been able to discover small terms, with coefficients not much exceeding 1″, in the same way. It is, therefore, simply an impossibility that such a large correction can be needed; and this consideration ought to have made Mr. Stockwell doubt the legitimacy of his mode of treating this inequality.

We next notice our author's treatment of the inequality depending on the angular distance between the lunar and solar perigees. He says, p. XXVI, that "the inequalities of long period are determined with great facility" by his equations [23] and [24]. The latter, indeed, are much simpler than any of those which have been hitherto employed to discover the value of this inequality. But truth must not be sacrificed to convenience. We cannot find any attempt at proof of these equations, either in the present volume, or elsewhere in Mr. Stockwell's writings. He seems to have adopted them quite arbitrarily. Now the theory of dimensions of units shows us that the equations [23] are inconsistent with each other. For the first member of the first is evidently of one dimension less as to the linear unit than the first member of the second; hence it is impossible that the second members can have coefficients (denoted by h) equal to each other. We

might suppose that perhaps h denoted, in general, a constant coefficient, but we are debarred from this, by observing that, in [24], it stands outside as a common factor of both terms. On turning to page 356, where Mr. Stockwell makes an application of these equations, however, we learn that the two h's are not necessarily equal. But, this allowed, equation [24] is still out of harmony with the law of dimensions; it directs us to do something very like adding miles to square miles. It is also singular in giving an infinitely great coefficient to the inequality, when the disturbing force is infinitely small. It is almost needless for us to state that this equation is incorrect, and that the large coefficient 108″.53, found by it for the inequality considered, is quite wrong. The reader, who wishes to see how the inequality ought to be determined, will find in Pontécoulant, Tom. IV, pp. 463–5, probably as brief an investigation as can be made.

The author states, p. XXVII, that he has "discovered that there is a small secular equation of the longitude arising from the oblateness of the earth" and that its value is $+ 0''.1979i^2$, i being the number of centuries elapsed since 1850. But this equation does not exist. The way the mistake arises is this:—the differential equations of the moon's motion, which are usually given, and which our author employs, suppose that the planes of reference are fixed. So that when longitude and latitude are employed, they must be referred to the fixed ecliptic and equinox of some date. Now the obliquity of the equator to a fixed ecliptic varies very slowly and proportionally to the square of the time. Peters gives as the formula, when 1800 is taken as the epoch,

$$\varepsilon_1 = \varepsilon_0 + 0''.00000735t^2.$$

This ought to have been employed instead of the author's

$$\varepsilon = \varepsilon_0 - 0''.48970t - 0''.0000012t^2.$$

And afterwards, if we desire to have the longitude and latitude referred to the mean ecliptic and equinox of date, all we have to do is to substitute the results obtained in the well-known formulæ which give the precession in longitude and latitude.

Mr. Stockwell gives, in equation [25], p. XXVIII, the value of the secular acceleration which arises from the variation of the solar eccentricity. It lies between the old value of Damoiseau and Plana and the new one of Adams and Delaunay, and does not agree with the value obtained by the author in his previous papers. Why he abandons his old value for that given here, he does not inform his readers. By elimination between eq'ns (672) and (676), p. 358, we discover that the equation employed for determining it, is

$$\frac{d.\delta v}{dt} = 2n\frac{a^2}{\mu}\left(\frac{dR}{dr}\right).$$

But this gives correctly only the first term of the coefficient, and when we wish to go as far as terms multiplied by the square and cube of the solar disturbing force, it is necessary to employ two equations, such as Prof. Cayley has given (R. A. S. Mon. Not., Vol. XXII, p. 177).

Mr. Stockwell, having calculated the coefficient of the largest inequality in the longitude, due to the figure of the earth, and finding that his result does not agree with that of Laplace, says, p. XXX, he will give "what seems to be an entirely satisfactory explanation" of this discordance. For this purpose he follows Laplace's procedure correctly and obtains his equation [43]; and remarks that the value of the right member is only one-third of that of the corresponding term in (647). Now the truth is, [43] is exact and (647) is erroneous. However the author will explain this. He says Laplace has given, in a certain chapter of his work, a general method of integration, "and that the method followed by him (Lapl.) in his investigation of the effects of the earth's oblateness does not seem to be in accordance with it". But the fact is, the chapter contains two distinct methods, the first attributes the perturbations to the coordinates, the second, to the elements; and the first is the one used by Laplace in his treatment of the present subject. Yet we find Mr. Stockwell, immediately afterward, quoting, from another chapter in Laplace, a process which is only permissible in the second method, to justify himself in putting $g = 1$ in a certain equation. Now to put $g = 1$, is to insist that the node shall not move, and thus the argument shall remain invariable. Any one can understand that a coefficient determined under such a forced condition, not in accordance with the real state of things, for we know the node does move, would very likely be widely different from that which actually has place. Consequently all the equations from [44]to [47] are quite wrong.

Mr. Stockwell now investigates a second term of Laplace, and he finds, in his equation [52], that it is the exact negative of a term in the disturbing function, which had been previously considered. He therefore fears it is some unwarranted duplicate of the latter, and calls it "the reaction of the force expended by the sun in giving motion to the moon's nodes". How such a strange designation can be applied to it, when it vanishes together with the non-sphericity of the earth, he does not inform us.

But the origin of this term is easily explained. The sun acts upon the moon in the place where the latter actually is, and not in the place where it would have been without the action of the non-sphericity. The difference of the two actions is expressed by this term. This difference was not taken into account, when equation [28], which expresses only the action of the non-sphericity, was written. Consequently it is a legitimate additional term,

which cannot be set aside, as Mr. Stockwell proposes. And his remark, "it is evident that the whole value of δv must be derived from the value of R in [5362]", cannot be admitted.

Mr. Stockwell adds, "But, in this second part of his work, Laplace seems to have committed a grave oversight, for he has treated his equation [5372], in the construction of [5373], as though δs were constant; whereas it is a function of r and v according to [5376], which he afterwards uses in his reductions". Now this remark has no bearing on the matter in hand; for all the partial differentiations and the integration are to be executed before the variation, expressed by the symbol δ, is taken. Thus R, to the degree of approximation adopted by Laplace, having no variables but the coordinates of the moon, we must have $\int dR = R$, and $r\left(\frac{dR}{dr}\right) = 2R$, consequently

$$3\int dR + 2r\left(\frac{dR}{dr}\right) = 7R,$$

and, taking the variation,

$$3\int \delta dR + 2\delta . r\left(\frac{dR}{dr}\right) = 7\delta R.$$

Hence to get [5373] from [5372], we have only to multiply by 7.

Mr. Stockwell next notices the correction given by Laplace to reduce the inequality from the plane of the orbit to the plane of the ecliptic, and says, "It is apparent, however, that this correction is not required, for Laplace has shown in [923′], etc., where this subject is first treated, that this correction is of the order of the square of the disturbing force; and as terms of that order have not been considered, it is evident that the value of that correction, which he has given in [5385], is erroneous". This statement is incorrect in every point. What Laplace has really shown in [923′] is that the reduction of the longitude from one plane to another is a quantity of the order of the square of the disturbing force, *when the inclination of the planes is a quantity of the order of the disturbing force.* But the inclination of the moon's orbit to the ecliptic is not a quantity of the order of the disturbing force, consequently Laplace's remark has no application in the present case. And had Mr. Stockwell taken the trouble to work out [5385] from the equations from which it is derived, he would have found it exact.

There is nothing in the present volume, nor anything in his previous publications, that substantiates the author's assertion "that the existing theories, instead of being correct to terms of the seventh order, are really erroneous in terms of the third order". It is a matter for regret that so much persevering labor, enthusiastically followed, for so many years, should have been given to the production of this book, since, directed in less ambitious channels, it might have brought both honor to the author and profit to science.

NOTE ON PROBLEM 374.

BY PROF. ASAPH HALL.

IF the solution of this problem given on p. 30 of the ANALYST, Vol. IX, be correct, it furnishes an easy method for improving numerical tables. Thus, we have only to interpolate the table into the middle, as the phrase is, and then laying aside the original table we have one of equal intervals but more accurate. Again interpolating into the middle we get a table still more accurate than the second one, and having the same arguments as the original, and so on *ad infinitum*. But the solution seems to me erroneous.

If we have a series of values v, v', v'', &c., all of which have the common probable error r, then the probable error of the sum or difference of any two of these values is $r.\sqrt{2}$. Since in interpolating we multiply the difference, $v'-v$, by a factor, $\frac{t}{10}$, the probable error of the correction $= \Delta$, that we add to v, is $\frac{rt.\sqrt{2}}{10}$. Now the probable error of $v+\Delta$ is

$$r.\left(1 + \frac{2t^2}{100}\right)^{\frac{1}{2}}.$$

Hence the interpolated value is less correct than the tabular one, and our method of improving the tables fails.

PLANETARY MASS AND VIS VIVA.

BY PLINY EARLE CHASE, LL. D.

ALL persistent oscillations in elastic media, whether luminous, electric, thermal, atomic, molecular or cosmical, MUST BE harmonic.

The fundamental harmonies of oscillatory movement in the luminiferous æther, must involve simple functions of the velocity of light.

In applying the oscillatory equation,

$$t = \pi\sqrt{\frac{l}{g}},$$

at the centre of gravity of a stellar system, let t represent the duration of an oscillation or half-rotation, g the acceleration of gravity at the stellar equatorial surface, $\pi^2 l$ the stellar modulus of light or the height of a homogeneous æthereal atmosphere which would propagate undulations with the velocity of light. Then, if the stellar rotary oscillation is due to the reaction of cosmical inertia against ætherial influence, *gt is equivalent to the velocity of light*, v_λ.

Coulomb's torsional formula may be applied to Sun, by taking Sun's equatorial semidiameter, r_0, as the radial unit:—

$$f = \frac{m}{2} = \frac{W}{2} \cdot \frac{\pi^2 a^2 r}{gt^2};\ \pi^2 a^2 r_0 = \pi^2 l = gt^2;\ gt = v_\lambda. \qquad (1)$$

If sun's apparent semidiameter is 961″.83, Earth's semi axis major (ζ_3) $= 214.45 r_0$; g at Sun's equatorial surface (g_0) $= .0000003909446 r_0$; $v_\lambda = \zeta_3 \div 497.827 = .4307721 r_0$. Therefore, from (1), $t = 1101875$ sec.; solar rotation $= 2t = 2203750$ sec. $= 25.506$ days.

Vis viva may be represented by orbital areas, as well as by distances of projection against uniform resistance. The virtual areas of synchronous planetary reaction, or the mean instantaneous areas which a particle at Sun's surface *tends* to describe about any given planet, vary as $\sqrt{(mr)}$. The doctrine of conservation of energy supplements Laplace's two laws of constancy by a third, viz.:—*The sum of all the instantaneous virtual areas in a system will always remain invariable.*

The laws of the chief centre of condensation, Earth, which is the central and controlling planet in the dense belt, exerts important harmonic influences. If Earth were rotating with the speed which a coincidence of Laplace's limit with its equatorial surface would give, its time of rotation would be $2\pi \sqrt{(r \div g)} = 5073.8$ seconds. Its coefficient of orbital retardation is, therefore, $x = 86164.1 \div 5073.8 = 16.9822$. In an expanding or condensing nebula, the atmospheric radius varies as the $\frac{4}{3}$ power of the nucleal radius; $x^{\frac{4}{3}} = 43.651$.

Herschel's locus of incipient subsidence, in the controlling two-planet belt, or Saturn's secular aphelion, is 1.0843289 times the outer limiting locus of the belt (Stockwell, *Smithson. Contrib.*, 232, p. 38); $x^{\frac{4}{3}} \div 1.0843289 = 40.256$, which is, approximately, the ratio of the instantaneous virtual area at the inner locus of the controlling belt, to the virtual area at the chief centre of condensation. The tendency of exponents, in elastic media, to become coefficients of harmonic *vis viva*, is shown in the following table:—

	Harmonic Areas.	Mean Virtual Areas.		Difference.
$\alpha = x^{\frac{4}{3}}$	40.256	Jupiter	40.587	—.331
$\beta = \frac{3}{4}\alpha$	30.192	Saturn	30.063	+.129
$\gamma = \frac{3}{4}\beta$	22.644	Neptune	22.675	—.031
$x = \frac{3}{4}\gamma$	16.983	Uranus	16.782	+.201
ε	1.000	Earth	1.000	.000
$\zeta = \frac{3}{4}\varepsilon$	.750	Venus	.749	+.001
$\delta = \frac{3}{4}\zeta$	.562	. . .	. . .	. .
$\eta = \frac{3}{4}\delta$	.422	Mars	.404	+.018
$\theta = \frac{3}{4}\eta$	.316	. . .	. . .	. .
$\iota = \frac{3}{4}\theta$	.237	. . .	. . .	. .
$\lambda = \frac{3}{4}\iota$	.178	Mercury	.162	+.016

The percentage of difference between the harmonic and virtual areas is, respectively, $\frac{5}{6}$ of .01, $\frac{3}{7}$ of .01, $\frac{1}{7}$ of .01, $\frac{6}{5}$ of .01, $\frac{1}{7}$ of .01, .045, .099. In testing the combined harmonic influences of a *vera causa*, subject to internal perturbations, there is room for a possible deviation of 50 per cent, and a probable deviation of 25 per cent. The combined probability of the dependence of the above approximations upon æthereal influence is, therefore, $(25 \div \frac{5}{6} = 30) \times \frac{175}{3} \times 175 \times \frac{125}{6} \times 175 \times \frac{50}{9} \times \frac{250}{99} = 15664091727 : 1$. The intermediate harmonic areas between Venus and Mars, and between Mars and Mercury, may, perhaps, be partly distributed among the asteroids, zodiacal meteoroids, the intra-Mercurial harmonic nodes, and the special requirements of approximately synchronous rotation in the dense planetary belt.

If we let m_0, m_3, m_5 represent the masses of Sun, Earth, Jupiter; v_3, Earth's aphelion orbital velocity; the value of m_0, in units of m_3, which would satisfy central requirements of subsidence, linear oscillation, conical oscillation, and the conversion of subsident into orbital velocity, is $(2 \times 3 \times 4)^4 \doteqdot 331776$. The photodynamic *vis viva* at the chief centre of condensation, $\frac{1}{2}(m_3 v_\lambda^2)$, furnishes the following approx. harmonic proportionality:–

$$m_0 v_3 v_\lambda : m_3 v_\lambda^2 :: m_3 v_\lambda^2 : m_5 v_3 v_\lambda. \qquad (2)$$

For, $\zeta_3 = (331776)^{\frac{1}{3}} \times 3962.8 \times (31558149 \div 5073.8)^{\frac{2}{3}} = 92783200$ miles; $v_\lambda = \zeta_3 \div 497.827 = 186376$ miles; $v_3 = 2\pi\zeta_3 \div (1.01677 \times 31558149) = 18.1683$ miles. Substituting these values in (2) we get, $m_5 = 317.18$; $m_0 = 1046.02$. A harmonic approximation which is, probably, still closer, is found by taking r_0 as the locus of the secular perihelion centre of gravity of the two controlling masses, m_0 and m_5; Stockwell's estimate (*loc. cit.*), with the British Nautical Almanac estimate of Sun's semidiameter, gives $.9391726 \times 5.202798 \times 214.45 = 1047.872$.

The maximum and mean planetary accelerations and retardations at secular and mean apsides, produce æthereal disturbances with secondary nodal tendencies, which modify the harmonic areas in various ways. The masses of the principal planets in the extra-asteroidal and intra-asteroidal belts, Jupiter and Earth, are determined, as we have seen, by simple harmonic relations to Sun. Their companion planets, Saturn and Venus, show the nodal influence of the principals; Jupiter and Saturn varying directly as the mean orbital *vis viva*, and inversely as the mean subsidence potential, of particles in their respective orbits, while Earth and Venus represent the maximum disturbance of the mean orbital *vis viva* of Earth's particles upon the orbital *vis viva* of the particles of Venus. Uranus shows the combined influence of apsidal nodes of Earth, Jupiter and Uranus; Neptune, the combined influence of apsidal nodes of Earth Uranus and Neptune.

Let α, β, γ, δ, ε represent, respectively, secular aphelion, mean aphelion, mean, mean perihelion, secular perihelion; subscript 0, 2, 3, 5, 6, 7, 8, respectively, Sun, Venus, Earth, Jupiter, Saturn, Uranus, Neptune. Then the harmonic mass-ratios will be represented by the following proport'ns:—

$$m_3 : m_2 :: \gamma_3 : \alpha_2$$
$$m_0 : m_3 :: 24^4 : 1$$
$$m_0 : m_5 :: \varepsilon_5 : \varepsilon_0$$
$$m_5 : m_6 :: \delta_6\gamma_6 : \delta_5\gamma_5$$
$$m_7 : m_3 :: (\beta_7 - \varepsilon_3)\gamma_7 : (\beta_7 + \delta_5)\gamma_3$$
$$m_8 : m_3 :: (\delta_8 - \alpha_3)\gamma_8 : (\delta_8 + \alpha_7)\gamma_3.$$

The accordance between the nodal and computed values is shown in the following table:—

		Nodal.	Computed.	Authority.
Venus	$m_0 \div m_2$	428417	427240	Hill.
Earth	$m_0 \div m_3$	331776	331776	Chase.
Jupiter	$m_0 \div m_5$	1047.872	1047.879	Bessel.
Saturn	$m_0 \div m_6$	3503.22	3501.6	Bessel.
Uranus	$m_0 \div m_7$	22643	22600	Newcomb.
Neptune	$m_0 \div m_8$	19428	19380	Newcomb.

The following points of symmetry and alternation may be noted in the nodal mass-factors of the outer planets:—

1. The tendency to equality of mean orbital *vis viva* in Earth, Uranus and Neptune, as indicated by the factors γ_3, γ_7, and γ_8.
2. The nodal modification of Neptune's mass by Earth's secular aphelion, and of the mass of Uranus by Earth's secular perihelion.
3. The nodal modification of Neptune's mass by its own mean aphelion, and of the mass of Uranus by its mean aphelion.
4. The modification of Uranus by Jupiter, and the corresponding modification of Neptune by Uranus.

NOTE ON DIRECTION.

BY PROFESSOR T. M. BLAKSLEE.

[Continued from page 16.]

WHAT has preceded relative to the point as generatrix and line as path, may assist in giving a clear idea of direction, which is rendered necessary by the increased use of the term in Geometry and Quaternions.

The following additional definitions and propositions are intended to further illustrate the subject.

1. *Def.* The St. surface having the St. line as its G. is called *a plane*.

a. A surface, such that if any two of its points be joined by a St. line the line will lie wholly in the surface, is determined by three of its points; for, revolve it about the St. line joining two of the points till it coincides with the third. If now it be revolved it will no longer contain the third point; ∴ it is determined by three points, or two St. lines, ∴ it is a plane; and since but one plane can be passed through two St. lines, therefore—

b. If any two points of a plane be joined by a St. line, the line will lie wholly in the surface.

2. A regular polygon is one which is both equilateral and equiangular.

3. A *circle* is the path of a point the direction of whose motion is uniformly varied. The circle is evidently the limit of a regular polygon.

4. All the vertices of a regular polygon are equally distant from a point called the centre of the polygon; for, bisect the angles, the resulting triang's are isosceles, and since their bases are equal, the bisectors intersect in a common point.

5. From 3 and 4, the circle may be defined as the path of a point moving so as to remain at a constant distance from a fixed p't called the centre.

6. A line drawn from the centre to any point of the circumference of a circle is called the radius at that point.

7. (As in Roberval's method of tangents) The St. p. of instantaneous motion of the generatrix at any G. of a curved path is called the tangent path at that G.

8. In a circle, the radius is perpendicular to the tangent.

Proof.—Decompose the motion in the direction of the radius and at right angles to it. The component in the direction of the radius must be zero, or the radius would not be constant, as it is by definition.

Prop. —If from any point in one terminus of an angle, and at a distance h from the vertex, a perpendicular, p, be let fall on the other terminus, the distance, b, from vertex to foot of perpendicular, is called the *base*, b and p being the projections of h on the corresponding indefinite St. lines.

Since but one perpendicular can be drawn from a point to a line, a right angled triangle is determined by the hypothenuse and the angle at the base.

If the length of h be denoted by h_1, and the corresponding values of p and b, by p_1 and b_1, then will

$$\frac{p}{h} = \frac{p_1}{h_1} = \text{sine, and } \frac{b}{h} = \frac{b_1}{h_1} = \text{cosine}$$

of angle at base.

CORRESPONDENCE.

Editor Analyst:

A copy of the ANALYST for July 1881 has just fallen into my hands in which I find "An Investigation of the Mathematical Relations of Zero and Infinity", by Professor Judson. In this article the author makes use of some equations, and the solution of the same, from Thomson and Quimby's Collegiate Algebra, as follows:—

"Messrs. Thomson and Quimby give the following illustration of a false interpretation of $a \div 0 = \infty$. (Algebra, Art. 348, p. 146.)

'Given $x^2 + xy = 10$ (1), and $xy + y^2 = 15$ (2), to find x and y.

Let $x = zy$. Then, from (1),

$$y^2 = \frac{10}{z^2 + z} \text{ (5), and from (2), } y^2 = \frac{15}{z+1} \text{ (6).}$$

From (5) and (6) $10z + 10 = 15z^2 + 15z$ (7); $\therefore z = \frac{2}{3}$ or -1.

Substituting -1 for z in (5) or (6) we have $y = \pm\infty$, $\therefore x = \mp\infty$; hence $\qquad \infty^2 - \infty^2 = 10$ and $\infty^2 - \infty^2 = 15$'.

That these results are incorrect is manifest; for if we eliminate y between (1) and (2) we have an equation of the second degree, which should have two roots only. But if $\pm\infty$ be roots, then an equation of the second deg. may have four roots. Adding (1) and (2), $x^2 + 2xy + y^2 = 25$, $\therefore x + y = \pm 5$. By substituting in (1), $\pm 5x = 10$, or $x = \pm 2$. Substituting in (2), $\pm 5y = 15$, $\therefore y = \pm 3$; and these are the only roots.

The correct interpretation of this example is, since when $z = -1$, $y^2 = 15 \div 0$, $\therefore z = -1$ is an impossible value for (1) and (2)."

In reply to Professor Judson, I will say:—

1. The elimination of y between (1) and (2) gives an equation of the 4th degree which must have four roots.

2. Hence it follows that $x = \pm 2$ and $y = \pm 3$ are not the only roots but $x = \pm\infty$ and $y = \mp\infty$ are also roots.

3. I have never before heard that $y^2 = 15 \div 0$ indicated impossibility except the impossibility of expressing the result in terms of a finite unit. If Prof. Judson will refer to the Algebra above named he will find that *zero* is defined as *an infinitesimal*, and the book does not recognize a zero which means nothing. The Professor is certainly wrong when he says that infinity cannot be a root of simultaneous equations. The equations above are each the equation of an hyperbola, and the two hyperbolas have common points at $x = \pm 2$, $y = \pm 3$, and also have a common asymptote, and therefore have infinity as roots.

E. T. QUIMBY.

Hanover, N. H.

Editor Analyst:

The problem on p. 17 of the ANALYST of January 1882, "On the Computation of the Eccentric Anomaly", etc., reminds me of the method of solving numerical equations which, some years ago (Sept. 1876), I contributed to the ANALYST, and of which nobody seems to have taken notice.

This method, giving an acceleration of the third order, applies also to transcendental equations that have no multiple roots. I beg to subjoin its application to the above mentioned problem.

If $f(x) = 0$ be the equation to be solved, and $f'(x)$, $f''(x)$ denote the first and second differential quotient; if further the assumed initial value of the root be x_0, then is the corrected value

$$x_1 = x_0 - \frac{f(x_0)f'(x_0)}{f'^2(x_0) - \frac{1}{2}f(x_0)f''(x_0)}.$$

Applied to our problem, we have:

$$f(E) = E - e\sin E - m,$$
$$f'(E) = 1 - e\cos E,$$
$$f''(E) = e\sin E.$$

Now as $e = 0.2056$, by mere inspection we see that $E = 150°$ is a good initial value; for $\sin 150° = \sin 30° = \frac{1}{2}$, and the term $e\sin E = \frac{1}{2}(0.2056) = 0.1028$; the length of the arc of $1° = (\pi \div 180) = 0.0174533$, so that 0.1028 translated into degrees will give an arc of between $5°$ and $6°$; as $m = 143°$, the value $150°$ will give a small number for $f(E)$. The computation is now as follows:—

$$f(150°) = 150° - 5°.89003 - 143° = 1°.10996, \text{ in degrees of arc,}$$
$$= 0.01937 \text{ in length of arc.}$$
$$f'(150°) = 1 + 0.2056 \times 0.86603 = 1.178056;$$
$$f''(150°) = \tfrac{1}{2}(0.2056) = 0.1028.$$

$$\text{Now } E_1 = 150° - \frac{1°.10996 \times 1.178056}{(1.178056)^2 - \frac{1}{2}(0.01937 \times 0.1028)}$$
$$= 150° - 1°.10996 \times 0.84946 = 149°.05714$$
$$= 149°.\ 3'\ 25''.7$$

This value substituted in f will make $f(E_1) = -0.000009$
$$= 1''.85 \text{ in arc.}$$

It seems to me that this method is recommendable, especially in those cases where the several differential quotients admit of simple expressions.

DR. H. EGGERS.

Milwaukee, Wisconsin.

Editor Analyst:

Since the publication of the article on the Solution of Equations in the last ANALYST, I think I have discovered "the weak link in the chain" of its logic, as applied to the solution of the equation of the 5th degree.

It lies in the statement, near the bottom of page 5, that in the supplemental equations "one letter may be interchanged with another without disturbing any relations". If, in brief, numerical values be assigned to the letters, an interchange in the order of their arrangement materially disturbs the relations.

Thus if $x=\sqrt[5]{a^5}+\sqrt[5]{b^5}+\sqrt[5]{c^5}+\sqrt[5]{d^5}=1+3+5+7$, the arrangement of $x=1+3+7+5$ furnishes, when the substitution is made, a *different* equation from the one first given. As these letters are capable of twenty-four permutations, the result is the production of twenty-four general eq'ns of the 5th degree. But as only one is given for solution, an additional element is unfolded in the eq'n of the 5th degree, namely, a dependence of its solution, not only upon the value of the four letters a, b, c and d, but also upon the *order* of their arrangement. As these letters cannot be thus permuted and satisfy the given equation, it would appear that instead of one final equation being produced by elimination, "whichever three of the four unknown quantities are eliminated", four final equations would be produced, one each in a, b, c and d. It remains to determine by algebraic analysis, if possible, the character of the roots of these final equations, their mutual relations, the effect of permissable permutations, and whether they can be separated into groups, so as to permit of the reduction of the equations to the 4th degree; or, on the other hand, to determine, by algebraic methods within the comprehension of the ordinary algebraist, the impossi'ty of such reduct.

I regret that pressure of professional labors renders it impossible for me to further prosecute the work at present; but I shall be pleased if, per chance, I shall have contributed my mite towards awakening a renewed interest in the subject, trusting that further research will yet result in the development of a perfect and complete Theory of Equations.

T. S. E. DIXON.

Chicago, Ill., Feb. 13, 1882.

SOLUTION OF 393 BY R. J. ADCOCK.—"Two particles of masses m and m' respectively, are connected by a string passing through a small fixed ring and are held so that the string is horizontal; their distances from the ring being a and a', they are let go. If ρ and ρ' be the initial radii of curvature of their paths, prove that

$$\frac{m}{\rho}=\frac{m'}{\rho'},\text{ and }\frac{1}{\rho}+\frac{1}{\rho'}=\frac{1}{a}+\frac{1}{a'}."$$

Let the origin of rectangular coordinates be the fixed point thro' which the string passes, the axis of y vertical and positve downwards, $\sqrt{(x^2+y^2)} = r$, $\sqrt{(x'^2+y'^2)} = r'$, their distances from the origin at any time t after the motion begins, $T =$ the tension of the string, $g =$ the force of gravity per unit of mass, $\frac{(dx^2+dy^2)^{1/2}}{dt} = \frac{ds}{dt} = v$, $\frac{(dx'^2+dy'^2)^{1/2}}{dt} = \frac{ds'}{dt} = v'$, their velocities. Then the sum of the horizontal, and the sum of the vertical components of the forces on m and m', per unit of mass, are, respectively,

$$\frac{d^2x}{dt^2} = -\frac{Tx}{mr}, \qquad \frac{d^2y}{dt^2} = -\frac{Ty}{mr} + g,$$

$$\frac{d^2x'}{dt^2} = -\frac{Tx'}{m'r'}, \qquad \frac{d^2y'}{dt^2} = -\frac{Ty'}{m'r'} + g. \quad \text{Hence,}$$

$$\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right\} \div \left\{\left(\frac{d^2x'}{dt^2}\right)^2 + \left(\frac{d^2y'}{dt^2}+g\right)^2\right\}$$

$$= \left\{\left(\frac{-Tx}{mr}\right)^2 + \left(\frac{-Ty}{mr}\right)^2\right\} \div \left\{\left(\frac{-Tx'}{m'r'}\right)^2 + \left(\frac{-Ty'}{m'r'}+g\right)^2\right\}. \quad (1)$$

Since $r+r' = a+a'$, therefore

$$\frac{d^2r}{dt^2} + \frac{d^2r'}{dt^2} = \frac{v^2}{r} - \frac{T}{m} - \frac{dr^2}{rdt^2} + \frac{v'^2}{r'} - \frac{T}{m'} - \frac{dr'^2}{r'dt^2} = 0, \text{ and}$$

$$T = \frac{m\,m'}{m+m'}\left[g\left(\frac{y}{r}+\frac{y'}{r'}\right) + \frac{v^2}{r} + \frac{v'^2}{r'} - \frac{dr^2}{rdt^2} - \frac{dr'^2}{r'dt^2}\right]. \quad (2)$$

Since $\frac{1}{\rho} = \left(\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}\right) \div v^3$, $\frac{1}{\rho} = \left\{ -\frac{m'y}{(m+m')r}\left[g\left(\frac{y}{r}+\frac{y'}{r'}\right)\right.\right.$

$$\left.\left.+\frac{v^2}{r}+\frac{v'^2}{r'}-\frac{dr^2}{rdt^2}-\frac{dr'^2}{r'dt^2}\right]\frac{dx}{dt} + \frac{m'x}{(m+m')r}\left[g\left(\frac{y}{r}+\frac{y'}{r'}\right)+\frac{v^2}{r}+\frac{v'^2}{r'}-\frac{dr^2}{rdt^2}-\frac{dr'^2}{r'dt^2}\right] \times \frac{dy}{dt}\right\} \div v^3.$$

When $t=0$, y and y' each is also zero. Hence by (1) and (2) the initial tatio of the forces, per unit of mass, on m and m', is unity, and therefore, since the bodies start from rest, the initial ratios, $v \div v' = 1$, $dy \div ds = 1$, $dx \div ds = -y \div r$, $x \div r = 1$, $y \div v^2 = 1 \div 2g$, $y \div y' = 1$, $dr^2 \div ds^2 = dr'^2 \div ds'^2 = 0$, and $\frac{1}{\rho} = -\frac{1}{2a} + \frac{3m'}{2(m+m')}\left(\frac{1}{a}+\frac{1}{a'}\right) = \frac{m'}{m+m'}\left(\frac{1}{a}+\frac{1}{a'}\right)$, for its initial val.

In like manner $\frac{1}{\rho'} = \frac{m}{m+m'}\left(\frac{1}{a}+\frac{1}{a'}\right)$. Therefore

$$\frac{m}{\rho} = \frac{m'}{\rho'}, \text{ and } \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{a} + \frac{1}{a'}.$$

NOTE ON THE SOLUTION OF PROB. 374, BY R. S. WOODWARD.—That the probable error in a system of errors of constant probability between the limits $\pm a$ is $\pm \frac{1}{2}a$, or one-half the maximum error, would appear to follow directly from the definition of the probable error, viz., that error which is as likely to be exceeded as not. The proof ordinarily given is this:

Let $\varphi(\varepsilon)$ = probability of error ε, and $\varphi(+\varepsilon)=\varphi(-\varepsilon)$. Then if all errors are included between the limits $+a$ and $-a$, and $\varphi(\varepsilon)=c$, a constant,

$$\int_{-a}^{+a}\varphi(\varepsilon)d\varepsilon=\int_{-a}^{+a}cd\varepsilon=1;\ \text{whence}\ c=\frac{1}{2a}.$$

The probable error for such a system of errors is the limit ρ determined by the equation $\int_{-\rho}^{+\rho}cd\varepsilon=\frac{1}{2}$. This gives $\rho=\frac{1}{4c}=\frac{a}{2}$.

That the probable error of an interpolated value is correctly given by the formula $0.25[1-2(\frac{t}{10}-\frac{t^2}{100})]^{\frac{1}{2}}$ may, however, be questioned. This expression is based on two assumptions, viz., 1st, that the errors of the tabular values are independent, 2nd, that the law of combination of these errors is the ordinary law of error.

The result, $\frac{3}{8}(0.25)^2$, for the probable error of an interpolated value is given without proof by F. G. W. Struve in *Arc Du Méridian, Tome I*, p. 94, with reference to 7 place logarithms.

Bessel, in *Astronomische Nachrichten*, No. 529, has also investigated this problem as applied to logarithms. His investigation assumes that the errors of the tabular values are independent; and the probable error of the interpolated value is made to depend on the special law of error which obtains in case two independent errors, each of constant probability between its limits of variation, conspire. Bessel's formulas make the probable error of an interpolated value less always than that of a tabular value.

[Because, in *writing* the interpolated value, there may be a maximum error of .5 of a unit of the last place figure, even when its calculated value is *exact*, and because the tabular values, from which the interpolated value is determined, may have a maximum error of .5 of a unit of the last place figure, which will, in general, cause the interpolated value to be in error as calculated which error may conspire with the error of writing; therefore the interpolated value, when incorporated into the table, must have a greater probable error than the tabular values from which it is deduced.—ED.]

NOTE ON THE SOLUTION OF 372, BY PROF. D. V. WOOD.—There are two positions of the state bordering on motion—one in which the force μR acts up the plane; the other in which it acts down. Hence we find

$$\sin\theta=\frac{8\mu\sin\beta}{3\sin(\alpha+\beta)\pm3\mu\cos(\alpha+\beta)}.$$

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in No. 1, have been received as follows:—

From R. J. Adcock, 386; Marcus Baker, 376; Alex. S. Christie, 375, 377, 379, 382; Prof. L. G. Barbour, 386; Prof. W. P Casey, 375, 376, 377, 279, 381; Prof. H. T. Eddy, 384, 385, 386; Dr. Eggers, 376; Prof. E. J. Edmunds, 375, 377. 380; George Eastwood, 386; H. B. Goodnow, 384; Prof. A. Hall, 384; H. Heaton, 376, 380, 382, 383, 384, 385; T. N. Haun, 375; W. E. Heal, 379, 380, 383, 384; Dr. A. B. Nelson, 375, 376, 377, 380, 381, 382; Paul Peltier, 379; Prof. E. B. Seitz, 375, 376, 377, 379, 380, 382, 383; Prof. J. Scheffer, 375, 376, 377, 380, 383; Prof. D. V. Wood, 380; E. Vansickel, Jr., 375; Alexander Ziwet, 379.

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375. *By W. B. Bates.*—"*A* and *B* enter into partnership and gain $200. Now six times *A*'s accumulated stock (capital and profit) equals five times *B*'s original stock, and six times *B*'s profit exceeds *A*'s original stock by $200. Required the original stock of each."

SOLUTION BY T. N. HAUN, GREENVILLE, E. TENNESSEE.

Let x = A's original stock. Then, by conditions of the problem.

$$\tfrac{1}{6}(x+200) = \text{B's gain};$$

$$\therefore\ 200 - \tfrac{1}{6}(x+200) = \tfrac{1}{6}(1000-x) = \text{A's gain, and}$$

$$6[x+\tfrac{1}{6}(1000-x)]\div 5 = x+200 \qquad = \text{B's original stock.}$$

Since their gains are proportional to their original stock, we have

$$x : x+200 :: \tfrac{1}{6}(1000-x) : \tfrac{1}{6}(x+200),$$

or

$$x : x+200 :: 1000-x : x+200;$$

$$\therefore (x+200)x = (1000-x)(x+200).$$

Whence we get $x = \$500$, A's stock, and

$$500+200 = \$700, \text{ B's stock.}$$

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376. *By Dr. Eggers.*—"Divide a right angle into three parts, such that the tangents of the several angles are proportional to three given numbers."

SOLUTION BY PROF. J. SCHEFFER.

Denoting two of the angles by α and β, and the given ratio by $m : n : p$, we obtain from the proportions

$$\tan\alpha : \tan\beta \qquad = m : n,$$

$$\tan\alpha : \cot(\alpha+\beta) = m : p,$$

$$\tan\alpha = \frac{m}{\sqrt{(mn+mp+np)}},\ \tan\beta = \frac{n}{\sqrt{(mn+mp+np)}},$$
$$\tan\gamma = \frac{p}{\sqrt{(mn+mp+np)}}.$$

If, therefore, we construct a triangle, the radii of the escribed circles of which are equal to m, n, p, the half-angles of this triangle will be the req'd parts; for, denote the radii of the escribed circles by r_1, r_2, r_3, we have the relation $\tan\frac{1}{2}A = r_1 \div \sqrt{(r_1r_2+r_1r_3+r_2r_3)}$.

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377. *By W. E. Heal.*—"If the equations,

$$x^2 + a\,x + b = 0$$
$$x^2 + a_1x + b_1 = 0,$$

have a common root, find the remaining roots."

SOLUTION BY PROF. E. B. SEITZ.

Let m, r be the roots of the first equation, and n, r those of the second equation. Then, by the theory of equations, $m+r = -a$ (1), $mr = b$ (2), $n+r = -a_1$, (3), and $nr = b_1$ (4).

Dividing the diff. of (2) and (4) by the diff. of (1) and (3), we find

$$r = -\frac{b-b_1}{a-a_1}.$$

Substituting the value of r in (1) and (3), we find

$$m = \frac{b-b_1}{a-a_1} - a, \text{ and } n = \frac{b-b_1}{a-a_1} - a_1.$$

The condition that the equations have a common root is $(b-b_1)^2 = (a-a_1)(a_1b-ab_1)$.

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378. No solution received. Prof. Casey says the proposition is not true.

379. *By Paul Peltier, A. M., Waterloo, Ill.*—"If any number of circles touch one another in one point, all their polars which correspond to a common pole, pass through a single point."

SOLUTION BY ALEXANDER ZIWET, DETROIT, MICH.

Take the common tangent of the circles for the axis of y, and the diameter to the point of contact for the axis of x; then the equation of any one of the circles is $x^2+y^2 = 2rx$; hence the equation of the polar is

$$x'x + y'y - r(x + x') = 0.$$

The lines represented by this equation for different values of r, i. e., the polars of the point (x', y') for the different circles, will all pass through the intersection of the lines $xx' + yy' = 0$, $x + x' = 0$, of which the former is perpendicular to the line joining (x', y') to the point of contact of the circles.

The problem is a special case of a more general problem, solved in Geo. Salmon's Conic Sections, Chapt. IX; as the common tangent may be regarded as the radical axis of the circles.

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380. *By Lieut. Chas. A. Stone, U. S. Naval Acad., Ann., Md.*—"Find the equation of the curve in which the tangent of the angle which the tangent line makes with the axis of X, increases proportionally to the length of the curve."

SOLUTION BY PROF. DE VOLSON WOOD.

We have $\frac{dy}{dx} = as$; $\therefore\ \frac{d^2y}{dx^2} = a\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}$, and

$$d\left(\frac{dy}{dx}\right) \div \sqrt{\left(1+\frac{dy^2}{dx^2}\right)} = ax.$$

Integrating

$$\log\left[\frac{dy}{dx} + \sqrt{\left(1+\frac{dy^2}{dx^2}\right)}\right] = ax + (C = 0).$$

Passing to exponentials, clearing of radicals, &c., and integrating, gives

$$y = \frac{1}{2a}\left(e^{ax} + e^{-ax}\right) + C,$$

which is the equation of the Catenary.

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381. *By Prof. W. P. Casey.*—"To find a point in a given line so that the rectangle contained by two lines drawn to it from two given points may be given or a minimum (without the aid of the Cassinian Ovals)."

SOLUTION BY PROF. W. P. CASEY.

Let FD be the given line, A and B the given p'ts and C the req'd point.

Analysis.— Draw AH perp. to FB; $\therefore$ H is a given point. Join BH, it is a given line, and its middle point L, is a given point, and the perpend. LY to HB is in position; $\therefore$ Y is a given point. Draw CM perp. to BH and CK perp. to AC meeting AH produced in K, and make $AH \times HS = 2HB \times LM$, draw SN perp. to AK meeting the perp. CN to FD in the p't N; join NY and produce it to meet the perp's AO, KR to AK in O and R.

Now $CH^2 - CB^2 = HM^2 - MB^2 = 2HB \times LM = AH \times HS$; $HC^2 - AH \times HS = CB^2$, that is, $AH \times SK = CB^2$, $\therefore$ $AC^2.AH.SK = AC^2.CB^2$,

which is given, $\therefore AC^2 \times AH \times SK$ is given and AH is a given line, $\therefore AC^2 \times SK$ is given, or $AH \times AK \times SK$ is given, whence $AK \times SK$ is given; but as $AH \times HS = 2HB \times LM$; $\therefore AH : 2HB :: LM : HS$ or CN a giv'n ratio, and $HL : HY :: LM : YC$ a given ratio; $\therefore$ the ratio of $NC : CY$ is given and the angle NCY is a right angle; whence the $\angle$s NYC, YNC are given, $\therefore NY$ is in position, and the p't O is given. The ratio of $NP : PR$ is also given being the same as $AC : CY$, $\therefore AK \times NP : AK \times PR$ is given; but $AK \times NP$ or SK has been shown to be given $\therefore AK \times PR$ is given, or the parallelogram $XORP$ is given, hence the locus of P is an hyperbola PZ, whose asymptotes are OX, OR, and $AH \times HK = KP^2$, i. e., a given line multiplied by $HK = KP^2$, $\therefore$ the locus of P is a parabola PH, hence P is a given point, and $\therefore$ the perp. PC is in position and C is a given point. When the two curves become tang't there will be only one point from which a perp. to FB can be drawn, and in that case the product will be a minimum. There are two points when the curves intersect. The construction is obvious from the analyis.

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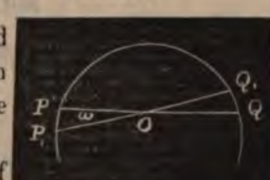
382. *By Thomas Spencer.* — "Prove in general that the chord drawn though a given point so as to cut off the minimum area from a given curve is bisected at that point."

SOLUTION BY ALX. S. CHRISTIE, U. S. COAST SURVEY.



Let $P_1 P Q_1 Q$ be any given curve, PQ a chord through the given point O cutting off minimum area on one side of the chord, and, if the curve be closed, a maximum area on the other side.

Then taking $P_1 O Q_1$ a consecutive position of the chord, ω infinitesimal, the areas OPP_1, OQQ_1 are, to terms of the 1st order, equal to $OP.OP_1.\omega$, $OQ.OQ_1.\omega$, respectively. But to the same order $OP = OP_1$, $OQ = OQ_1$; hence the element'y areas are $(OP)^2.\omega$, $(OQ)^2 \times \omega$; and for either a maximum or a minimum these areas are equal, or

$$(OP)^2.\omega = (OQ)^2.\omega, \therefore OP = OQ.$$

383.—*By Prof. Edmunds.*—"Solve and discuss:

$$\begin{cases} x^2 + y^2 = a^2, \\ \log x + \log y = n." \end{cases}$$

SOLUTION BY DR. A. B. NELSON.

Writing the second equation in the form $xy = e^n$, we readily find

$$x = \tfrac{1}{2}[\pm\sqrt{(a^2+2e^n)}\pm\sqrt{(a^2-2e^n)}],$$
$$y = \tfrac{1}{2}[\pm\sqrt{(a^2+2e^n)}\mp\sqrt{(a^2-2e^n)}].$$

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384. *By Prof. Asaph Hall.*—"Show that

$$\int_0^a dx\int_0^x \varphi(x, y)\,.\,dy = \int_0^a dy\int_y^a \varphi(x, y)dx.$$

(Dirichlet's theorem.)"

SOLUTION BY PROF. H. T. EDDY.

Consider the isosceles right angled triangle enclosed by the lines

$$x = y,\ x = a,\ y = 0.$$

The first member of the given equation extends the integration over area of this triangle by finding first an elementary strip parallel to y bet' $y = 0$ and $y = x$, and, secondly, taking sum of such strips between $x =$ and $x = a$.

The second member proposes to take first an elementary strip paralle x between $x = y$ and $x = a$, and then sum all such strips between $y =$ and $y = a$. These summations are evidently equal.

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385. *Selected by Prof. H. T. Eddy.*—"Show that

$$\int_{-\infty}^{+\infty} e^{\left(-x^2\cos 2\theta+\frac{a^2}{2x^2}\sin 2\theta\right)} \begin{matrix}\cos\\ \sin\end{matrix}\left[x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right]dx$$
$$= \pi^{\frac{1}{2}}e^{-a}\begin{matrix}\cos\\ \sin\end{matrix}\left[\theta + a\right]."$$

SOLUTION BY HENRY HEATON.

We have given the definite integral

$$\int_{-\infty}^{+\infty} e^{-(y^2+c^2y^2)}dy = \sqrt{\pi}.e^{-2c}.$$ (See Todhunter's Int. Calc. Art. 2

Put $y = x(\cos\theta+i\sin\theta)$ and $c = \frac{1}{2}a(1+i)$, where $i = \sqrt{-1}$. The

$$\int_{-\infty}^{+\infty} e^{-[x^2\cos 2\theta+x^2 i\sin 2\theta+(a^2 i+2x^2)\cos 2\theta+(a^2+2x^2)\sin 2\theta]}dx$$
$$= \sqrt{\pi}.e^{-a}e^{-ia}(\cos\theta-i\sin\theta$$

But $e^{-i[x^2\sin 2\theta+(a^2\div 2x^2)\cos\theta]} = \cos[x^2\sin 2\theta+(a^2\div 2x^2)\cos 2\theta]$
$-\,i\sin[x^2\sin 2\theta+(a^2\div 2x^2)\cos 2\theta$, and
$e^{-ia} = \cos a - i\sin a$. Hence

$$\int_{-\infty}^{+\infty} e^{-[x^2\cos 2\theta+(a^2\div 2x^2)\sin 2\theta]}\left\{\cos\left(x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right)\right.$$

$$\left.-i\sin\left(x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right)\right\}dx = \sqrt{\pi}.e^{-a}[(\cos\theta\cos a - \sin\theta\sin a) -$$

$i(\cos\theta\sin a+\sin\theta\cos a)]$. Separating the possible and impos. p'ts we have

$$\int_{-\infty}^{+\infty} e^{-[x^2\cos 2\theta+(a^2\div 2x^2)\sin 2\theta]}\cos\left(x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right)dx = \pi^{1/2}e^{-a}\cos(\theta+a),$$

$$\int_{-\infty}^{+\infty} e^{-[x^2\cos 2\theta+(a^2\div 2x^2)\sin 2\theta]}\sin\left(x^2\sin 2\theta+\frac{a^2}{2x^2}\cos 2\theta\right)dx = \pi^{1/2}e^{-a}\sin(\theta+a).$$

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386. *By George Eastwood.*—"Integrate the equation

$$\frac{d^2\phi}{dt^2}\cdot\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dt}\cdot\frac{d\phi}{dx}\right)^2 = 0."$$

SOLUTION BY PROF. L. G. BARBOUR.

Let $\frac{d\phi}{dt} = p$, $\frac{d\phi}{dx} = q$; then is $\frac{dp}{dt}\cdot\frac{dq}{dx} = p^2q^2$. It is allowable and suff. to write $\frac{dp}{dt} = p^2$, and $\frac{dq}{dx} = q^2$, $\therefore$ $\frac{dp}{p^2} = dt$; $-\frac{1}{p} = t+c$; $d\phi = -\frac{dt}{t+c}$ $\phi = -\log(t+c)+\log c'$, $\therefore$ $e^{\phi} = c'\div(t+c) = c'''\div(x+c''$, $\therefore$ $x = tc_4+c_5$.

PROBLEMS.

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387. *By Prof. L. G. Barbour.*—Given the length of each side of any quadrilateral, and the distance from the middle point of any side to that of the side opposite. Required the distance from the middle point of one of the other sides to that of the side opposite.

388. *By Prof M. L. Comstock.*—F and F' being the foci of an ellipse and P a point on the curve, FD is drawn perpendic'r to FP meeting $F'P$ in D. Find the locus of D: (1) when $b > c$, (2) when $b = c$, (3) when $b < c$, if b = semi-minor axis and c = distance from the centre to either focus.

389. *By Prof. W. W. Johnson.*—If three triangles have a common axis of homology when taken in pairs, the three centres of homology are in a straight line: and reciprocally if three triangles have a common centre of homology when taken in pairs, the three axes of homology pass through a common point.

390. *By Prof. W. P. Casey.* — In a triangle ABC, BD is perpendic'lr to the base AC, and O is the center of gravity of the triangle. Join AO, DO and CO. Given the base AC and the $\angle$s AOD, AOC, to construct the triangle ABC.

391. *By Prof. Asaph Hall.*— Given

$$\log. 91 = 1.95904 \pm r,$$
$$\log. 92 = 1.96379 \pm r,$$

find log. 91.5 to five decimals, by simple proportion from the difference; and find the probable error of this logarithm.

Query by Prof. H. T. Eddy. — When two determinants of the same order have the same algebraic value, show whether it is always possible to transform the one into the other by mere combinations of rows and colu'ns; and if possible transform the two following values of $2bc\cos A - b^2 - c^2$, the one into the other:

$$\begin{vmatrix} 0, & b, & c, \\ b, & 1, & \cos A, \\ c, & \cos A, & 1, \end{vmatrix}, \quad \begin{vmatrix} 2bc\cos A, & b, & c, \\ b, & 1, & 0, \\ c, & 0, & 1, \end{vmatrix}$$

Correction of "Barlow's Tables of the Squares and Cubes of Numb's", De Morgan's Edition. London. 1875. Communicated by Prof. A. Hall.
Page 42: for the square of 2059, instead of 4230481 read 4239481.

PUBLICATIONS RECEIVED.

A List of Writings on Determinants. By Thomas Muir, M. A., F. R. S. E. 50 pages. 8vo. [Extracted from *The Quarterly Journal of Pure and App'd Math.* Vol. XVIII, No. 70.]

Four-place Tables. By W. Beebe, Yale College. *A Pocket Edition.* Retail price 25 cents. Henry H. Peck, Publisher, New Haven, Conn.

The Mathematical Magazine: A Journal of Elementary Mathematics. Edited and bublish'd by Artemas Martin, M. A. Erie, Pa. 16 pp. 4to. Quarterly. $1.00 per year.

ERRATA.

In Index to Vol. VIII, line 17, for "3, 73", read 3, 137.
On page 41, line 5, for "Mr. Stockwell", read Mr, Stockwell.

THE ANALYST.

VOL. IX. MAY, 1882. No. 3.

LAW OF ERROR IN THE POSITION OF A POINT IN SPACE.

BY E. L. DE FOREST.

[Continued from page 40.]

WHEN the positions of the coefficients in the given polynomial and its expansion are referred to any arbitrarily chosen rectangular axes, the differential equations (15) will still be those of the limiting function sought, provided that the constants α, β, &c., are understood to refer to the axes thus adopted, each L receiving new sub-indices equal to its new coordinates, expressed in the unit of measure $\Delta x = \Delta y = \Delta z$. For convenience, we shall henceforth assume the free axes of the system of weights L as the coordinate axes, because this, as already stated, reduces α_1, α_2, α_3 and γ_1, γ_2, γ_3 to zero, so that the equations (15) take the simple form

$$\beta_1 d_x w = \left(\frac{-i}{n+1}\right) w, \quad \beta_2 d_y w = \left(\frac{-j}{n+1}\right) w, \quad \beta_3 d_z w = \left(\frac{-k}{n+1}\right) w. \quad (19)$$

Substituting for i, j, k their equivalents from (13), and using n instead of $n+1$, which we may do because n is infinite, we get

$$\frac{d_x w}{w} = \frac{-x dx}{n\beta_1 (dx)^2}, \qquad \frac{d_y w}{w} = \frac{-y dy}{n\beta_2 (dy)^2}, \qquad \frac{d_z w}{w} = \frac{-z\, dz}{n\beta_3 (dz)^2}. \quad (20)$$

If we also write

$$r_1^2 = \beta_1 (dx)^2, \qquad r_2^2 = \beta_2 (dy)^2, \qquad r_3^2 = \beta_3 (dz)^2, \quad (21)$$

$$h_1^2 = 1 \div 2nr_1^2, \qquad h_2^2 = 1 \div 2nr_2^2, \qquad h_3^2 = 1 \div 2nr_3^2, \quad (22)$$

then (20) will stand

$$\frac{d_x w}{w} = -2h_1^2 x dx, \quad \frac{d_y w}{w} = -2h_2^2 y dy, \quad \frac{d_z w}{w} = -2h_3^2 z dz. \quad (23)$$

From the significations of β_1, β_2, β_3, as stated in connection with (15), and the fact that $\Sigma L = 1$, it appears that r_1, r_2, r_3 in (21) represent the "radii of gyration" of the system of coefficients or masses L with respect to the free-axial planes of YZ, XZ and XY, in the sense in which that nomenclature was used by me in ANALYST, May, '80, p. 80.

The squared radius of gyration of a system of masses of material points, with respect to a plane, was there defined as the result obtained by multiplying the mass of each point into the square of its perpendicular distance from the plane, adding all the products together, and dividing their sum by the sum of the masses. The squared radius of such a system with respect to a line or point is found in like manner, using the distance of each mass from the line or point. When the points and their masses represent points of error and their probabilities, the radius of gyration of the system of masses with respect to a plane, a line, or a point, is identical with the *quadratic mean error* or q. m. deviation of the syst. of points, from such plane, line or point. It will be seen that in forming the mean of the squares of the errors, each error is taken with a weight proportional to the probability of its occurrence. It was shown in my article that when two entire polynomials and their product are arranged each in block form, and coordinate planes are passed through the centre of gravity of the eoefficients in each and parallel to the sides of the block, the square of the radius of gyration of the system of coefficients in the product, with respect to any one of these coordinate planes, is equal to the sum of the squares of the radii for the two factors, with respect to their corresponding planes. It is not material, however, whether the rectangular coordinate planes are parallel to the sides of the block, or whether they have other directions, for, as we have seen, the system of coefficients in a block-formed entire polynomial can be replaced, to any desired degree of approximation, by those of another block whose centre and sides have any required position and directions, the intervals being supposed equal and as small as we please, while the exponents are changed so that in any term they shall represent the number of new intervals from the coefficient of that term to the coordinate planes respectively.

A relation between distances, such as lever arms or radii of gyration, holds true without regard to the magnitude of the common unit employed to measure such distances. If any two polynomials have had their coordinate planes of reference changed as above, the coefficients in their product are not changed either in magnitude, or in their relative positions when set so that the coordinates of each are represented by the corresponding exponents.

If the axes in each factor are taken through the centre of gravity of the coefficients, the origin or place of exponent zero in the product will be the centre of gravity of its system of coefficients. The proposition then holds good, that the square of the radius of gyration of the system of coefficients in the product, with respect to any one of the coordinate planes, is equal to the sum of the squares of the corresponding radii in the two factors. If such a polynomial is raised to the nth power, the square of the radius of gy-

ration of the coefficients in the expansion, with respect to any one of the coordinate planes, is n times the square of the corresponding radius in the first power. This shows that in (22), nr_1^2, nr_2^2 and nr_3^2 represent the squared radii of gyration of the system of coefficients l in the expansion, with respect to the planes of YZ, XZ and XY. In other words,

$$\varepsilon_1 = r_1\sqrt{n}, \qquad \varepsilon_2 = r_2\sqrt{n}, \qquad \varepsilon_3 = r_3\sqrt{n}, \tag{24}$$

are the quadratic mean errors or q. m. deviations of the coefficients in the expansion, from the coordinate planes which pass through the free axes.

When two polynomials referred to rectangular planes through their centres of gravity are multiplied together, their product being likewise referred, the sum of the squares of the two radii of gyration of the system of coefficients in the product, with respect to the ZX and the ZY planes, is equal to the sum of the squares of the radii in the two factors, with respect to the same two planes in each. But the square of the distance of any point from the axis of Z is equal to the sum of the squares of its distances from the ZX and ZY planes, so that the square of the radius of gyration of any system of coeff's, about the Z axis, is equal to the sum of the squares of its radii with respect to the ZX and ZY planes. Hence the square of the radius of gyration of the coeff's in the product of the two polynomials, about the Z axis of the product, is equal to the sum of the squares of the radii for the two factors, about their Z axes. The like will evidently be true for radii about the X or Y axes. If a polynomial thus referred is raised to the nth power, the square of the radius of gyration for the product, about any one of the coordinate axes, is n times the square of the radius for the first power, about the same axis. It is a known mechanical property of the free axes, in bodies of three dimensions, that out of all axes of rotation which can be passed through the centre of gravity, the one which renders the radius of gyration a minimum is one of the free axes, and the one which renders it a maxim'm is another free axis. If therefore a given polynomial is raised to successive powers, making $n = 2$, $n = 3$, &c., and the polynomial and its powers are all referred to the same rectangular axes arbitrarily taken through the centre of gravity of the coefficients in the first power, then in the nth power the centre of gravity remains at the origin, and the radius of gyration about each axis is $\sqrt{n}$ times what it was in the first power, for any given directions of the axes, so that it is still a minimum when taken about one of the free axes of the first power, and a maximum about another. Hence, the free axes retain a constant position in all the successive powers. The free axes of the nth system of coefficients are the same as those of the first, and the squares of the radii of gyration about them are

$$n(r_2^2+r_3^2), \qquad n(r_1^2+r_3^2), \qquad n(r_1^2+r_2^2). \tag{25}$$

The squared radius of gyration of the coefficients in the nth power, with respect to the origin or centre of gravity, is

$$E^2 = n(r_1^2 + r_2^2 + r_3^2). \tag{26}$$

We now proceed to integrate the eq's (23), and find that the func. sought is

$$w = ce^{-(h_1^2x^2+h_2^2y^2+h_3^2z^2)}, \tag{27}$$

from which (23) can be derived by differentiating with respect to the variables separately, and dividing by w. To find the value of c we consider that since the sum of the coefficients in any power of the polynomial is unity, we shall have at the limit,

$$\frac{1}{dx\,dy\,dz}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w\,dx\,dy\,dz = 1, \tag{28}$$

which is equivalent to

$$\frac{c}{dx\,dy\,dz}\int_{-\infty}^{+\infty} e^{-h_1^2x^2}\,dx \int_{-\infty}^{+\infty} e^{-h_2^2y^2}\,dy \int_{-\infty}^{+\infty} e^{-h_3^2z^2}\,dz = 1.$$

The known values of the three definite integrals here are

$$\sqrt{\pi} \div h_1, \qquad \sqrt{\pi} \div h_2, \qquad \sqrt{\pi} \div h_3,$$

so that we have

$$\frac{c}{dx\,dy\,dz}\left(\frac{\pi^{\frac{3}{2}}}{h_1h_2h_3}\right) = 1, \tag{29}$$

which determines c, and the complete equation of the limiting funct. stands

$$w = \frac{h_1h_2h_3\,dx\,dy\,dz}{\pi^{\frac{3}{2}}}\,e^{-(h_1^2x^2+h_2^2y^2+h_3^2z^2)}. \tag{30}$$

If we assign any constant value to w, the exponent of e becomes constant, so that

$$h_1^2x^2 + h_2^2y^2 + h_3^2z^2 = C. \tag{31}$$

This is the equation of an ellipsoid whose centre and axes coincide with the origin and coordinate axes. The axes of the ellipsoid lying in the x, y, z directions are inversely proport'al to h_1, h_2, h_3, and therefore by (22) are directly propor'l to r_1, r_2, r_3. Since any change in w, and consequently in C, causes all three axes to vary in like proportion, it appears that the loci of points of error whose probabilities w are the same, are ellipsoidal surfaces which are all similar and similarly situated and concentric. From their symmetry with respect to the coordinate axes, it follows that these are the free axes of the system of expanded coefficients or probabilities w, regarded as masses in space, so that the given probabilities L, and the resultant probabilities w, have both the same free axes. This accords with what has been already found in connection with (25). The value of w is a maximum at the origin, which is the centre of gravity of the whole system of masses w, as it also is that of the masses L. The function (30) represents the law of

facility of deviation of the centre of gravity of a large number n of similarly observed points of error, from its most probable place. For w is the probability that the sums of the deviations in the x, y, z directions will be $x = idx$, $y = jdy$, $z = kdz$ respectively, and this is the probability that their arithmetical means will be $x \div n$, $y \div n$, $z \div n$ respectively.

If all the possible points of error are set closer together, so that the intervals dx, dy, dz are reduced to $dx \div n$, $dy \div n$, $dz \div n$, the distribution of the probabilities or masses w will represent the law of facility of deviation of the mean position or centre of gravity of the n observed points, from its most probable place. Thus, whatever may be the actual law for a single observed point, as expressed by the distribution of the probabilities L in (3), the law of facility of error of the centre of gravity of a large number n of similarly observed points is always expressed by the same exponential form of function as in (30). This function, then, seems to be a typical form of the law of probability of errors in space, and may reasonably be assumed as the most plausible law for all such observations, in the absence of any previous knowledge to the contrary. Although the law has hardly any direct practical applications, it is important in theory. Its properties are quite analogous to those of the law of errors in two dimensions, as discussed by me in ANALYST, May, '81. Indeed the law for three dimensions is the general law of probability, which includes the laws for one and two dimensions as special cases under it.

If an unlimited number of equidistant planes are supposed to be drawn parallel to the coordinate planes, dividing space into elementary cubes $dxdy dz$, so situated that the origin is at the middle of one of them, then (30) may be regarded as giving the probability w that a point of error which occurs will fall within the cube whose centre is at the point x, y, z.

The like holds true approximately when any arbitrary but small finite values are assigned to dx, dy and dz, for the probability that an error will fall within any small space whose location is given is approximately proportional to the size of that space. From observed positions of a considerable number of points of error, regarded as material points whose masses are equal, we can find the positions of the free axes of the system of points, and then compute the quadratic mean deviation of the system from each of the free-axial coordinate planes. These are approximately the values of ε_1, ε_2, ε_3 in (24), from which with (22) we find

$$h_1^2 = 1 \div 2\varepsilon_1^2, \qquad h_2^2 = 1 \div 2\varepsilon_2^2, \qquad h_3^2 = 1 \div 2\varepsilon_3^2. \qquad (32)$$

It is not material what values are assigned to dx, dy and dz, nor whether they are taken equal to each other, only being represented as infinitesimals in the formula, they ought always to be quite small in comparison with the q. m. deviations ε_1 &c., which are finite. If the probability of deviation

from the most probable point is considered to be the same in all directions, we have $\epsilon_1 = \epsilon_2 = \epsilon_3$, and by (24) and (26) the q. m. deviation measured directly from the origin is

$$E = \epsilon_1 \sqrt{3}. \tag{33}$$

Consequently by (32),

$$h_1^2 = h_2^2 = h_3^2 = 1 \div \tfrac{2}{3} E^2, \tag{34}$$

and the function (30) becomes

$$w = dxdydz \left(\frac{h_1}{\sqrt{\pi}} \right)^3 e^{-h_1^2 r^2}, \tag{35}$$

where r is the radius vector or distance from the origin to the centre of the space $dxdydz$, and h_1 is computed by (34), from the observed value of E.

Returning now to the most general form of the function, let (30) be referred to polar coordinates by taking

$$x = r \cos\theta \cos\varphi, \qquad y = r \cos\theta \sin\varphi, \qquad z = r \sin\theta, \tag{36}$$

where θ is the angle which r makes with the XY plane and φ is the angle which its projection on that plane makes with the X axis. Then if we put

$$a^2 = h_1^2 \cos^2\theta \cos^2\varphi + h_2^2 \cos^2\theta \sin^2\varphi + h_3^2 \sin^2\theta, \tag{37}$$

(30) becomes

$$w = \frac{h_1 h_2 h_3 dxdydz}{\pi^{\frac{3}{2}}} e^{-a^2 r^2}. \tag{38}$$

This expresses the facility of error at any point along an unlimited radius vector whose direction is given. From the origin as a centre, suppose a series of spherical surfaces to be described at infinitely small intervals eq'l to

$$(dx\, dy\, dz)^{\frac{1}{3}}.$$

Let the origin be made the apex of a square pyramid whose axis coincides with r, and whose base, or intersection with the spherical surface of unit radius, is the infinitesimal $(d\psi)^2$, so that if the pyramid is extended to the dist. r its base is $(rd\psi)^2$. The section of the pyramid included between consecutive spherical surfaces at distance r is

$$(rd\psi)^2 (dx\, dy\, dz)^{\frac{1}{3}},$$

and $dxdydz$ is contained in this $(rd\psi)^2 \div (dx\, dy\, dz)^{\frac{2}{3}}$ times, so that the probability that a point of error which occurs will fall within this section of the pyramid, is

$$w (rd\psi)^2 \div (dxdydz)^{\frac{2}{3}},$$

and the probability that it will fall anywhere within the pyramid whose height is r will be

$$p = \frac{1}{dr} \int_0^r w \left(\frac{rd\psi}{(dx\, dydz)^{\frac{1}{3}}} \right)^2 dr,$$

where $dr = (dxdydz)^{\frac{1}{3}}$. Taking w from (38) we get

$$p = \frac{h_1 h_2 h_3 (d\psi)^2}{\pi^{\frac{3}{2}}} \int_0^r e^{-a^2 r^2} r^2 dr. \tag{39}$$

The probability P that the error will fall within the pyramidal space extended to infinity, is found by taking ∞ as the upper limit of the integral. If now it is required that the probability p for the finite space shall be a given fraction ν of the prob. P for the infinite space, we have the condition

$$\int_0^r e^{-a^2r^2}r^2dr \div \int_0^\infty e^{-a^2r^2}r^2dr = \nu. \tag{40}$$

The known value of the integral in the denominator is $\sqrt{\pi} \div 4a^3$. As for the numerator, we have

$$e^{-a^2r^2} = 1-\tfrac{1}{1}(ar)^2+\tfrac{1}{1.2}(ar)^4-\tfrac{1}{1.2.3}(ar)^6+\&c.,$$

$$\int_0^r e^{-a^2r^2}r^2dr = r^3\left\{\frac{1}{3}-\frac{1}{5.1}(ar)^2+\frac{1}{7.1.2}(ar)^4-\frac{1}{9.1.2.3}(ar)^6+\&c.\right\}.$$

Hence (40) becomes

$$\frac{4(ar)^3}{\sqrt{\pi}}\left\{\frac{1}{3}-\frac{1}{5.1}(ar)^2+\frac{1}{7.1.2}(ar)^4-\frac{1}{9.1.2.3}(ar)^6+\&c.\right\} = \nu. \tag{41}$$

If any particular value is assigned to ν, we have a numerical equation containing only one unknown quantity ar, the value of which may be found by known methods. If for instance we wish to find a limit such that it is an even chance whether the point of error will fall within or without it, taking $\nu = \frac{1}{2}$, and using 12 terms of the series, (41) gives

$$ar = 1.087652,$$

the last decimal figure being doubtful. This determines the distance r of the desired limit from the origin, in any given direction which determines a. To find the locus of the limit for all directions, since $-(ar)^2$ is the exponent of e in (30), we have

$$h_1^2x^2+h_2^2y^2+h_3^2z^2 = (1.087652)^2,$$

and assigning to h_1, h_2, h_3 their equivalents from (32),

$$\left(\frac{x}{1.53817\,\varepsilon_1}\right)^2+\left(\frac{y}{1.53817\,\varepsilon_2}\right)^2+\left(\frac{z}{1.53817\,\varepsilon_3}\right)^2 = 1. \tag{42}$$

This is the equation of an ellipsoid whose semi-axes coincide with the coordinate axes, and are equal to 1.53817 times the q. m. deviations from the YZ, XZ and XY planes. It is the *ellipsoid of probable error*, since there is an even chance that any error which occurs will fall within it.

If any three or more planes are made to meet at the origin, so as to include between them any polyhedral angle, there is an even chance whether a point of error which falls within this angle will fall inside or outside of the ellipsoid. If the probability of a given deviation or error is the same in all directions, we may introduce the direct central q. m. error E from (33), and (42) becomes the equation of a sphere,

$$x^2+y^2+z^2 = (.88806\,E)^2, \tag{43}$$

whose radius $.88806\,E$ is the probable error, or probable deviation of the observed point from its most probable place.

We can now make a comparison between the relations which the probable error bears to the central q. m. error, in space of one, two, or three dimensions. In the ordinary case of one dimension, it is well known that

$$\text{prob. error} = .67449\,(\text{q. m. error}).$$

From the equation of the "ellipse of probable error" in the case of two dimensions (ANALYST, May '81, p. 77), and the fact that when errors are equally probable in any direction the q. m. deviation from the origin is $\sqrt{2}$ times the q. m. deviation from either the X or the Y axis, it follows that

$$\text{prob. error} = 1.17741\,(\text{q. m. e.}) \div \sqrt{2} = .83255\,(\text{q. m. e.}).$$

Taking these two cases in connection with (43), we see that when the probability of an error of given amount is the same in all the possible directions in any one of the three cases, the ratio of the probable error to the q. m. error will be

$$.67449, \qquad .83255, \qquad .88806,$$

according as the errors are limited to space of one, two, or three dimensions.

The occurrence of errors in three dimensions can be illustrated by the drawing of balls from an urn, each ball being marked with three numbers representing the x, y and z coordinates of a possible point of error, referred to the true point, or place of zero error, as an origin. Each ball is restored to the urn as soon as drawn, and they are all mixed so as to keep the chances the same. Knowing the number of balls of each kind, we have the prob'ty of drawing one of that kind, and all these probabilities are coefficients L in a polynomial such as (3), the numbers marked on the balls becoming the exponents of ξ, η, ζ. When n drawings are made, the probability that the sums of the x-, y-, and z- numbers drawn, will be s, t and v respectively, is the coefficient of $\xi^s\eta^t\zeta^v$ in the expansion of the polynomial to the nth power.

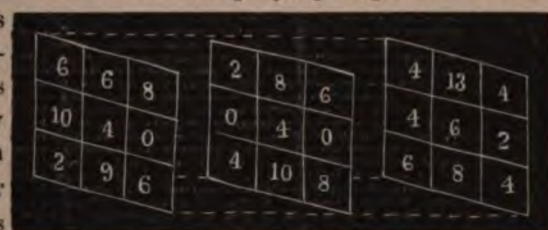
If n is a large number, the probability of any given result can be found by means of the limiting function (30). To obtain this, we should proceed in a manner analogous to that given for the case of errors in space of two dimensions, at p. 80 of my ANALYST article cited. It is easy to find the centre of gravity of the coefficients L, and their coordinates referred to axes passing through this centre and parallel to the original axes. Then known formulas such as are given in Poisson's treatise already cited enable us to calculate the positions of the free axes, or principal axes through the centre of gravity, and to find the radii of gyration r_1, r_2, r_3 of the system of coefficients L about these axes, from which h_1, h_2, h_3 are obtained by (22), and the limiting function (30) is determined. This function has its maximum value at the origin, which is the centre of gravity of the coefficients

in the expansion, and the relative position of this centre is readily found because its coordinates, referred to the original axes, are n times those of the centre of gravity of the coefficients L in the first power. By assigning the proper values to x, y, z in (30), the approximate value w of the coefficient of any term in the expansion may be found. It will be understood that $dx = dy = dz$ here represents an interval equal to the distance, in any original coordinate direction, between coefficients of terms whose corresponding exponents differ by unity. For convenience, we may reckon

$$dx = dy = dz = 1,$$

making this the unit of measure for x, y, z and r_1, r_2, r_3.

For the purpose of illustrating and verifying results contained in this paper, a polynomial of 27 terms has been constructed, whose coefficients arranged in block form are shown in the accompanying diagram. The sum of all the numbers in it is 144, so that the true coefficients L are the numbers in the diagram divided by 144, and the sum of them all is unity. For greater simplicity, the numbers have been so chosen that when regarded as the masses of material points, their centre of gravity and free axes coincide with the centre and axes of symmetry of the block. By the notation we have used, the three coefficients in the left hand column, for example, are



$$L_{-1,1,1} = \tfrac{6}{144}, \qquad L_{-1,1,0} = \tfrac{10}{144}, \qquad L_{-1,1,-1} = \tfrac{2}{144}.$$

These in the lower right hand row are

$$L_{-1,-1,-1} = \tfrac{6}{144}, \qquad L_{0,-1,-1} = \tfrac{8}{144}, \qquad L_{1,-1,-1} = \tfrac{4}{144}.$$

The whole polynomial is denoted by

$$\Sigma_{c=-1}^{c=1}\, \Sigma_{b=-1}^{b=1}\, \Sigma_{a=-1}^{a=1}\, (L_{a,b,c}\xi^a\eta^b\zeta^c).$$

When the polynomial is raised to the second power, and referred to the same axes as in the first power, we find that the coefficients, forming a block of 125 terms, have their centre of gravity and free axes coincident with the origin and axes of reference, and these are the centre and axes of symmetry of the block, just as they were in the first power. It is also found that the squared radius of gyration of the coefficients with respect to the origin, or with respect to any one of the three axes, or to any one of the three coordinate planes of reference passing through them, is twice as great in the sec'd power as it is in the first.

To obtain approximately the coefficients w in the expansion of the polynomial to any high power, say the nth, we can proceed thus. The coeff's L give

$$\beta_1 = \tfrac{76}{144}, \qquad \beta_2 = \tfrac{102}{144}, \qquad \beta_3 = \tfrac{114}{144},$$

and by (21), (22) and (13),

$$h_1 dx = 6 \div \sqrt{(38n)}, \quad h_2 dy = 6 \div \sqrt{(51n)}, \quad h_3 dz = 6 \div \sqrt{(57n)},$$
$$h_1^2 x^2 = 36i^2 \div 38n, \quad h_2^2 y^2 = 36j^2 \div 51n, \quad h_3^2 z^2 = 36k^2 \div 57n.$$

Substituting these in (30) we find

$$\log w = \bar{1}.06711 - \tfrac{3}{2}\log n - \frac{1}{n}(.41143\,i^2 + .30656\,j^2 + .27429\,k^2),$$

from which the values of w may be computed by assigning suitable values to i, j, k, which in this example are any whole numbers, either + or —. It would be interesting to have the polynomial raised algebraically to a power high enough to show the agreement between the true coefficients in the expansion, and their approximate values as given by the above formula; but this is impracticable, owing to the tedious length of the work required in forming the algebraic expansion.

The exponential function (30), regarded as the law of probability of error in space, has been reached by previous writers, but in ways quite different from ours. One of the early investigators of the law for space of two and three dimensions was Bravais, whose essay, *Sur les probabilité's des erreurs de situation d' un point*, may be found in the *Mémoires . . . par divers savans* . . . . , Inst. France, Vol. IX. (1846). His process treats the probability of error of an observed point in one direction as independent of its probability of error in another direction perpendicular to the first. This objectionable assumption, which has been made by other writers on the subject so far as I know, is avoided in our present method. (See ANALYST, May 1881, pp. 75 and 79.)

ON THE COMPUTATION OF PROBABLE ERROR.

BY T. W. WRIGHT, U. S. LAKE SURVEY, DETROIT, MICHIGAN.

IN computing the probable error of the determinations of an observed quantity two forms are in common use, Bessel's and Peters'. If with the ordinary notation we let

$m =$ the number of observations,
$r =$ the probable error of a single observation,
$r_0 =$ the probable error of the final result,
$\rho = 0.4769363$,
$v_1, v_2 \ldots =$ the residual errors of observation, $[vv]$ the sum of their squares and $[v$ their sum without reg'd to sign, then accord'g to Bessel's form,

$$r = \frac{\rho\sqrt{2}}{\sqrt{(m-1)}}\sqrt{[vv]} = \frac{0.6745}{\sqrt{(m-1)}}\sqrt{[vv]},$$

$$r_0 = \frac{\rho\sqrt{2}}{\sqrt{m(m-1)}}\sqrt{[vv]} = \frac{0.6745}{\sqrt{m(m-1)}}\sqrt{[vv]},$$

and according to Peters' form

$$r = \rho\sqrt{\left(\frac{\pi}{m(m-1)}\right)}.[v = \frac{0.8453}{\sqrt{m(m-1)}}.[v,$$

$$r_0 = \rho\sqrt{\left(\frac{\pi}{m^2(m-1)}\right)}.[v = \frac{0.8453}{m\sqrt{(m-1)}}.[v.$$

The latter form can be more rapidly computed and is usually close enough.

The labor of computing the factors $0.6745 \div \sqrt{m(m-1)}$, ... is considerable in any case. Accordingly in practice I have found it convenient to tabulate these quantities for values of m from 2 to 100. The formulæ for the probable error being now written in the simple forms

$$r = \lambda_1\sqrt{[vv]} \qquad r = \lambda'[v$$

$$r_0 = \lambda_2\sqrt{[vv]} \qquad r_0 = \lambda''[v,$$

all that we have to do is to enter the proper table with the argument m, take out the corresponding multiplier λ and perform a single multiplication.

For finding $\sqrt{[vv]}$ a close enough approximation to the square root can be taken out at sight from a table of squares, or if it be preferred the computation may be made logarithmically.

It often happens that we wish to find the probable error of a series of observations roughly and quickly. The following rule will be found convenient in such a case and also as a check on the values found by the preceding method.

A glance at the observed results will show the greatest and least, and their difference will be the *range* of the results. Then for the probable error (r) of a single observation we have, if the number of results is

2,	$r = \frac{1}{2}$	the range,
3,	$r = \frac{1}{3}$	" "
between 3 and 8,	$r = \frac{1}{4}$	" "
" 8 " 15,	$r = \frac{1}{5}$	" "
" 15 " 35,	$r = \frac{1}{6}$	" "
" 35 " 100,	$r = \frac{1}{7}$	" "

The probable error of the final result is found at once by dividing the probable error of a single observation by the square root of the number of observations.

In finding the range caution must be exercised with regard to abnormal results.

Table I.

m	λ_1	λ_2	m	λ_1	λ_2
			40	0.1080	0.0171
			41	.1066	.0167
2	0.6745	0.4769	42	.1053	.0163
3	.4769	.2754	43	.1041	.0159
4	.3894	.1947	44	.1029	.0155
5	0.3372	0.1508	45	0.1017	0.0152
6	.3016	.1231	46	.1005	.0148
7	.2754	.1041	47	.0994	.0145
8	.2549	.0901	48	.0984	.0142
9	.2385	.0795	49	.0974	.0139
10	0.2248	0.0711	50	0.0964	0.0136
11	.2133	.0643	51	.0954	.0134
12	.2029	.0587	52	.0944	.0131
13	.1947	.0540	53	.0935	.0128
14	.1871	.0500	54	.0926	.0126
15	0.1803	0.0465	55	0.0918	0.0124
16	.1742	.0435	56	.0909	.0122
17	.1686	.0409	57	.0901	.0119
18	.1636	.0386	58	.0893	.0117
19	.1590	.0365	59	.0886	.0115
20	0.1547	0.0346	60	0.0878	0.0113
21	.1508	.0329	61	.0871	.0111
22	.1472	.0314	62	.0864	.0110
23	.1438	.0300	63	.0857	.0108
24	.1406	.0287	64	.0850	.0106
25	0.1377	0.0275	65	0.0843	0.0105
26	.1349	.0265	66	.0837	.0103
27	.1323	.0255	67	.0830	.0101
28	.1298	.0245	68	.0824	.0100
29	.1275	.0237	69	.0818	.0098
30	0.1252	0.0229	70	0.0812	0.0097
31	.1231	.0221	71	.0806	.0096
32	.1211	.0214	72	.0800	.0094
33	.1192	.0208	73	.0795	.0093
34	.1174	.0201	74	.0789	.0092
35	0.1157	0.0196	75	0.0784	0.0091
36	.1140	.0190	80	.0759	.0085
37	.1124	.0185	85	.0736	.0080
38	.1109	.0180	90	.0713	.0075
39	.1094	.0175	100	.0678	.0068

TABLE II.

m	λ′	λ″	m	λ′	λ″
			40	0.0214	0.0034
			41	.0209	.0033
2	0.5978	0.4227	42	.0204	.0031
3	.3451	.1993	43	.0199	.0030
4	.2440	.1220	44	.0194	.0029
5	0.1890	0.0845	45	0.0190	0.0028
6	.1543	.0630	46	.0186	.0027
7	.1304	.0493	47	.0182	.0027
8	.1130	.0399	48	.0178	.0026
9	.0996	.0332	49	.0174	.0025
10	0.0891	0.0282	50	0.0171	0.0024
11	.0806	.0243	51	.0167	.0023
12	.0736	.0212	52	.0164	.0023
13	.0677	.0188	53	.0161	.0022
14	.0627	.0167	54	.0158	.0022
15	0.0583	0.0151	55	0.0155	0.0021
16	.0546	.0136	56	.0152	.0020
17	.0513	.0124	57	.0150	.0020
18	.0483	.0114	58	.0147	.0019
19	.0457	.0105	59	.0145	.0019
20	0.0434	0.0097	60	0.0142	0.0018
21	.0412	.0090	61	.0140	.0018
22	.0393	.0084	62	.0137	.0017
23	.0376	.0078	63	.0135	.0017
24	.0360	.0073	64	.0133	.0017
25	0.0345	0.0069	65	0.0131	0.0016
26	.0332	.0065	66	.0129	.0016
27	.0319	.0061	67	.0127	.0016
28	.0307	.0058	68	.0125	.0015
29	.0297	.0055	69	.0123	.0015
30	0.0287	0.0052	70	0.0122	0.0015
31	.0277	.0050	71	.0120	.0014
32	.0268	.0047	72	.0118	.0014
33	.0260	.9045	73	.0117	.0014
34	.0252	.0043	74	.0115	.0013
35	0.0245	0.0041	75	0.0113	0.0013
36	.0238	.0040	80	.0106	.0012
37	.0232	.0038	85	.0100	.0011
38	.0225	.0037	90	.0095	.0010
39	.0220	.0035	100	.0085	.0008

Example.—In the telegraphic determination of the longitude between St. Paul and Duluth, Minn., June 15, 1871, the following were the corrections found for chronometer No. 176, at 15h. 51m., sidereal time, from the observation of 21 time stars. (See Annual Report of Survey of N. & N. W. Lakes, 1872, p. 40.)

	Correction	v	vv
	s		
	−8.78	+ 0.04	0.0016
	.76	+ .02	4
	.85	+ .11	121
	.78	+ .04	16
	.51	− .23	529
	.64	− .10	100
	.68	− .06	36
	.63	− .11	121
	.58	− .16	256
	.80	+ .06	36
	.75	+ .01	1
	.78	+ .04	16
	.96	+ .22	484
	.64	− .10	100
	.65	− .09	81
	.83	+ .09	81
	.70	− .04	16
	.64	+ .10	100
	.79	+ .05	25
	.90	+ .16	256
	−8.93	+ .19	0.0361
Mean	−8.74	2.02	0.2756

Then

$$m = 21$$

$$[v = 2.02$$

$$[vv] = 0.2756$$

$$\sqrt{[vv]} = 0.525, \text{ from a table of squares.}$$

Hence from table I,

$$r = 0.525 \times 0.151 = 0.079$$

$$r_0 = 0.525 \times 0.033 = 0.017.$$

From table II,

$$r = 2.02 \times 0.041 = 0.082$$

$$r_0 = 2.02 \times 0.009 = 0.018$$

Approximate Method.—Range = 8.96 — 8.51 = 0.45

$$\therefore r = \frac{0.45}{6} = 0.075$$

$$r_0 = \frac{0.075}{\sqrt{21}} = 0.016.$$

So close an agreement among the determinations of the values of r and r_0 is not always to be expected.

LIMITS.

BY PROF. DE VOLSON WOOD, HOBOKEN, N. J.

THE arguments raised against the infinitesimal Calculus, in the last July ANALYST, have been considered by different writers at various times since the days of Leibnetz; and the philosophy of this method, as well as the Newtonian method of Limits, and of Lagrange's method is clearly set forth in Compte's *Philosophy of Mathematics* (Chap. III, Gillespie's translatson).

The Infinitesimal method is more arbitrary, and hence less inductive and apparently less philosophical, than the method of Limits; still it is none the less a valid *system*. It is not, however, just to that system to use it for a part of our argument and some other system for the remaining part. The argument must be self-consistent and in accordance with the principles of the analysis. We notice one point:—

This method asserts that the finite value of $1 + \circ$ is 1, where the horizontal zero is, according to Prof. Judson's notation, infinitesimal. A *supposed* fallacy in this statement is detected by showing that $[1 + (1 \div x)]^x$ is not 1 for x infinite, whereas all finite powers of 1 are 1; but the infinitesimal system asserts that $\infty \times 0 =$ *something*, and, in this example there will be an infinity of foctors when x is infinite, each of the form $1 + \circ$, and hence it may be possible that $(1 + \circ)^\infty$ will exceed unity; and hence the value found is not inconsistent with the principles of the system.

There is an element in the definition of the limit, which is worthy of notice. The definition given is—"The *limit* of a variable is a *constant* which the variable indefinitely approaches."

"Cor. I. The variable can never reach its limit, otherwise the approach would not be indefinite."

Many writers incorporate the corollary into the definition; and taken together they constitute the substance, and in many cases the language, of the definition of the limit as given by many modern writers. For our part we do not consider the corollary as a necessary consequence of the definition; or if incorporated with the definition it unnecessarily restricts the law of approach of the variable. Todhunter, in his Differential Calculus, asserts that a variable can not reach its limit, and yet, on the same page, asserts that any one of the values of the variable may be considered as a limit. Is it true that a variable cannot reach any one of its values? We admit that a variable may be subjected to such a law that to the human mind it will appear impossible for it to reach the limit, but we assert that it may also be subjected to such a law that it will reach it. Writers generally give illus-

trations of a law of approach (as, for instance, a descending geometrical progression approaching zero as a limit) according to which we are unable to see how the limit can be reached so long as the law is continuous; but they make no use of the illustration, or principle, in practice.

Indefinite approach may imply that the difference between the variable and its limit shall be less than any assignable quantity; and in this sense the difference may be absolutely zero without doing violence to the definition—in which case the limit will be reached.

But the argument sometimes employed to sustain the definition is open to objection. Thus, if a point moves from A towards B, moving over the first half of the distance in one-half of a minute, one-half of the remaining distance (or $\frac{1}{4}AB$) in one-fourth of a minute, and so on; when will it reach B? Some assert that, according to this law, B can never be reached. Professor Newcomb reasons thus (Alg., p. 212): "since the remainder is halved, if there was any movement that would overcome the entire remainder, the half of a thing would equal the whole, which is impossible". As forcible and apparently unanswerable as this argum't seems to be, it is shown to be fallacious by its proving too much. Thus, by precisely the same argument, it may be proved that the min. hand of a watch can never overtake the hour hand; that two intersecting right lines can never intersect; that bodies at rest can never move, &c.

In the above illustration, we are asked to admit the fundamental difficulty. It is stated that the point moves over one-half the distance, &c.; now explain *how* it moves over half the distance, and we will explain how it reaches B according to the same law.

We assert that there is no more difficulty, and precisely the same difficulty in reaching B that there is in leaving A. It cannot get any distance from A without first having passed over one-half of that distance; and since no distance is so small that it cannot be halved, the *smallest* movement would pass over the smallest space and its half at the sme time which is absurd; therefore it cannot leave A! But motion does occur; hence the logic must be fallacious. The trouble with the logic is, it does not include the *immeasurably* small quantities which make time and space absolutely continuous. Introduce the term *finite*, or *measurable*, and the logic is self consistent. We know nothing of the ultimate elements of time and space; but Newton considered time as *flowing* at a uniform rate, and that finite time may be considered as the sum of all the elements. This is a natural way of considering the subject, and we assert that it is possible to connect a variable to time in such a way that the variable shall reach the limit *under the operation of said law* in a finite time. Thus, for example, if the

apothem of an inscribed square should grow in length at a uniform rate, and if polygons of double the number of sides be conceived to be instantly described as the growing line becomes the apothem of an octagon, then of a polygon of sixteen sides, and so on, the inscribed polygons will reach the circle in a finite time.

This is an extreme case. It does not include the possibility of actually constructing the polygons, nor even the possibility of carrying out the conception; but simply that, under the operation of the law, the limit will be reached. But the Calculus does not determine the area of the circle in this way. The abscissa x may be conceived to grow uniformly, and thus to reach any assigned limit between 0 and r; and at the same time the ordinate and area will reach their limit. Definitions should not exclude a kn'n fact. We therefore suggest that the Corollary be excluded, and that the following be substituted for the definition:

The *limit* of a variable is a quantity which the variable approaches and from which it may be made to differ by less than any assignable quantity.

We have here used the term *quantity* instead of *constant* so as to make the definition more general.

In regard to *the law of continuity*, Prof. Price seems to consider it reasonable that it should be endless in extent. Thus an ellipse is a continuous curve, and the function $y = \frac{b}{a}\sqrt{(a^2-x^2)}$, is a continuous function of x, and real from $x = -a$ to $x = +a$. But x, as an independent variable, is continuous and real for all finite values, but the function becomes imaginary for all values of x greater than a, in which case it may be written

$$y = \frac{b}{a}\sqrt{(x^2-a^2)}\sqrt{-1}.$$

The *real* part of the preceding expression is the equation of an hyperbola, and hence the entire expression is called an imaginary hyperbola.

Prof. Price sought such an explanation of *imaginaries* and *infinities* as would make the locus continuous. For this purpose he considered that the effect of $\sqrt{-1}$ was to turn the locus through a right angle perpendicular to the plane of the ellipse and that in that plane the hyperbola would be real. The ellipse and hyperbola will then have a common axis and common vertices, but will be in mutually perpendicular planes. This assumption is arbitrary, but it serves the purpose of preserving *the law of continuity*.

We also find that $\tan x$ passes from $+\infty$ to $-\infty$ as x passes $\frac{1}{2}\pi$, hence $-\infty$ is considered as consecutive to $+\infty$. Similarly for other expres'ns.

Admitting that $a \div 0$ is impossible, but that $a \div \circ = \infty$ is rational, we suggest that the former may be expressed thus: $a \div 0 = 8$. But nothing substantial will be gained by multiplying symbols for this purpose.

ON MR. HILL'S REVIEW OF THEORY OF MOON'S MOTION.

BY JOHN N. STOCKWELL.

In the March number of this Journal, Mr. G. W. Hill, of the Nautical Almanac Office, has given a somewhat extended "review" of the "Theory of the Moon's Motion, &c.," prepared and published by myself. Mr. Hill is a profound mathematician, and has rendered valuable service to astronomy in more than one department; his opinions, therefore, on subjects which he has thoroughly investigated, are entitled to great weight. Unfortunately, the "review" bears evidence of having been hastily prepared; for it is difficult to believe that a mathematical astronomer of Mr. Hill's attainments would have made statements so palpably untrue, had he taken the time to inform himself in regard to the subject in all its various bearings. It appears also to have been prepared in an unfriendly spirit; and it cannot properly be called an "impartial review", since the reviewer is content with the slightest allusion to its acknowledged merits, while he bears with tremendous power on what he fancies to be defects. This is to be regretted, because it shows the "reviewer" and the "reviewed" in an unfavorable light, and cannot fail to have less influence over the unprejudiced reader, than it would, had it been written in a spirit of friendly criticism. To correct for any false impression which the "review" may produce, has made it incumbent upon me to prepare the following reply; but no notice would have been taken of the "review" had it emanated from a less distingu'ed source.

On reading the very first paragraph of the "review" one is reminded of the celebrated answer of the caliph *Omar*, to the inquiry of *Amrou*, as to what disposition should be made of the Alexandrian library. "If the writings agree with the Koran", says *Omar*, "they are useless and need not be preserved; if they disagree, they are pernicious, and ought to be destroyed". In like manner, according to Mr. Hill, although the greater part of my results agree with what other calculators have found, and only a very few differ; those which agree are useless, and those that differ *must be wrong*; so that in either case, although obtained by a different method from what other calculators have employed, the putting of the results in a permanent form is to be regretted, and the labor bestowed on its preparation cannot be considered as much better than wasted. The motives which prompted the preparation of the work are said by the reviewer to be a "desire to controvert certain of the results arrived at by his predecessors; results, too, which have long been regarded by astronomers as definitely settled and acquired to seience".

Now, just how much has been "definitely settled and acquired to science" can be best determined by referring to the history of scientific development. And we need go back no farther than the time of Sir Isaac Newton for a starting point. Newton computed the mean motions of the node and perigee of the moon's orbit, and also several of the periodic inequalities in the motion of the moon; and the results obtained by him were "definitely settled and acquired to science", during a period of more than sixty years, and were confirmed by the most celebrated analysts about the middle of the last century. But notwithstanding these calculations of Newton, and their confirmations by his successors, Clairaut demonstrated that Newton's value for the motion of the perigee was not the true value due to gravitation, but was only about one-half of it. I am not, however, sufficiently acquainted with the periodical literature of that time to be able to state that the announcement, by Clairaut, that Newton had not correctly computed the motion of the perigee, created a flurry among his scientific brethren, and caused some of them to accuse him of seeking notoriety by controverting the results long since "acquired to science". On the other hand, I suspect they were profoundly grateful for his having so happily removed the greatest obstacle to the development of the lunar theory.

The next result of especial prominence to which I shall call attention, relates to the secular acceleration of the moon's mean motion in longitude. This was calculated by La Place as long ago as 1787, and confirmed by La Grange and afterwards by Damoiseau, Plana, Pontécoulant and others who busied themselves with the subject; so that it was "definitely settled and acquired to science" during more than half a century. And yet, notwithstanding all these confirmations, Mr. Adams tells us that the result thus "acquired to science" is wholly wrong, and our reviewer accepts the statement without question.

It is unnecessary to multiply examples in which results "definitely acq'd to science" have been just as definitely abandoned, when the phenomena sought to be explained have become more perfectly known, and the *modus operandi* of the acting forces been brought into subjection to mathematical laws. Indeed the history of science is made up, in great measure, with the accounts of controversies about theories long since exploded; and the best names in science are often associated with theories that are no longer accepted. Thus, Newton and La Place both advocated the emission theory of light, and opposed the undulatory theory; but the emission theory, though once well established, has been pushed into the background, and at present is interesting only as a fiction of the past. These subjects are alluded to here, not by way of justification for any mistakes that I may have committed in the work under review, but to show the reader that mistakes have

been made by the most eminent men in every department of physical science; and I can recall no names of men who have attained to eminence in either theology, literature or science, who have not confessed to having made mistakes in the treatment of even simple questions; and I hope to show that my reviewer is no exception to the general rule.

I shall now pass from these general considerations to some of the particular cases mentioned by Mr. Hill. And first in regard to his statement that the value "$m = 0$, implies that we have either an infinitely short month or an infinitely long year; that is, the semi-axis major of the moon's orbit is infinitely small or the semi-axis major of the sun's orbit is infinitely great. Hence when we put $m = 0$, to be consistent, we are obliged to put $a \div a' = 0$". I cordially agree with Mr. Hill in this statement of general principles; and am profoundly grateful for his having stated the case so clearly. This is also valued the more highly from the fact, that, some years ago, he declined to discus the lunar theory at all, "because", says he," there is not a *single point of agreement* between us on which to base an argument".

And I will also venture to believe that there is one other point in connection with the lunar theory, on which we shall agree perfectly; and that is, that the solutions of the problem of the lunar perturbations, by La Place, Plana, Pontećoulant and Delaunay are entirely general in their nature, and are not restricted to the particular values of the elements of the sun and moon; except in the parts where the analytical values of the inequalities are reduced to numbers. If this be so, it is an easy matter to determine the perturbations the moon's motion would suffer if subjected to disturbing forces of different intensities. In fact the principal utility of a general solution of a problem consists in the facility with which the results may be applied to all problems of a similar character.

Now for greater facility in printing, we may suppose the sum of the masses of the moon and earth to be represented by unity, and then we may write equation (217) of the "Theory of the Moon's Motion", as follows,

$$m^2 = m' \frac{a^3}{a'^3}. \qquad [1]$$

In this equation a denotes the moon's mean distance from the earth, a', the sun's distance; m' denotes the sun's mass, and m the disturbing funct'n. Now we may vary m in three ways without changing the moon's distance First we may suppose that the sun's distance is changed while its mass remains the same; or second, we may suppose it to have a different mass while its distance remains unchanged; and finally we may suppose both m' and a' to vary, without altering the nature of the problem.

This being premised, let us take the equation corresponding to the third term of the second member of equation (B), page 72 tome II, of "*Théorie*

du Mouvement de la Lune, by Plana. By changing the notation of the argument so as to conform as much as possible to that used in my own work, it becomes

$$\frac{d^2\delta\mu}{dv^2}+(1-\tfrac{3}{2}\mu^2)\delta\mu+e\gamma^2(-\tfrac{21}{4}m^2+\tfrac{171}{128}m^3)\cos(v+\omega-2\Omega)=0. \qquad [2]$$

For the integral of this equation, Plana gives on page 76 the following

$$\delta\mu=e\gamma^2(-\tfrac{7}{8}+\tfrac{135}{64}m)\cos(v+\omega-2\Omega). \qquad [3]$$

This is wholly due to perturbation, because there is no equation depending on this argument in the elliptical value of μ.

Let us now inquire what would be the value of this equation, provided the sun's mean distance were *four* times its present distance. In this case the disturbing function would be $\frac{1}{64}$ of the actual value, and the year would be 8 times its present length. Equation [3] would become

$$\delta'\mu=e\gamma^2(-\tfrac{7}{8}+\tfrac{135}{512}m)\cos(v+\omega-2\Omega). \qquad [4]$$

Again, suppose the sun to be at *sixteen* times its present distance; the disturbing function would be $\frac{1}{4096}$ of its present value, and the year would be 64 times its present length; but for this case equation [3] would become

$$\delta''\mu=e\gamma^2(-\tfrac{7}{8}+\tfrac{135}{4096}m)\cos(v+\omega-2\Omega). \qquad [5]$$

In these two hypotheses the value of the disturbing function scarcely approaches the limiting value, zero; in fact the smaller value corresponds to the case of a planet and its satellite revolving around the sun nearly midway between the centre and outer limit of the solar system. It is therefore a legitimate hypothetical case, and will answer the purpose of comparison as well as though it were real.

If we now compare the values of $\delta\mu$, $\delta'\mu$ and $\delta''\mu$, omitting the term multiplied by m, which is equivalent to neglecting the square of the disturbing force, we find 1st, that the disturbance of the moon's distance from the earth would be the same as at present, if the disturbing force were only $\frac{1}{64}$th of its present value; and 2nd, that it would still be the same if the disturbing force were only the $\frac{1}{4096}$th of its present value. The most obvious conclusion to be drawn from these very curious results of analysis, is, that the disturbance is independent of the disturbing force, and would still remain the same were the disturbing force to vanish; in which case we shall have $m=0$; and the term depending on the given argument in equation [2] would vanish, and we should have $\delta\mu=0$. [6]

If we now take account of the terms depending on m in equations [3], [4] and [5], another very remarkable result follows; namely,

$$\delta\mu<\delta'\mu<\delta''\mu; \qquad [7]$$

that is, *the less the disturbing force the greater the disturbance!*

Were this very extraordinary result of analysis to be maintained by any body else than a profound mathematical astronomer, his sanity would at once be called in question.

According to Mr. Hill, this term expresses the transition from an orbit in motion to one at rest. If this be so, the transition becomes more violent in proportion as the disturbing force becomes less.

Mr. Hill next refers to my notice of Plana's calculation of the same inequality, by means of the variation of the elements; and becomes quite excited because I stated that a certain conclusion is not satisfactory. "Why a conclusion", says Mr. Hill, "legitimately deduced from correct principles, should be thrown aside at a mere *arbitrium*, certainly surpasses our powers of explanation." Now without reflecting at all on Mr. Hill's "powers of explanation", it is at least charitable to suppose that he is acquainted with the principle of the *reductio ad absurdum*, so much employed in geometry.

If a conclusion supposed to be legitimately deduced from correct principles, leads to absurd or impossible results, it is fair to believe either that the *principles* are *not* correct, or else that the *conclusion* is *not legitimately* deduced. Take a case in point: La Place deduces certain conclusions respecting the tides, which conclusions are confirmed by Mr. Airy, by means of a somewhat different method of investigation; but these conclusions were not satisfactory to Mr. Ferrel, because they involved the absurd consequence that *very high tides* would be produced even when the dist'bing force vanished. But it often happens that conclusions may be legitimately deduced from correct principles, and yet not be applicable to the problem which gave rise to them. Mr. Hill certainly knows, that in the computation of the orbit of a planet, the equation by which the distance of the planet from the sun is determined is of the eighth degree, and consequently may be satisfied by eight different values of the unknown quantity. But only one of them is applicable to the particular planet whose orbit is sought to be determined.

How, then, are we to discriminate between so many values, and fix upon the correct one? Or are there really eight different positions of the planet at the same time, in order that Mr. Hill's powers of explanation may not be overtaxed? Surely not. In order, then, to discriminate between the different values of the unknown quantity and fix upon the correct one, we test them, and find which ones lead to absurd or impossible conditions in regard to the place of the planet. For example, one value might give a *negative* radius vector to the planet, another might give a *negative distance* from the earth, and these must of course be rejected; still another might bring the planet even *into the eye of the observer*, which would of course be the wrong place for a planet. And by continuing in this way till we find one value which will satisfy both *physical* and *analytical* conditions we ac-

cept it as the correct value of the unknown quantity, and do not hesitate to reject the seven values which merely satisfy the analytical conditions.

Returning now to the particular case under consideration, we find that Plana has given the two following expressions for the differential variations of the eccentricity and perigee of the moon's orbit. See tome I, p. 97.

$$\frac{de}{dv} = \tfrac{21}{8}m^2e\gamma^2\sin 2(\omega-\Omega), \qquad \frac{d\omega}{dv} = \tfrac{21}{8}m^2\gamma^2\cos 2(\omega-\Omega). \qquad [8]$$

These are strictly secular equations, since they depend wholly on the variations of the elements of the moon's orbit, and are independent of the posi'n of the disturbing body. Now Plana has integrated them as if they were what are called periodic equations; simply because the perigee and node complete a revolution in a comparatively short period of time. It is found that the quantity $2(\omega-\Omega)$ varies by the quantity $3m^2v$. Now if we put the variable part of the angle $2(\omega-\Omega)$ equal to $3m^2v$, and call the integrals of equations [8], δe and $\delta\omega$ we shall have,

$$\delta e = -\tfrac{7}{8}e\gamma^2\cos 2(\omega-\Omega), \qquad \delta\omega = \tfrac{7}{8}\gamma^2\sin 2(\omega-\Omega). \qquad [9]$$

The principal term in the value of μ is

$$\mu = e\cos(v-\omega); \qquad [10]$$

and the variation of μ arising from any finite variations of e and ω, will be given by the equation

$$\delta\mu = \left(\frac{d\mu}{de}\right)\delta e + \left(\frac{d\mu}{d\omega}\right)\delta\omega = \cos(v-\omega)\delta e + \sin(v-\omega)e\delta\omega. \qquad [11]$$

If we substitute the preceding values of δe and $\delta\omega$, in this equation, it will become

$$\delta\mu = -\tfrac{7}{8}e\gamma^2\cos(v+\omega-2\Omega); \qquad [12]$$

which is the same as the first term of equation [3].

In the case of the moon disturbed by the sun, the argument of δe and $\delta\omega$, requires three years to complete a revolution. Were the sun placed at *four times* his present distance, the period of the argument would be 192 years; but the values of δe and $\delta\omega$ would remain the same, since the time *increases* in the same ratio as the force diminishes. Again, were the sun *sixteen times* his present distance, the period of the argument would be more than 12000 years, and yet the variations of δe and $\delta\omega$ would remain unchanged, since the diminution of the force would still be compensated by the increase of the time. But whatever be the amount of these variations of δe and $\delta\omega$, their substitution in equation [11] always gives the same value of $\delta\mu$.

If, then, the value of $\delta\mu$ remains the same whether the perigee and node move much or little, it is obvious that it is independent of this change of the elements, and would still subsist were the elements constant, in which case we should have $\delta e = 0$, $\delta\omega = 0$, and then equation [11] would give $\delta\mu = 0$, the same as before found.

It is easy to trace these curious results to the values of the integral given by equation [9]. If we suppose that at a particular epoch, the eccentricity and longitude of the perigee are denoted by e and ω, it is evident that at that epoch we should have $\delta e = 0$, $\delta\omega = 0$; but equations [9] show this to be impossible, because when δe vanishes $\delta\omega$ is a maximum, and *vice versa.*

The maximum value of δe in equation [9] amounts to 80″, and has a period of three years, or 40 revolutions of the moon. Now in general, the variations of the elements, in orbits of small eccentricity, are much greater than the variations of the coordinates; but according to Plana's calculations we have here a *monthly* equation amounting to 114″ growing out of a small secular equation whose period is 40 months. Were we to apply the same principle to the perturbations of the earth by Venus, we should find the following values $\delta e' = 449''\cos(\omega''-\omega')$, $e'\delta\omega' = 449''\sin(\omega''-\omega')$, ω'' denoting the longitude of the perihelion of Venus; and these quantities would give for the perturbation of the earth's longitude

$$\delta v' = 898''\sin(n't-\omega'').$$

That is to say, a small secular inequality, in the elements of the earth's orbit having a period of 90000 years, gives rise to an *annual* equation of 898″ in the earth's longitude. Now there is no such inequality in the earth's longitude, and consequently any method of computation which gives such an inequality must be erroneous. But if the principle is not applicable to the motion of the earth, it is not applicable to the motion of the moon; and the results derived from its application must be erroneous.

There is another equation of the moon's longitude, of considerable importance in the lunar theory, and which has for its argument $nt - \omega'$, or the moon's distance from the sun's perigee. This arises from the motion of the moon's perigee, and like the one we have been considering, it increases in magnitude as the disturbing force diminishes;—the distance of the disturbing body being supposed to remain unchanged.

There is also an important equation of the moon's latitude, which is produced by the secular variations of the node and inclination of the moon's orbit, the argument of which is $nt-2\omega+\Omega$. It is also subject to the same peculiarities as those already mentioned; and it is unnecessary to enter more into the details of the question.

Our reviewer next insinuates that the terms depending on the square of the disturbing force annoy me very much. But in this he is mistaken. These terms are simply the perturbations arising from the previous perturbations produced by the first power of the disturbing force; and in order to correctly compute them, the terms depending on the first power must be correctly computed. But the terms arising from the first power may be accurately computed without any reference to those arising from the second and higher powers of the disturbing force.

Mr. Hill next states that the largeness of the error which my results would imply as existing in the lunar tables ought to have led me to suspect the legitimacy of my own conclusions. I cheerfully accept this statement of my reviewer; and will only reply to it as Prof. Adams replied to a similar objection from his opponents in the controversy about the moon's secular acceleration; namely, that it is purely a question of theory, with the decision of which observation has nothing whatever to do.

But perhaps nothing can better illustrate the supreme efforts Mr. Hill has made to become acquainted with the work he has attempted to review, than his remarks about the inequality depending on the angular distance between the perigees of the sun and moon. After admitting that my equation [24] is simpler than any which have before been applied to the computation of such inequalities, he says that he cannot find any proof of these equations; and charges me with having adopted them quite arbitrarily.

Now I admit that the demonstration of these equations has not required a separate chapter, as is usual in most works on the lunar theory; and this will probably account for his not being able to find it; but the demonstration is, nevertheless, in the work under review. And as to his statement that equations [23] are inconsistent with each other, I will only say in explanation, that that part of the work was designed more for the non-techn'al reader, and that I did not wish to complicate it with abstruse mathematical formulæ. But I now perceive that it was an oversight, as it has had the effect of misleading so good a mathematician as Mr. Hill. In the body of the work, however, the formulas are all right. And as to his statement that the formula gives an infinitely great coefficient to the inequality, when the disturbing force is infinitely small, it is sufficient to say that the formula meets all the requirements of the case. If the disturbing mass is infinitely small, the value of h will be infinitely small, and the inequality would be small; but if a is small, the period of the inequality would be long, and the mere novice in science would be able to understand that a very small force acting during a very long time is sufficient to accomplish very considerable work. In fact, were the perigees of the sun and moon to remain stationary, the moon would be acted upon by a constant tangential force (unless their longitudes were the same or diametrically opposite); and it is evident that the resulting inequality would ultimately become infinite.

Mr. Hill next says that the secular equation depending on the oblateness of the earth, and arising from the diminution of the obliquity of the ecliptic to the equator does not exist, because the inclination of the equator to the fixed ecliptic of any given date varies very slowly and proportionally to the square of the time. Now the fact that the inclination of the moon's orbit to the apparent ecliptic remains constant, was one of La Place's happiest

discoveries; and so long as the apparent ecliptic approaches the equator, the moon's orbit must approach it also. The fixed ecliptic has, therefore, no more to do with the problem than has the plane of Jupiter's orbit.

Lastly, in regard to the secular inequality, I would say that I have never before attempted a thorough investigation of that subject. It is true, however, that some fifteen years ago I published a pamphlet in which I attempted to show that the new terms found by Mr. Adams, had no existence; and as yet I have seen no reason to change the views there expressed in regard to that matter.

In general, we may say that small secular equations of the elements of both, planets and moon, are produced by the large periodic inequalities to which these bodies are subjected; but for a large periodic inequality to be produced from a small secular inequality, is inconsistent with both reason and correct calculation. And from whatever point of view we approach the subject, it becomes more and more apparent that our lunar tables in use at present are based on very defective theories; and the only wonder is, that they can be made to represent the moon's motion as well as they do. I can, therefore, as yet, see no reason for recalling or modifying my statement that our present lunar tables are really erroneous by some of the smaller terms of the *third order*, instead of being correct to terms of the *seventh order* as has heretofore been supposed.

[*Correct'n.*—In the foregoing paper, for μ read u, except in first parentheses of line 4, p. 85.]

NOTE BY THE EDITOR. —At the time Prof. Wood's article on Limits (see p. 80) was put in type, we had not seen Newcomb's Algebra to which reference is there made. As it seemed improbable that Professor Newcomb would pursue the line of argument there attributed to him, we have since procured a copy of the Algebra alluded to and find that Prof. Wood has (unintentionally no doubt) misrepresented Prof. Newcomb's argument. As stated by Prof. Newcomb, the argument is entirely legitimate and the conclusion is *unquestionably correct.* The argument, as stated by Prof. Newcomb, is as follows:—

"Suppose AB to be a line of given length. Let us go one-half the dist. from A to B at one step, one-fourth at the second, one-eighth at the third, etc. It is evident that, at each step, we go half the distance which remains. Hence the two principles just cited apply to this case. That is,

"1. We can never reach B by a series of such steps, because we shall always have a distance equal to the last step left.

"2. But we can come as near as we please, because every step carries us over half the remaining distance."

INTEGRATION OF TWO DIFFERENTIAL FORMS.

BY FERDINAND SHACK, ESQ., NEW YORK CITY.

To integrate $x^m \sin x\,dx$ and $x^m \cos x\,dx$. Because $\int u\,dv = uv - \int v\,du$; let $u = \cos x$ and $dv = x^n dx$, then

$$\int x^n \cos x\,dx = \frac{1}{n+1}x^{n+1}\cos x + \frac{1}{n+1}\int x^{n+1}\sin x\,dx;$$

$$\therefore\ \int x^{n+1}\sin x\,dx = -x^{n+1}\cos x + (n+1)\int x^n \cos x\,dx. \qquad (1)$$

Again, let $u = \sin x$ and $dv = x^n dx$,

$$\int x^{n+1}\cos x\,dx = x^{n+1}\sin x - (n+1)\int x^n \sin x\,dx. \qquad (2)$$

Let $n = 0$ and substitute in (1) and (2),

$$\int x \sin x\,dx = -x\cos x - \sin x,$$
$$\int x \cos x\,dx = x\sin x - \cos x.$$

Let $n = 1$ and substitute in (1) and (2),

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$
$$= -x^2\cos x + 2x\sin x - 2\cos x,$$
$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$
$$= x^2\sin x + 2x\cos x - 2\sin x.$$

Let $n = 2$ and substitute in (1) and (2),

$$\int x^3\sin x\,dx = -x^3\cos x + 3\int x^2\cos x\,dx$$
$$= -x^3\cos x + 3x^2\sin x + 3.2x\cos x - 3.2\sin x$$
$$\int x^3\cos x\,dx = x^3\sin x - 3\int x^2\sin x\,dx$$
$$= x^3\sin x + 3x^2\cos x - 3.2x\sin x - 3.2\cos x.$$

Let $n = 3$ and substitute in (1) and (2), &c.

The laws of the series are thus determined, and may be expressed:

$$\int x^m\sin x\,dx = -x^m\cos x + mx^{m-1}\sin x + m(m-1)x^{m-2}\cos x$$
$$-m(m-1)(m-2)x^{m-3}\sin x - ++ --,$$
$$\int x^m\cos x\,dx = x^m\sin x + mx^{m-1}\cos x - m(m-1)x^{m-2}\sin x$$
$$-m(m-1)(m-2)x^{m-3}\cos x ++ --.$$

[These integrals may be found at p. 265 of Hirsch's Integral Tables, but they were computed by Mr. Shack without suspecting that they were known forms.—Ed.]

NOTE ON THE SOLUTION OF PROB. 373.—We have received from Prof. H. T. Eddy, a brief and elegant solution of (373), in which several errors which occur in the published solution are pointed out and corrected. We have also received from Mr. Adcock the following corrections of his solution of that problem (see p. 55):

"I find two errors in the MS. of my solution of (373). In the printed solution on page 56, second line, strike-out the words, "and positive downwards", and, in fourth line from bottom, for "$dx \div ds = -y \div r$", read $2\rho dx = ds^2$. From these two corrections there follow two resulting corrections, viz.; in lines 8 and 9, change the signs of the second member in the right-hand equation in each; and in line 3 from bot., for $-1 \div 2a$ read $-1 \div 2\rho$.

"*Errors in printing.* Strike-out g in line 10 and insert $+ g$ in the 2nd parentheses in line 11; and in line 16 insert $+ g$ immediately before the first bracket".

SOLUTIONS OF PROBLEMS IN NUMBER TWO.

Solutions of problems in No. 2 have been received as follows:

From Prof. L. G. Barbour, 387, 390; Prof. W. P. Casey, 387, 388, 389, 391; Prof. A. B. Evans, 387; George Eastwood, 390; W. E. Heal, 387, 388, 389; Prof. A. Hall, 391; Prof. J. Scheffer, 387; Isaac H. Turrell, 390.

Prof. Casey should have been credited, in No. 2, for a solution of 383.

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387. *By Prof. L. G. Barbour.*—"Given the length of each side of any quadrilateral, and the distance from the middle point of any side to that of the side opposite. Required the distance from the middle point of one of the other sides to that of the side opposite."

SOLUTION BY W. E. HEAL.

Let EF, GH be the lines joining the middle points of opposite sides of the quadrilateral $ABCD$, and LK the line joining the middle points of the diagonals.

Put $AB = 2a$, $BC = 2b$, $CD = 2c$, $DA = 2d$, $EF = k$, $GH = h$.

Then $FL = EK = a$, $GH = HL = b$, $EL = FK = c$, $GL = HK = d$.

And, since $GHLK$ and $EFLK$ are parallelograms,

$$h^2 + (LK)^2 = 2(a^2 + c^2),$$
$$k^2 + (LK)^2 = 2(b^2 + d^2).$$
$$\therefore h^2 - k^2 = 2(a^2 - b^2 + c^2 - d^2).$$

388. *By Prof M. L. Comstock.*—"F and F' being the foci of an ellipse and P a point on the curve, FD is drawn perpendic'r to FP meeting $F'P$ in D. Find the locus of D: (1) when $b > c$, (2) when $b = c$, (3) when $b < c$, if b = semi-minor axis and c = distance from the centre to either focus."

SOLUTION BY PROF. W. P. CASEY.

Let x, y be the coordinates of D to the axes AB, OR, O being the center of the ellipse. Then $FS = c+x$ and $SF' = c-x$, $FP = a+ex$ and $F'P = a-ex$; and $F'D = [(c-x)^2+y^2]^{\frac{1}{2}}$. $\therefore PD = [(c-x)^2+y^2]^{\frac{1}{2}}+a-ex$ and $FD^2 = (c+x)^2+y^2$. $\{[(c-x)^2+y^2]^{\frac{1}{2}}+a-ex\}^2-(a+ex)^2$, or $(c-x)^2+y^2+2(a-ex)[(c-x)^2+y^2]^{\frac{1}{2}}+(a-ex)^2-(a+ex)^2 = (c+x)^2+y^2$. Therfore $2(a-ex)[(c-x)^2+y^2]^{\frac{1}{2}} = 4cx+4aex = 8cx$, or $[a-(c\div a)x]^2[(c-x)^2+y^2] = 16c^2x^2$, the equation of the curve which is the locus of the point D; and taking $b > c$, $= c$ and $< c$, we find each curve.

---

389. *By Prof. W. W. Johnson.*—"If three triangles have a common axis of homology when taken in pairs, the three centres of homology are in a straight line: and reciprocally if three triangles have a common centre of homology when taken in pairs, the three axes of homology pass through a common point."

SOLUTION BY W. E. HEAL.

It is only necessary to prove the first part of the theorem, from which the second part follows by reciprocation.

Let ABC, $A'B'C'$, $A''B''C''$ be the given triangles whose corresponding sides meet in three points O, O', O'' lying on the common axis of homol'y.

The lines AA', BB', CC' meet in O the center of homology of the triangles ABC, $A'B'C'$; $A'B''$, $B'B''$, $C'C''$ meet in O', the center of homology of $A'B'C'$, $A''B''C''$; and $A''A$, $B''B$, $C''C$ meet in O'' the center of homology of $A''B''C''$, ABC.

The triangles $AA'A''$, $BB'B''$, $CC'C''$, taken in pairs, have either O, O', O'', for a center of homology, and therefore the intersections of corresponding sides are in a straight line; but these are the points O, O', O'', the centers of homology of ABC, $A'B'C'$, $A''B''C''$.

390. This is the same as 356, and its solution will be found at page 163 of Vol. VIII. It was inserted in No. 2 by an oversight.

391. "Given

$$\log. 91 = 1.95904 \pm r,$$
$$\log. 92 = 1.96379 \pm r,$$

find log. 91.5 to five decimals, by simple proportion from the difference; and find the probable error of this logarithm."

ANSWER BY PROF. ASAPH HALL.

If f be the interpolating factor we have $f\Delta = 237.5$, and the value of log 91.5 is 1.96142, or 1.96141.

To find the probable error of this value, let r_1, r_2, be the errors of the two logs, and r_3 the error made in stopping the product $f\Delta$ at the given decimal; then the real error of the interpolated value is

$$(1 - f) . r_1 + f . r_2 + r_3.$$

Assuming f constant, the probable error is found from the mean value of some power of the real error; — r_1, r_2, r_3 being independent variables between the limits ± 0.5. The following are the results given by Bremiker, the editor of our best logarithmic tables: Since $r = 0.25$,

$f =$ 0.0	: 0.1	: 0.2	: 0.3	: 0.4	: 0.5	: 0.6	etc.
Pr. Er.= 0.293	: 0.279	: 0.270	: 0.263	: 0.262	: 0.261	: 0.262,	etc.

The method that I gave in the preceding No. of the ANALYST is incorrect. In fact, the law of error is not that which is assumed in the method of least squares.

QUERY BY PROF. H. T. EDDY.—"When two determinants of the same order have the same algebraic value, show whether it is always possible to transform the one into the other by mere combinations of rows and colu'ns; and if possible transform the two following values of $2bc \cos A - b^2 - c^2$, the one into the other:

$$\begin{vmatrix} 0, & b, & c, \\ b, & 1, & \cos A, \\ c, & \cos A, & 1, \end{vmatrix} , \quad \begin{vmatrix} 2bc \cos A, & b, & c, \\ b, & 1, & 0, \\ c, & 0, & 1, \end{vmatrix} ."$$

Prof. Casey answers the above query as follows:

"The two determinants are equal; but by no combination of rows or columns can $2bc \cos A$ be factored so as to transform this determinant into the other. Neither can the first be transformed into the second, as far as I can see, by any of the known laws of determinants."

PROBLEMS.

392. *By George Eastwood, Saxonville, Mass.*—In a triangular pile of round shot, each shot rests upon three other shot, thus forming an empty space. It is required to find the ratio of the capacity of all the spaces to the capacity of all the balls.

393. *By J. M. Boorman, Esq., New York City.*—State the general equation of the 4th degree in terms whose coefficients shall be real and direct functions of its roots and admit a solution showing the root's *real* nature—no root to be directly *expressed* by a letter.

394. *Id.*—Show that the general equation of the 4th degree has its companion biquadrate, and state it and the respective relat'ns of their roots.

395. *By C. O. Boije of Gennas, Gothenburg, Sweden.*—Determine the law of density of a sphere in order that its centre of gravity may be coincident with the centre of gravity of the half sphere cut off from the sphere.

396. *Selected by Prof. H. T. Eddy.*—A smooth horizontal disk revolves with the angular velocity $\sqrt{\mu}$ about a vertical axis at which is placed a material particle attracted to a certain point of the disk by a force whose acceleration is $\mu \times$ distance; prove that the path on the disk will be a cycloid. (Routh's Rigid Dynamics, p. 163.)

397. *By Prof. J. M. Rice, U. S. Naval Academy.*—Given

$$\varphi(x^2)\,\varphi(y^2) = \varphi(x'^2)\,\varphi(y'^2)$$

and

$$x^2 + y^2 = x'^2 + y'^2,$$

to determine the form of the function denoted by φ.

398. *By Wm. Hoover, A. M., Dayton, Ohio*—An angular velocity having been impressed upon a heterogeneous sphere, about an axis, perp. to the vertical plane which contains its center of gravity G and geometrical center C, and passing through G, it is then placed on a smooth horizontal plane; to find the magnitude of the impressed angular velocity that G may rise into a point in the vertical line SCK through C, and there rest; the angle GCS being α at the beginning of the motion, c, the radius, and φ the req'd angular velocity.

399. *By E. J. Esselstyn, New Haven, Conn.*—Given two points A and B, and a circle K having its centre at O. Let any circle L be drawn thro' A and B so as to cut the circumf. of the circle K in two variable points m and n. Show that the circle through O, A and B is cut by the variable circle through O, m and n, in a fixed point P.

400. *By Prof. Asaph Hall.*—In a plane passing through the centre of the sun, 12 right lines are drawn from this centre making an angle of 30° with each other. On each of these lines, three homogeneous spherical bodies are placed at distances respectively of 10, 20 and 30 from the centre of the sun; the distance from the earth to the sun being the unit of distance.

The mass of each of these bodies being equal to that of the sun, what will be the velocity of a particle that starts from an infinite distance and moves in a right line towards the centre of the sun, and perpendicular to the plane of the bodies, when the particle is at a distance of 0.01 from the centre of the sun; the law of attraction being that of Newton?

QUERY BY GEORGE EASTWOOD.—In the solution of problem 389, show *why* "It is allowable and sufficient to write

$$\frac{dp}{dt} = p^2, \text{ and } \frac{dq}{dx} = q^2\text{"}.$$

PUBLICATIONS RECEIVED.

Theory of the Moon's Motion. Deduced from the Law of Universal Gravitation. By JOHN N. STOCKWELL, PH. D. 378 pp. Large 8vo. Philadelphia: J. B. Lippincott & Co. 1881.

A Sequel to the first Six Books of the Elements of Euclid, Containing an easy Introduction to Modern Geometry, with numerous examples. By JOHN CASEY, LL. D., F. R. S. Dublin: Hodges, Figgis, & Co. London: Longmans, Green, & Co. 1882.

A Treatise on the Theory of Determinants. By THOMAS MUIR, M. A., F. R. S. 240 pp. 8vo. Macmillan & Co. London.

I.—The Law of Extensible Minors in Determinants: II.—On some Transformations connecting General Determinants with Continuants. By THOMAS MUIR, M. A. 14 pp. 4to.

Micrometrical Measurement of 455 *Double Stars*, under the Direction of ORMOND STONE, A. M., Astronomer. 69 pp 8vo. Cincinnati. 1882.

A Value of the Solar Parallax from Meridian Observations of Mars. By PROF. J. R. EASTMAN. 43 pp. 4to. Washington. 1882.

The American Journal of Mathematics, Vol. IV, No. 2.

This number of the Journal is wholly devoted to the publication of the late PROFESSOR BENJ'N PEIRCE'S *Linear Associative Algebra.* With Notes and Addenda by C. S. PEIRCE, Son of the Author.

The Mathematical Visitor, Vol. II, No. 1. By ARTEMAS MARTIN, M. A., Erie, Pa.

ERRATA.

On page 49, line 15, for "laws", read locus.
" " 50, " 22, insert m_5 after 1046.02.
" " 85, " 1, insert " after "*Lune*".

THE ANALYST.

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COMPLEMENTARY DIVISION.

BY LEVI W. MEECH, A. M.

The object of the following investigation is to extend the well known facility of dividing by 10, 100, or 1000 etc., as far as practicable, to 9, 99, 999, and other adjacent divisors. For illustration, let 996 be the divisor, where $996 + 4 = 1000$. It will be seen presently that the smaller complement or 4 can be used in place of the larger 996; and this has suggested the name of the method.

Introductory to the Rule, a first Example is selected, in which the carrying figures from one period to the next, are known to be 0. After separating the dividend into periods by the Rule, we simply bring down the first left hand period 023 or 23 into the quotient, *multiply it by the complement* 4, *and add the product* 23×4 *to the next period* 068, writing the sum 160 underneath; and so on. Under the last period, we have $160 \times 4 + 069 = 709$ the Remainder.

In Example 2, the carrying figures 1, 4, 2, 0 have been computed mentally in advance. We add each to its period, and proceed with the sum as in Example 1, only omitting the same 1, 4, 2 when reproduced, since each was added previously. The power of 10 nearest the divisor is 1000.

Example 1.			Example 2.	
Divisor	996	023′ 068′ 069 Dividend.	996	328′ 571′ 953′ 274
Complem't	4	+0 +0 +0	4	+1 +4 +2 +0
	+	$23160\frac{709}{996}$ Quotient.	+	$329891519\frac{350}{996}$.

It should be observed in Example 2, that, after adding the 1, 4, 2, successively, the standard process, which gave these again with the quotient figures, is a full repetition of the mental process previously employed or abbreviated to find the same carrying figures 1, 4, 2.

DEMONSTRATION.—In Example 2, let the divisor 996 take the form of $1000-4$. And let c denote a correction such that $328+c$ shall express the first three figures of the quotient. Multiplying divisor by quotient, or $1000-4$ by $328+c$ and subtracting the product, the first period of the dividend evidently disappears. Also by making the total carrying figure $1=c$, the terms $c-1$ disappear, leaving the remainder $571+312+4c$ or 887.

The next step including the sum of 953 and 3548, that is 4′501, requires that $c'=4$; so that $501+4c'$ gives the 517 remainder. To this we bring down the next period 274, and proceed as before.

$$
\begin{array}{lr|r|l}
 & 996 & 328'571'953'274 & 328+c=329 \\
 & 1000-4 & c-1\ 312-4c & \\
\cline{3-3}
 & & 887\ 953\ 274 & 887+c'=891 \\
c=1; & & c'-3\ 548-4c' & \\
c'=4; & & 517\ 274 & 517+c''=519 \\
c''=2. & & c''-2068-4c'' & \\
\end{array}
$$

$$\overline{329891519\tfrac{350}{996}}, \text{ Quotient.}$$

Equivalent process.

996	328	571	953	274	Dividend.
4		8+1	87+4	7+2	Correct'n.
+	329	891	519	$\frac{350}{996}$	Quotient.

A.	*B*, ment'y	*C.*	*D.*	*E.*	
8	32×4+57	8+1	329×4+571	87	etc.
32	= 1(85)	329	= (1)887	8	

RULE I. FOR DIVISION NEAR 100, OR SOME OTHER POWER OF 10.

Rule.—Under the Divisor, write its Complement, such that the Algebraic sum of the two shall equal the nearest integer power of 10. Also, commencing on the right, separate the Dividend into Periods of as many figures each as there are cyphers in the chosen power of 10. Then bring down the first one or two digits of the left hand period into the Correction line, and the rest of the period into the lower line of the quotient, as shown at *A*, or *E*.

When any period of the Quotient is thus incomplete, multiply its two left hand figures (or cyphers, if the period is deficient) by the Complement, and mentally add the product to the corresponding figures in the next period of the dividend, as shown at *B*. Write the carrying figure of the period so found, on the correction line; also add it to the preceding figure, and write the sum in the quotient, as shown at *C*.

As soon as each period in the Quotient line is completed, multiply it by the Complement as indicated in *D*, and, adding the product to the next period of the dividend, write the sum underneath in the form *E* or *A*, but omit the final carrying figure or correction, which has been added previously.

NOTE 1. Should this true carrying figure found from the full period in D, be 1 more than the deficient figure previously written in the correction line, erase and correct accordingly. And correct, when there is 1 to carry in adding to form the quotient.

NOTE 2. At the end, the last period of the Dividend which gives the Remainder, has the carrying figure or correction always 0, as shown in the demonstration.

NOTE 3. In the case of Divisors like 103, 104, 1006, which exceed the nearest power of 10, we multiply by the Complement regarded as negative, and (instead of adding) subtract the products and the correction or carrying figure of the period.

EXAMPLES.

997	86 512 743 295 Dividend.	991	328 571 953
3	6+0 0+3 2+0	9	28+3 0+5
+	86 773 062 $\frac{481}{997}$ Quotient.	+	331 555 $\frac{348}{991}$.

96	3 28 57 19 53 Dividend.	9995	314 1592 6535
4	3+00+25+13+1	5	4+0 2+2
+	3 42 26 24 $\frac{49}{96}$ Quotient.	+	314 3164 $\frac{2355}{9995}$.

93	46 09 16 12 57	999	123* 456 789 Dividend
7	6+32+48+08+5	1	3+0 9+1
+	49 56 08 73 $\frac{68}{93}$	+	123 580 $\frac{369}{999}$ Quotient.

By Note 1, erase (5) (6) (7)

998) 23025850929 98) 434294482

For verification of the quotients so easily found in practice, we employ from Arithmetic, the proof by excess of 9's. By supposing the digits of the dividend to be all 9's, we learn that whenever the mental process of the Rule gives a correction that is equal to the Complement of the divisor, such correction is exact, being the maximum value: also that the correction is exact when its omitted right hand digit is less than the right hand digit of the Divisor.

In the next Examples, where the Complement is *subtractive*, it will be most convenient to extend the process of "borrowing 10 units and paying 1 ten", to borrowing 20, 30 or 40 etc. units, and paying 2, 3 or 4 etc. tens, respectively. Thus, in the first Example below, for $872—67\times 5$, we begin with $2—7\times 5$ or $2—35$ and, borrowing 40 units, say $42—35 = 7$, remainder. Then paying 4 tens to the next product 6×5, it becomes 34. And $87—34$ leaves 53, so that 537 is the whole remainder.

Divisor. 1005	67 872 531 624 Div.	103	23 02 58 50 93
Comp't. 5	7—0 7—361—4	3	3—16—13—27—2
—	67 534 857 $\frac{339}{1005}$ *Q*'t.	—	22 35 51 95 $\frac{08}{103}$.

EXAMPLE.

Divisor		9876543	6437695	Quotient.
Complement		123457	6358217483	Dividend.
		+	740742	6×Complement.
Continued.			493828	4 "
	...338338	Quotient	370371	3 "
123457	3341615	Remain.	600334	New Dividend.
3833	37037		864199	7×Complement.
	3704		740742	6 "
	987		1111113	9 "
Contraction.	3343		617285	5 "
	37		3341615	Remainder.

Here subtracting the right hand figure of the Divisor from 10, and the others from 9, we obtain the complement 123457. Multiplying this by 6, the left figure common to the Dividend and the Quotient, we write the first figure, 2 of the product, seven places to the right of the 6, since there are seven places in the whole Divisor. Then the next figure of the Dividend, 3, increased by 1, carrying figure, gives 4 in the quotient. Continuing the process as described in the Rule, we obtain 6437695 Quotient, and 3341615 Remainder.

Contraction.—As shown on the left, above, the process is continued by writing the next quotient figure, 3, both in the Quotient, and under 5, the second figure of the Complement, to apply contracted multiplication. And thus we obtain further 338338, making these thirteen places in the Quotient, 6437695338338.

By the the usual operation of Long Division, each quotient figure would cost the writing of sixteen other figures; so that 208 other figures would be required with the corresponding mental labor to determine the quotient 6437695338338; where the Complementary process properly requires only 78 figures, and the mental labor is much shorter and easier. In general, the more 9's on the left of the Divisor, the smaller the Complement and the easier the process of Division.

With respect to the preliminary Multiplier of the Rule, since the product of the Divisor by its reciprocal is always 1, the required Multiplier will be a little less than the reciprocal of the Divisor, and may thus be found from the published Tables of Reciprocals.

Less exact Multipliers are given in the following outline Table which may serve for illustration. The values can be easily verified by the common Table of Reciprocals.

Preparatory Multiplier for Rule IV.

First Divisor	Multiplier.	First Divis.	Multiplier.	First Divis.	Multiplier.	First Divis.	Multiplier.	First Divis.	Multiplier.
100	9×11	141	7	201	7×7	304	4×8	589	4×4
104	8×12	150	6×11	205	6×8	323	3	626	3×5
110	9	154	8×8	218	5×9	345	4×7	667	2×7
113	8×11	157	7×9	223	4×11	358	3×9	779	12
118	7×12	164	6	233	6×7	385	$\frac{1}{4}$	834	11
122	9×9	176	7×8	244	4	401	4×6	869	$11\frac{1}{2}$
124	8	179	5×11	257	38	435	2×11	918	9×12
129	7×11	182	6×9	271	6×6	455	3×7	944	$10\frac{1}{2}$
132	$\frac{3}{4}$	189	$5\frac{1}{5}$	278	5×7	477	2	975	102
137	8×9	197	5	295	3×11	527	3×6	990	1

The following Divisor and Dividend are first multiplied by 8, since 124 in the Divisor indicates this Multiplier. The new Dividend is written one space lower to make room for the Quotient over it; and after the first addition, Contraction is applied. In the quotient, the more full carrying figure once changes 8 to 9: *Example:*

First Divisor,	124689236	) 4791562834	First Dividend.
Multiplier,	8	8	
Divisor,	997513888	38428039081	Quotient.
Complement,	2486112	38332502672	Dividend.
From Quo't,	18093082	7458336	
		19888896	
		9944448	
		279693728	
		497222	
		198889	
		746	
		224	
		809	
		2	

Example for Practice,	3521837	) 8263541783×4	
	14087348	) 33054167132×7	
	98611436		Quotient.
Complement,	1388564	231379169924	Dividend.

The Rule can also be applied to Divisors beginning with 100 or 101, by omitting the 100 or 10 on the left, and taking the remaining part as the negative Complement, whose products or the sum of products by quotient figures are to be subtracted from the Dividend.

ON THE SOLUTION OF EQUATIONS.

BY JOSEPH B. MOTT, ESQ., WORTHINGTON, MINNESOTA.

THE irrational roots of equations of all degrees may be determined by series. Thus, if

$$a=x+bx^2+cx^3+dx^4+ex^5+fx^6+gx^7+\&c. \qquad (G)$$

we may find

$$\begin{aligned} x=a-ba^2&+(2b^2-c)a^3-(5b^3-5bc+d)a^4+(14b^4-21b^2c+6bd+3c^2-e)a^5 \\ &-(42b^5-84b^3c+28b^2d+28bc^2-7be-7cd+f)a^6 \\ &+(132b^6-330b^4c+120b^3d+180b^2c^2-36b^2e-72bcd+8bf+8ce-12c^3 \\ &\qquad\qquad +4d^2-g)a^7-\&c., \qquad (R)\end{aligned}$$

by assuming $x=a+Aa^2+Ba^3+Ca^4+Da^5+Ea^6+Fa^7+\&c.$, in (G).

Now let us take the following equations, viz.;

$$a=x+0x^2+0x^3+0x^4+\&c.,\ \text{an eq'n of the 1st degree.}$$

$$a=x+bx^2+0x^3+0x^4+\ \text{" " " " " 2nd " .}$$

$$\left.\begin{array}{l} a=x+bx^2+cx^3+0x^4+\text{ "} \\ a=x+0x^2+cx^3+0x^4+\text{ "} \end{array}\right\}\ \text{eq'ns " " 3rd " .}$$

$$\left.\begin{array}{l} a=x+bx^2+cx^3+dx^4+0x^5+\&c., \\ a=x+0x^2+cx^3+dx^4+0x^5+\&c., \\ a=x+0x^2+0x^3+dx^4+0x^5+\&c., \end{array}\right\}\ \text{eq's of 4th degree.}$$

$$\left.\begin{array}{l} a=x+bx^2+cx^3+dx^4+ex^5+0x^6+\&c., \\ a=x+bx^2+cx^3+0x^4+ex^5+0x^6+\&c., \\ a=x+0x^2+cx^3+0x^4+ex^5+0x^6+\&c., \\ a=x+0x^2+0x^3+0x^4+ex^5+0x^6+\&c., \end{array}\right\}\ \text{eq'ns of 5th deg., \&c.}$$

From which it appears that equation (G) is a complete expression for an equation of any degree, and that its series may be made to vanish at any term, so as to give any degree, by giving zero values to such of the coefficients b, c, d, &c. as the case may require.

The series in (R), however, will not end after the first degree, but its terms may be greatly contracted, in many cases, by cancelling all of those having zero coefficients. Thus, if in both (G) and (R) we make $c=0$, $d=0$, $e=0$, &c., we get,

$$a=x+bx^2, \qquad (G')$$

$$x=a-ba^2+2b^2a^3-5b^3a^4+14b^4a^5-42b^5a^6+132a^7-\&c. \qquad (R')$$

1. Given $x^2+10x=1$, or $\frac{1}{10}=x+\frac{1}{10}x^2$. Put $a=\frac{1}{10}$ and $b=\frac{1}{10}$ in (R'), then $x=\frac{1}{10}-\frac{1}{10^3}+\frac{2}{10^5}-\frac{5}{10^7}+\&c.=.0990195$

2. $x^2+10x=5$, or $\frac{1}{2}=x+\frac{1}{10}x^2$. Here $a=\frac{1}{2}$, $b=\frac{1}{10}$; therefore

$$x=\frac{1}{2}-\frac{1}{10.2^2}+\frac{2}{10^2.2^3}-\frac{5}{10^3.2^4}+\frac{14}{10^4.2^5}-\frac{42}{10^5.2^6}+\frac{132}{10^6.2^7}=.477225$$

3. $15x^2-50x+7=0$, or $\frac{7}{50}=x-\frac{3}{10}x^2$. Here $a=\frac{7}{50}$, $b=-\frac{3}{10}$;

$$\therefore x=\frac{7}{50}+\frac{3.7^2}{10.50^2}+\frac{2\,3^2.7^3}{10^2.50^3}+\frac{5.3^3.7^4}{10^3.50^4}+\&c.$$

4. Given $x^2+x=1$. Here $a=1$ and $b=1$; $\therefore x=1-2+5-14$, &c.

5. $x^2+x=100$. $a=100$, $b=1$; $x=100-2.100^2+5.100^3-$&c.

From Examples 1, 2 and 3, we see that the values of a and b in formula (R') when applied should be small fractions or the series will not converge rapidly, but may diverge as in Ex's 4 and 5. Hence direct application of the formula would only be practically useful in comparatively few cases.

There is, however, an indirect method, as presented in my Math. Key, which makes the formula applicable to all cases. Thus, in the equation $x^2+x=100$, we may soon find by trial that x lies between 9 and 10, then put $x=\frac{19}{2}+y$, and we have $(\frac{19}{2}+y)^2+(\frac{19}{2}+y)=100$, or $\frac{1}{4.20}=y+\frac{1}{20}y^2$;

$$\therefore x=\frac{19}{2}+y=\frac{19}{2}+\frac{1}{4.20}-\frac{1}{4^2.20^3}+\frac{2}{4^3.20^5}-\text{\&c.}=9.51249.$$

6. Given, $x^2+6x=8$, to find x. As the value is near 1, let $x=1+y$. Then the eq. may be written $\frac{1}{8}=y+\frac{1}{8}y^2$, and $x=1+y=1+\frac{1}{8}-\frac{1}{8^3}+\frac{2}{8^5}$ &c.

7. Required the square root of 2, or the value of x in the eq'n $x^2=2$. Let $x=1.4+y$, as 1.4 is near the value of x. Then $(1.4+y)^2=1.96+2.8y+y^2=2$, or $\frac{1}{70}=y+\frac{5}{14}y^2$. Developing y by (R') as before, we find

$$x=\frac{14}{10}+\frac{1}{70}-\frac{5}{14.70^2}+\frac{2.5^2}{14^2.70^3}-\text{\&c.}$$

In all the examples here given other series might be found that would converge much faster, by taking a nearer value of x before transforming.

In the solution of cubic equations, we may consider the values of all the letters zero, except a, b, c and x, in both (G) and (R), and by then cancel'g all zero terms we have $a=x+bx^2+cx^3$; and for its root,

$$x=a-ba^2+(2b^2-c)a^3-(5b^3-5bc)a^4+(14b^4-21b^2c+3c^2)a^5-\text{\&c.};\quad (R'')$$

or, when $b=0$, $\quad a=x+cx^3$, and

$$x=a-ca^3+3c^2a^5-12c^3a^7+\text{\&c.}\qquad (R''')$$

8. Given $x^3-3x^2+10x=1$, or $\frac{1}{10}=x-\frac{3}{10}x^2+\frac{1}{10}x^3$. Let $a=\frac{1}{10}$, $b=\frac{3}{10}$ and $c=\frac{1}{10}$ in (R''); then $x=\frac{1}{10}+\frac{3}{10^3}+\frac{8}{10^5}-\frac{15}{10^7}-$&c. $=.1031$, nearly.

9. Given, $x^4-3x^2+75x=10000$, to find one root of the equation.

As upon trial x is found nearly equal 10, let $x=9.9+y$, and the equation becomes, $-.0139676=y+.15y^2+$&c. Developing by (R') we get $x=9.9+y=9.9-.0139676-.0000293-.0000001-$&c. $=9.8860027$.

When the fractions of the new equation have large terms it is best to reduce to decimals, as in the last case.

10. Given, $x^5-30x^4+340^3-1800x^2+4384x=3841$, to find all the r'ts.

As the several values of x are near to 2, 4, 6, 8 and 10, respectively; $\therefore$ substitute successively $x_1=2+y$, $x_2=4+y$, $x_3=6+y$, $x_4=8+y$ and $x_5=10+y$. The new equations will then be

$$\frac{1}{4.96} = y - \frac{100}{96}y^2 + \frac{35}{96}y^3 - \frac{5}{96}y^4 + \frac{1}{4.96}y^5,$$

$$-\frac{1}{96} = y - \frac{40}{96}y^2 - \frac{20}{96}y^3 + \frac{10}{96}y^4 - \frac{1}{96}y^5,$$

$$+\frac{1}{64} = y + 0\ y^2 - \frac{20}{64}y^3 + 0\ y^4 + \frac{1}{64}y^5,$$

$$-\frac{1}{96} = y + \frac{40}{96}y^2 - \frac{20}{96}y^3 - \frac{10}{96}y^4 - \frac{1}{96}y^5,$$

$$\frac{1}{4.96} = y + \frac{100}{96}y^2 + \frac{35}{96}y^3 + \frac{5}{96}y^4 + \frac{1}{4.96}y^5.$$

If we now develop each of these equations by (R) and add to the results the corresponding integers, 2, 4, 6, 8 and 10, we shall have the five roots of the equation. Thus,

$$x_1 = 2 + \frac{1}{4.96} + \frac{100}{4^2.96^3} + \frac{16640}{4^3.96^5} + \ldots = 2.002611,$$

$$x_2 = 4 - \frac{1}{96} + \frac{40}{96^3} - \frac{5120}{96^5} + \ldots = 3.989591,$$

$$x_3 = 6 + \frac{1}{64} + \frac{20}{64^4} + \frac{1136}{64^7} + \ldots = 6.015626,$$

$$x_4 = 8 - \frac{1}{96} - \frac{40}{96^3} - \frac{5120}{96^5} - \ldots = 7.989568,$$

$$x_5 = 10 + \frac{1}{4.96} - \frac{100}{4^2.96^3} + \frac{16640}{4^3.96^3} - \ldots = 10.002597.$$

In like manner all irrational roots may be found by the rule expressed by (R) which may therefore be called the Root Theorem.

GEOMETRICAL DETERMINATION OF THE AREA OF THE PARABOLA.

BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLAND.

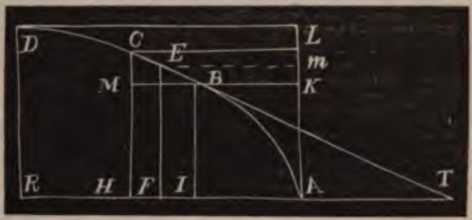
LET AR represent the principal axis of a parabola, so placed as to form one of the equal sides of an isosceles triangle ABR, with the base BR, equal to the corresponding ordinate of the given parabola, and at R erect a perpendicular RP to intersect AB produced in P.

Suppose the cone $PBDC$ completed by revolving PBR about PR; then will CPB, also, be isosceles, and similar to RAB, hence the line AR is parallel to PC, and AB is half of PB.

Along the line AR pass a plane with its cutt'g

edge at right angles with CB, and the section produced, AED, is the given parabola, RB being equal to RD. Through A pass a plane parallel to the base CBD cutting the circle AHF. Since PIA and PRB are similar triangles, $IA = \frac{1}{2}RB$.

Suppose the conical surface to be divided into an infinite number of triangles, as Prs which is supposed to terminate at the point D on the curve, with its base parallel to CB. On RB and RD form the square $RBGD$, and on AI and IH, radii at right angles, form the square $IAKH$, and complete the prismoid $HKGDRBIA$. The face $HKDG$ will be an expansion of the plane Prs, and the plane that cuts the parabola must also cut the prismoid from the corner K to the corner D, hence the line that is tangent at the point D of the parabola will pass through K. Prolong the axis RA to meet DK produced in T. Because $AK\,(=AI)$ by construct'n is parallel to $RD\,(=RB)$, and because it has been shown that $IA = \frac{1}{2}RB$, $\therefore AK = \frac{1}{2}RD$; $\therefore RA = \frac{1}{2}RT$, or the subtang't is bisected at A. The same relation can be shown for the tang't of any other point of the curve by erecting a cone in like manner on its coordinates, and demonstrating as above. When the ordinate exceeds the abscissa a cone with elliptic base must be used.



Let the parabola $ABCD$ (pr'l axis AR) be divided into an infinite number of parts, BC being one of these parts, and BI and CH ordinates at B and C, resp'y. From E, the middle point of BC, let fall the perpendic'lr EF, and draw the tangent ET, then is $FT = 2FA$. On DR and RA complete the parallelogram $DRAL$ and draw CL, Em and MBK parallel to RA; then is $Lm = mK$.

The triangles CMB and EFT are similar, hence $CM : MB :: EF : FT$;

$$\therefore CM \times FT = MB \times EF. \tag{1}$$

The trapezoid $CBHI$ is equal to $MB \times EF$ [by (1)] $= CM \times FT$. (2)

The trapezoid $CBKL$ is equal to $CM \times Em$, or since $Em = FA = \frac{1}{2}FT$,

$$\therefore CBKL = CM \times \tfrac{1}{2}FT. \tag{3}$$

Hence, from (2) and (3), the trapezoid $CBHI = 2CBLK$. And as the same relation holds for all similar trapezoids drawn in the parallelogram $DRAL$, it follows that the area of the parabola is two thirds of its circumscribed parallelogram.

ON A REMARKABLE PROPERTY BELONGING TO SOME CUBICS.

BY C. O. BOIJE AF GENNAS, GOTHENBURG, SWEDEN.

THE well known theorem, Eucl. III, 31, can be enunciated as follows:

If through a point on the circumference of a circle two chords be drawn, making with each other a right angle, the straight line joining the extremities of the chords will pass through a fixed point, the centre of the circle.

Can a Cubic be found having the same, or analogous property? This question will be partially answered in the following discussion.

A straight line drawn through a point, or a cubic, will generally meet the curve in two other points. Let the investigation therefore be limited to that class of *symmetrical* cubics which are represented by the equation

$$y^2 = x^2\,\frac{Ax+B}{Cx+D}, \tag{1}$$

the point through which the chords are to be drawn being the origin.

The equation of a straight line passing through the origin, and making an angle θ with the axis of x is

$$y = x \operatorname{tang} \theta.$$

The coordinates of the point of intersection between (1) and (2) are

$$\left.\begin{aligned} x_1 &= \frac{D\tan^2\theta - B}{A - C\tan^2\theta}, \\ y_1 &= x_1 \tan\theta. \end{aligned}\right\} \tag{3}$$

Let a be the angle included between the two chords, the coordinates of the point of intersection between the other end and the cubic are

$$\left.\begin{aligned} x_2 &= \frac{D\tan^2(a-\theta) - B}{A - C\tan^2(a-\theta)}, \\ y_2 &= -x_2 \tan(a-\theta). \end{aligned}\right\} \tag{4}$$

The equation of the straight line passing through (3) and (4) is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \tag{5}$$

If this line shall pass through a fixed point, that point must be on the axis of x because of the symmetrical form of the cubic. Then, putting $y = 0$ in the equation (5), we find the intersection between that line and the axis of x to be given by the abscissa

$$x = \frac{x_1y_2 - x_2y_1}{y_2 - y_1} = \frac{x_1x_2[\tan\theta + \tan(a-\theta)]}{x_1\tan\theta + x_2\tan(a-\theta)},\ (6),\ =$$

$$\frac{[D\tan^2\theta - B][D\tan^2(a-\theta) - B]}{AD[\operatorname{tg}^2\theta - \operatorname{tg}\theta\operatorname{tg}(a-\theta) + \operatorname{tg}^2(a-\theta)] - CD\operatorname{tg}^2\theta\operatorname{tg}^2(a-\theta) + BC\operatorname{tg}\theta\operatorname{tg}(a-\theta) - AB}$$

In order that this value may be independent of θ, $\frac{dx}{d\theta}$ must be equal to zero. Substituting

$$\tan\theta\tan(a-\theta) = t, \tag{7}$$

we get

$$\frac{dx}{d\theta}=\frac{[t^2(3D^2+2D^2\mathrm{tg}^2a-BD\,\mathrm{tg}^2a)+(2Dt-B)(B-D\,\mathrm{tg}^2a)](BC-AD)}{[t^2(AD\,\mathrm{tg}^2a-CD)+t(BC-2AD\,\mathrm{tg}^2a-3AD)+(AD\,\mathrm{tg}^2a-AB)]^2}\cdot\frac{dt}{d\theta}, \tag{8}$$

where the value of $dt\div d\theta$ is given by the equation

$$\frac{dt}{d\theta} = \frac{\sin(2a-2\theta)-\sin 2\theta}{2\cos^2\theta\cos^2(a-\theta)}. \tag{9}$$

Rejecting the solutions which will transform (1) into the equation of a conic, we see that $dx\div d\theta$ can be made equal to zero, first, if

$$t^2(3D^2+2D^2\tan^2a-BD\tan^2a)+(2Dt-B)(B-D\tan^2a) = 0, \tag{10}$$

which implies that the following eq's must be simultaneously satisfied;

$$\left.\begin{aligned}3D^2+2D^2\tan^2a-BD\tan^2a&=0\\ D(B-D\tan^2a)&=0\\ B(B-D\tan^2a)&=0\end{aligned}\right\} \tag{11}$$

The only acceptable solution of these equations is

$$\left.\begin{aligned}B &= 3D,\\ a &= 60^\circ\ (120^\circ).\end{aligned}\right\} \tag{12}$$

Then the equation of the cubic will be

$$y^2 = x^2\frac{Ax+3D}{Cx+D}, \tag{A'}$$

and having drawn two chords through the origin, making with each other an angle of 60° (120°), the straight line joining the extremities of the ch'ds will pass through a fixed point on the axis of x situated at a distance from the origin, which from equation (6) we find to be

$$x = \frac{8D}{C-3A}. \tag{B'}$$

Secondly, $dx\div d\theta$ can be made equal to zero, if

$$\frac{dt}{d\theta} = 0. \tag{13}$$

Hence it follows that

$$a = 90^\circ. \tag{14}$$

From (6) we then deduce

$$x = \frac{B^2-BD(\tan^2\theta+\cot^2\theta)+D^2}{AD(\tan^2\theta+\cot^2\theta-1)-CD-AD+BC}, \tag{15}$$

and

$$\frac{dx}{d\theta} = \frac{D(AD-BC)(D-B)(\sec^2\theta-\mathrm{cosec}^2\theta)}{[AD(\tan^2\theta+\cot^2\theta-1)-CD-AB+BC]^2}. \tag{16}$$

Rejecting as in the former case solutions which will transform (1) into a conic, we see that $dx \div d\theta$ is equal to zero if

$$D = B. \tag{17}$$

The eq'n of the cubic will then be

$$y^2 = x^2 \frac{Ax + B}{Cx + D}, \tag{A''}$$

and the straight line joining the extremities of two chords, drawn through the origin at right angles to each other, will pass through a fixed point on the axis of x, the abscissa of which, from eq'n (6) or (15) we find to be

$$x = -\frac{B}{A}. \tag{B''}$$

The fixed point can of course lie at an infinite distance from the origin; the right line joining the extremities of the chords will then be parallel to the axis of x.

Concerning the conics, where the constant angle is 90° and the point from which the chords are to be drawn can be taken anywhere on the curve, the investigation is of such a special interest that it deserves a separate treatise, which we hope to give in another number of this periodical.

INTEGRATION OF THE GENERAL EQUATION OF MOTION.

BY DR. J. MORRISON, NAUTICAL ALMANAC OFFICE, WASH., D. C.

The first difficulty which the student encounters in reading Gauss's *Theoria Motus Corporum Celestium*, is found in Article 3, where

$$\alpha x + \beta y + r = \gamma$$

is given as the general equation of the conic sections.

The equation is, I believe, due to La Place, who gave a demonstration of it in the Me'canique Ce'leste, Book II, Chap. III; and therefore, for want of a better name, I shall take the liberty of calling it La Place's Equation to the Conic Sections. I purpose in this brief paper to give a short and easy demonstration of this equation and to discuss it with the view of ascertaining the significance of the constants α, β, γ.

Let us assume the general equations of motion which are

$$\frac{d^2x}{dt^2} = -F\frac{x}{r} \ldots (1), \text{ and } \frac{d^2y}{dt^2} = -F\frac{y}{r} \ldots (2),$$

where F is the force. If F vary as $1 \div r^2$, as is the case in nature, then $F = \mu \div r^2$, where μ is the unit of force at the unit of distance or the absolute force, and (1) and (2) become

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} \ldots (3), \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3} \ldots (4).$$

Multiply (3) by y and (4) by x, subtract the former from the latter and we get

$$x\,\frac{d^2y}{dt^2} - y\,\frac{d^2x}{dt^2} = 0.$$

Integrating we have

$$x\,\frac{dy}{dt} - y\,\frac{dx}{dt} = \text{a constant}$$
$$= h, \text{ suppose,} \qquad (5)$$

and expressing this in polar coordinates we have the well known equation

$$r^2\,\frac{d\theta}{dt} = h, \qquad (6)$$

where h is twice the area described by the radius vector in a unit of time.

Again, $x \div r = \cos\theta$;

$$\frac{d}{dt}\left(\frac{x}{r}\right) = -\sin\theta\,\frac{d\theta}{dt} = -\frac{r\sin\theta}{r^3}\,r^2\,\frac{d\theta}{dt} = -\frac{y}{r^3}\,.\,h; \text{ whence}$$

$$\frac{y}{r^3} = -\frac{1}{h}\cdot\frac{d}{dt}\left(\frac{x}{r}\right).$$

Similarly

$$\frac{x}{r^3} = \frac{1}{h}\cdot\frac{d}{dt}\left(\frac{y}{r}\right).$$

Substituting in (3) and (4), we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{h}\cdot\frac{d}{dt}\left(\frac{y}{r}\right),$$
$$\frac{d^2y}{dt^2} = \frac{\mu}{h}\cdot\frac{d}{dt}\left(\frac{x}{r}\right).$$

By the integration of these we have

$$\frac{dx}{dt} = -\frac{\mu}{h}\left(\frac{y}{r} + \beta\right),$$
$$\frac{dy}{dt} = \frac{\mu}{h}\left(\frac{x}{r} + \alpha\right).$$

Substituting in (5) we get

$$\frac{\mu x}{h}\left(\frac{x}{r} + \alpha\right) + \frac{\mu y}{h}\left(\frac{y}{r} + \beta\right) = h,$$

which is easily reduced to

$$\alpha x + \beta y + r = \frac{h^2}{\mu}, \qquad (7)$$

the equation required.

By comparing this with La Place's Equation we see that γ, the absolute term, is necessarily positive and equal to $h^2 \div \mu$; we shall also see presently

that γ is an important element in the conic section which the above equation represents. Eliminating r from (7) by means of the eq'n $x^2 + y^2 = r^2$ and reducing we have

$$(1 - \alpha^2)x^2 + (1 - \beta^2)y^2 - 2\alpha\beta xy + 2\alpha\gamma x + 2\beta\gamma y - \gamma^2 = 0, \qquad (8)$$

the general equation to a conic section, the origin being in the present case at the focus. It now remains to ascertain the significance of the constants which enter into this equation, and in order to do this it will be necessary to express the general equation to the conic in the following form.

Let RAQ be a portion of the curve, F the focus and origin, HE the directrix, FX and FY the axes and BF the axis of the curve.

Put $\omega =$ the angle BFX, $p = FB$ $e = PF \div PC$ and x and y the coordin's of P; then we have $FD^2 + PD$ $= PF^2 = e^2\,PC^2$, or $x^2 + y^2 = e^2PC$ $= e^2(p - x\cos\omega - y\sin\omega)^2$, or

$$(1 - e^2\cos^2\omega)x^2 + (1 - e^2\sin^2\omega)y^2 - 2e^2xy\sin\omega\cos\omega + 2e^2px\cos\omega + 2e^2py\sin\omega - e^2p^2 = 0. \quad (9)$$

By comp'ng (8) and (9) we see that $\alpha^2 = e^2\cos^2\omega$ and $\beta^2 = e^2\sin^2\omega$, whence

$$e = \sqrt{(\alpha^2 + \beta^2)} \ldots (10), \qquad \tan\omega = \beta \div \alpha \ldots (11).$$

If $\omega = 0$, the axis of X coincides with the axis of the curve, but by (11) when $\omega = 0$, $\beta = 0$ and therefore putting $\beta = 0$ in (8) it becomes

$$(1 - \alpha^2)x^2 + y^2 + 2\alpha\beta x - \gamma^2 = 0;$$

and if $x = 0$, then $y = \gamma$, or the simi latus rectum $= \gamma = h^2 \div \mu$, . . . (12) as was shown above.

By comparing the absolute terms of (8) and (9) we have

$$p = \frac{\gamma}{e} = \frac{\gamma}{\sqrt{(\alpha^2 + \beta^2)}}. \qquad (13)$$

The eq. to the directrix EH is, in terms of ω and p, $x\cos\omega + y\sin\omega = p$. Eliminating ω and p by (11) and (13) we get $\alpha x + \beta y = \gamma$ . . . . . (14) for the equation to the directrix in terms of α, β, γ.

The eq'n $\alpha x + \beta y + r = \gamma$ will of course represent an ellipse, a parabola or a hyperbola according as $\sqrt{(\alpha^2 + \beta^2)}$ is less than, equal to, or greater than 1.

In the special case of α and β being both equal to zero, we have $e = 0$, which characterizes a circle whose eq'n will therefore be $r = \gamma = h^2 \div \mu$.

When $\beta = 0$, $\alpha = e$ and the eq'n bec's $ex + r = \gamma = h^2 \div \mu$ and $\alpha = r\cos\theta$,

$$\therefore r = \frac{h^2 \div \mu}{1 + e\cos\theta},$$

the well known polar equation to the conic sections.

CORRESPONDENCE.—*Editor Analyst:* I see from the note on page 90 of the last ANALYST, that the editor did not refer to the paragraph in Newcomb's Algebra that I intended to. He takes his extract from near the bot. of the page, while I intended to refer to one near the top of the same page. The error was natural, for the author's illustration in the extract referred to by the editor was nearly the same as the one I was using, while in the other case the author was using a quantity of debt by way of illustration, and in order to adapt the reasoning to my example it was necessary to change some of the words, but if I changed the *reasoning* in any sense it was done inadvertantly. The author said "For since the debt is *halved* at every payment, if there was any payment which discharged the whole remaining debt, the half of a thing would be equal to the whole of it, which is impossible."

I said, "Since the remainder is halved, if there was any movement that would overcome the entire remainder, the half of a thing would equal the whole, which is impossible." If I have not correctly represented the *reasoning* I will be pleased to be corrected.

The reference was not made to Newcomb for the sake of criticising him or his book, but because he had put the syllogism in such a terse form.

The point to which I referred involves subtelties of great diffic'ty, which have engaged the attention of philosophers for centuries. In that regard, my plea was for such definitions and such logic as would not provoke vain questionings, nor lead to results contradicted by nature.

DE VOLSON WOOD.

[We have not, now, Prof. Newcomb's Algebra at hand, but believe the above statement of Prof. Wood is substantially correct. We cannot perceive, however, that it, in any way, conflicts with the statements embraced in our note alluded to.

Prof. Newcomb has given two illustr'ns; in the first he supposes one-half of a given debt to be paid at the first pay't, one-fourth at the second and so on; while in the second he supposes one-half of a given line to be traversed at the first step, one-fourth at the second, and so on. The argu't in both is exactly the same, so that our Note applies alike to either, and the principles sought to be established by Prof. Newcomb are equally established by either.

Hence, as stated in our note, Prof. Newcomb's conclusions, based on his hypothesis, are unquestionably true; but if we assume with Prof. Wood that the velocity is uniform, or that the frequency of the payments increases as the value of the successive payments diminishes, then it is equally certain that the total distance will be traversed, or the whole debt will be paid, in a given time. This conclusion, however, does not indicate a fallacy in Professor Newcomb's argument, for the assumption is incompatible with, and virtually ignores, Prof. Newcomb's assumed law.—Ed.]

REMARKS ON THE DOCTRINE OF LIMITS.

BY PROFESSOR SIMON NEWCOMB.

THE very able papers on the doctrine of limits by Professors Judson and Wood in the ANALYST for July 1881 and May 1882 are suggestive of the necessity of a more complete examination of the subject if all possible misapprehension is to be avoided. In the use of any system of mathematical nomenclature a great number of unexpressed hypotheses have to be understood as limiting the statements, and the difficulty is to have these hypotheses equally understood by the writer and the reader. There is a kind of mathematical common sense which has to be taken as the basis of such a discussion, but it is very difficult to see just how far we are to assume that this sense will lead us without reasoning

In the paper referred to it appears to me that the propositions laid down by the respective writers are all correct when correctly interpreted but that their criticisms upon others arise in part from misapprehension. Professor Judson's first five propositions appear not open to any criticism and the following three to be only a definition of what shall be understood by a variable decreasing without limit.

But when, page 109, he shows that Whewell's axiom, *Whatever is true up to the limit is true at the limit,* is unsound, he makes too sweeping a use of the word *whatever*.

The axiom is obviously intended to apply only to finite quantitative relat'ns and therefore does not apply to any of the cases which he assumes. These case are descriptive, not quantitative.

Professor Wood's ingenious *reductio ad absurdum* of my reasoning on the sum of an infinite geometrical progres'n admirably illustrates what I wish to say on the unexpressed hypotheses which lie at the basis of the propositions of limits. He takes my reasoning that a debt can never be wholly disch'd by continually paying one half of it, changes the reading of it so as to make it apply to a prob. of his own in which a point moves a distance one half in half a minute, a distance one fourth in one fourth of a minute, and so on, and puts this changed reading into quotation marks in such a way as wo'ld lead the reader to suppose that I had reasoned in this way about this very problem*. No doubt Professor Wood considers the two problems perfectly analogous and to show their difference we must take up the unexpressed hypotheses. Let us first consider some definitions of the term limit. I take the following two.

*The writer learns that the change of words alluded to was accidental. But the fact remains that Prof. Wood's problem is one that was never considered by the writer.

Wood. "The limit of a variable is a quantity which the variable approaches and from which it may be made to differ by less than any assignable quantity." (Analyst, p. 81.) This definition is too sweeping since it includes every possible value which the variable may take. If we suppose an angle to increase from 0° to 90° and if we assume any quantity k not greater than 1, the sine will in the course of the increase approach this quantity and differ from it by less than any assignable quantity. Hence k is a limit of the sine.

Wentworth. "When it can be shown that the value of a variable, meas'd at a series of definite intervals, can by indefinite continuation of the series be brought to differ from a given constant by less than any assigned quantity, without ever exceeding the constant, that constant is called the *Limit* of the variable, and the variable is said to *approach indefinitely to its limit*." (*Geom.* p. 87.)

It seems to me that the word "exceeding" in this def. should be replaced by *equalling*, because the values of the variable may be alter'ly greater and less than the limit. Otherwise the def. has the adv. of being stated in such a manner as to indicate at once one of the hyp's commonly assumed in silence.

Now a variable quantity may in general have any value whatever. What then do we mean by speaking of a quantity which the variable can never reach or never exceed? Manifestly we must think of it as determined under certain conditions or subject to certain limitations, and the use of the word *limit* must relate to these conditions or relations.

The great object of the method of limits is *to determine quantities in cases where the ordinary operations with finite quantities fail in consequence of some of the quantities of the problem becoming infinite in number, or of zero or infinity entering in such a way that the operations shall not be rigorously practicable.* Thus, the sum of an infinite series would require an infinity of additions and the calculation of the area of a circle by increasing the number of sides of a polygon would also require an infinity of operations. But if we transform the problem so as to avoid an infinity of operations it is a perversion of speech to apply the word limit to the result. But this is just what Professor Wood has done. In the case he supposes, of a point mov'g half the distance in half a minute, half the remaining distance in one fourth of a minute and so on, that is, of moving with a uniform motion, he entirely removes the problem from the class to which the method of limits applies. Of course the ambiguity of the word *never* comes in as a source of perturbation. Define the sense in which it is used, and the *reductio ad absurdum* becomes applicable only to Prof. Wood's way of considering the problem. When used in the method of limits it is understood to imply an indef. continuat'n of some series of operations which cannot be executed in any finite time, and of which we therefore say they can *never* be completely executed.

ANSWER TO QUERY. (See p. 64 or 94.) *By Prof. C. A. Van Velzer, University of Wisconsin, Madison, Wis.*—In the determinant

$$\begin{vmatrix} 0 & b & c \\ b & 1 & a \\ c & a & 1 \end{vmatrix}$$

sub't from the third column $a \div b$ times the first column, and there results

$$\begin{vmatrix} 0 & b & c \\ b & 1 & 0 \\ c & a & 1 - ac \div b \end{vmatrix}.$$

In this determinant add to the first column $2ab$ times the third and we have

$$\begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c + 2ab - 2a^2c & a & 1 - ac \div b \end{vmatrix}.$$

In this determinant add to the third row $a \div b$ times the first, and subtract $2a$ times the second and we get

$$\begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix};$$

the required determinant, where, throughout, a is used for short in place of $\cos A$.

The transformation from this last determinant back to the first is also easy to effect.

Any determinant may in general be transformed into any other equivalent determinant of the same order. Let

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{vmatrix}$$

and let us represent the first of these by $\triangle_a$ and the second by $\triangle_x$.

To the fourth column of $\triangle_a$ add l_1 times the first column, m_1 times the second and n_1 times the third, where l_1, m_1, n_1 are chosen to satisfy the eq's

$$l_1a_1 + m_1b_1 + n_1c_1 = w_1 - d_1,$$
$$l_1a_2 + m_1b_2 + n_1c_2 = w_2 - d_2,$$
$$l_1a_3 + m_1b_3 + n_1c_3 = w_3 - d_3.$$

By this means $\triangle_a$ is changed into

$$\begin{vmatrix} a_1 & b_1 & c_1 & w_1 \\ a_2 & b_2 & c_2 & w_2 \\ a_3 & b_3 & c_3 & w_3 \\ a_4 & b_4 & c_4 & \delta_4 \end{vmatrix}.$$

In this determinant add to the third column l_2 times the first, m_2 times the second and n_2 times the fourth, where l_2, m_2 and n_2 are chosen to satisfy the three equations

$$\begin{aligned} l_2a_1 + m_2b_1 + n_2w_1 &= z_1 - c_1, \\ l_2a_2 + m_2b_2 + n_2w_2 &= z_2 - c_2, \\ l_2a_3 + m_2b_3 + n_2w_3 &= z_3 - c_3. \end{aligned}$$

By this means we obtain the determinant

$$\begin{vmatrix} a_1 & b_1 & z_1 & w_1 \\ a_2 & b_2 & z_2 & w_2 \\ a_3 & b_3 & z_3 & w_3 \\ a_4 & b_4 & \gamma_3 & \delta_4 \end{vmatrix}.$$

In this determinant add to the second column l_3 times the first, m_3 times the third and n_3 times the fourth, where l_3, m_3, n_3 satisfy the equations

$$\begin{aligned} l_3a_1 + m_3z_1 + n_3w_1 &= y_1 - b_1, \\ l_3a_2 + m_3z_2 + n_3w_2 &= y_2 - b_2, \\ l_3a_3 + m_3z_3 + n_3w_3 &= y_3 - b_3. \end{aligned}$$

We thus arrive at the determinant

$$\begin{vmatrix} a_1 & y_1 & z_1 & w_1 \\ a_2 & y_2 & z_2 & w_2 \\ a_3 & y_3 & z_3 & w_3 \\ a_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix}.$$

Now add to the first column l_4 times the second, m_4 times the third and n_4 times the fourth where l_4, m_4 and n_4 satisfy the equations

$$\begin{aligned} l_4y_1 + m_4z_1 + n_4w_1 &= x_1 - a_1, \\ l_4y_2 + m_4z_2 + n_4w_2 &= x_2 - a_2, \\ l_4y_3 + m_4z_3 + n_4w_3 &= x_3 - a_3. \end{aligned}$$

This gives the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ a_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix}.$$

To the fourth row of this determinant add l_5 times the first, m_5 times the second, and n_5 times the third row, where l_5, m_5, n_5 satisfy the equations

$$\begin{aligned} l_5x_1 + m_5x_2 + n_5x_3 &= x_4 - a_4, \\ l_5y_1 + m_5y_2 + n_5y_3 &= y_4 - \beta_4, \\ l_5z_1 + m_5z_2 + n_5z_3 &= z_4 - \gamma_4. \end{aligned}$$

By this means the elements a_4, β_4, γ_4 are changed into x_4, y_4, z_4, and since the resulting determinant must equal $\triangle_x$, it follows that the element δ_4 will be changed into w_4 and the determinant $\triangle_x$ is thus arrived at.

The process here employed is evidently applicable to determinants of any order. The process is a definite one and leads to definite results unless some of the quantities l, m, n become infinite through the vanishing of the determinant of their coefficients, which, in every case, is a first minor of $\triangle_a$ or $\triangle_x$ or one of the intermediate determinants. Should this thing happen

we must transform $\triangle_a$ not directly into $\triangle_x$ but into a determinant prepared from $\triangle_x$ so that the difficulty will be avoided. One case of difficulty, however, cannot be avoided by changing $\triangle_x$ beforehand and that is where

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

In this case, before we apply the general method we must change $\triangle_a$ so that the first minor corresponding to d_4 shall not vanish.

I believe that, in every case where the difficulty here spoken of arises, $\triangle_a$ or $\triangle_x$ or both may be changed, before applying the process of this paper, into determinants say $\triangle'_a$ and $\triangle'_x$ so that $\triangle'_a$ may be transformed into $\triangle'_x$ and therefore $\triangle_a$ into $\triangle_x$. If so, the transformation can *always* be effected; if not, it can *generally* be effected.

I have not yet found time to consider this last question.

[Professor L. G. Barbour answers Mr. Eastwood's Query (see p. 96) as follows:]

* * * "For much the same reason that it is allowable and sufficient to write $\log(1+y) = My+Ny^2+Py^3$ &c. It would not do to write $\log y = My+Ny^2+Py^3$ &c., nor $\log(1+y) = Ny^2+Py^3$ &c. And it it would not do to write $dp\div dt = q^2$, $dq \div dx = p^2$. At least there seems to be no antecedent likelihood that the former of these diff. coefficients could have been obtained from a combination of p, q and t. Nor could we write

$$\frac{dp}{dt} = pq, \frac{dq}{dx} = pq, \text{ for then } \frac{dp}{dt} \text{ must} = \frac{dq}{dx},$$

which is not a condition given. We then *assume* (if this term be prefer'd)

$$\frac{dp}{dt} = p^2, \frac{dq}{dx} = q^2,$$

which certainly fulfils the most obvious condition, viz.; that

$$\frac{dp}{dt}\cdot\frac{dq}{dx} = p^2q^2,$$

and does not present any difficulty on the threshhold. The integration procedes easily, and without encountering any unforeseen obstacle, to the complete primitives

$$e^{\psi} = \frac{c'}{t+c}; \; e^{\psi} = \frac{c'''}{x+c''};$$

$$\therefore x = tc_4+c_5.$$

In the part of my solution omitted in printing, it was proven that there could be no singular solution. Hence it seems to the writer that the guarded phrase 'allowable and sufficient', instead of the customary 'necessary and sufficient', is sustained by analogy and confirmed by the result."

A PENDULUM WHOSE TIME OF OSCILLATION IS INDEPENDENT OF THE POSITION OF ITS CENTRE OF GRAVITY.

BY R. J. ADCOCK, ROSEVILLE, ILL.

If a rigid body be suspended from fixed points in a horizontal plane by parallel rods of equal length, and then made to vibrate, the rods moving in parallel vertical planes, the time of a vibration is the same as that of a simple pendulum whose length is equal to that of the rods, provided the weight of the rods is small in comparison with that of the body. For the points of the body describe equal similarly situate arcs of equal circles.

Suspension Scales. — Let the horizontal platform be suspended from points in a horizontal plane, by parallel rods of equal length, such that, when the platform is drawn say 6 inches to one side, by a horizontal force appl'd near the middle of one side; the vertical distance between the turning p'ts of suspension and those where the rods connect with the platform may be say 10 ft, then $\frac{1}{2}:10::$ horiz'l force : weight of platform and load, that is, if the horizontal force be 150 lbs, the weight will be 3000 lbs.

The horiz'l force may be app'd by connecting the platform with the vertical arm of a bent lever turning about a fulcrum at its right angle while the horizontal graduated arm carries a movable weight, in the usual manner, showing the weight in lbs of the matter to be weighed.

Notice of Dr. Casey's Geometry. By George Eastwood.—Dr. Casey, Professor of Higher Mathematics, and Mathematical Physics, in the Catholic University of Ireland, has recently brought out a new edition of his unique Sequel to the First Six Books of Euclid, in which he has contrived, within the space of 158 pages, to arrange and pack more geometrical gems than are to be found in any single text book on Geometry that has appear'd since the days of the self-taught Thomas Simpson. "The principles of Modern Geometry, contained in the work, are, in the present state of Science, indispensable in Pure and Appl'd Mathematics, and Mathematical Physics; and it is important that the Student should become early acq'd with them."

Eleven of the sixteen sections, into which the work is divided, exhibit most excellent specimens of geometrical reasoning and research. These will be found to furnish very neat models for systematic Methods of study.

The other five sections comprise 261 choice problems for solution. Here the earnest student will find all that he needs to help bring himself abreast with the amazing developments that are being made, almost daily, in the vast regions of Pure and Applied Geometry. Here, also, the pains-taking Teacher and the faithful Professor will find a rich store of exercises from which to test the *grit* and proficiency of their pupils.

TABLE OF SQUARE AND CUBE ROOTS.

BY J. M. BOORMAN, ESQ., NEW YORK CITY.

$\sqrt{2}$ = 1.414213 562373 095048 801688 724209 698078 569671 875376
948073 176679 737990 732478 462107 038850 387534 327641
572735 013846 230912 297024 924823 405585 073721 206441

$\sqrt[3]{2}$ = 1.259921 049894 873164 767210 607278 228350 570251 464701
507980 081974 923167 206844 7

$\sqrt{3}$ = 1.732050 807568 877293 527446 341505 872366 942805 253810
380628 055806 979451 933016 908800 037081 146186 740794
093003 721807 125250 514518 309014 808658

$\sqrt[3]{3}$ = 1.442249 570307 408382 321638 310780 109588 391869 253499
35041+

$\sqrt[3]{4}$ = 1.587401 051968 199474 751705 639272 308260 391493 327899
865

$\sqrt{5}$ = 2.236067 977499 789696 409173 668731 276235 440618 359611
525724 270897 245410 520925 637804 899414 414408 378782
274869 508164 746826 651038 3—

$\sqrt[3]{5}$ = 1.709975 946676 696989 353

$\sqrt{6}$ = 2.449489 742783 178098 197284 074705 891391 965947 480656
670128 432692 567250 960377 457315 7

$\sqrt{7}$ = 2.645751 311064 590590 501615 753639 260425 710259 183082
450180 368334 459201 068823 230283 627760 39286

$\sqrt{8}$ = 2.828427 124746 190097 603397 448419 396157 139343 750753
896146 353359 475981

$\sqrt{10}$ = 3.162277 660168 379331 998893 544432 718533 719555 139325
216826 857504 852792 562815 862633 537558 833813 594245
29355[illegible] 405638 666

$\sqrt{11}$ = 3.316624 790355 399849 114932 736670 686683 927088 545589
353597 058682 147322 530020 920098 337296 091667 01

$\sqrt{13}$ = 3.605551 275463 989293 119221 267470 495946 251296 574399
946408 935682 178239

$\sqrt{14}$ = 3.741657 386773 941385 583748 732316 549301 756019 807778
726946 303735 467320 035126 307339 07

$\sqrt{15}$ = 3.872983 346207 416885 179267 399782 399610 832921 705291
590826 587573 766113 483079

$\sqrt{17}$ = 4.123105 625617 660549 821409 855974 077025 147199 225373
620434 289492 541828 604496 127332 312551 147476 621448
569577 078826 466444 645428 74.

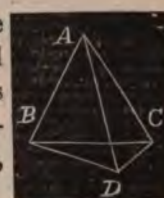
SOLUTIONS OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received as follows:

From J. M. Boorman, 393, 394; Prof. W. P. Casey, 399; Geo. E. Curtis, 395; George Eastwood, 392; William Hoover, 398; Henry Heaton, 399; Prof. Asaph Hall, 397, 400; Thos. Spencer, 397, 399; M. Updegraff, 399; Prof. C. Van Velzer, 397.

392. *By George Eastwood, Saxonville, Mass.*—"In a triangular pile of round shot, each shot rests upon three other shot, thus forming an empty space. It is required to find the ratio of the capacity of all the spaces to the capacity of all the balls."

SOLUTION BY THE PROPOSER.



In the annexed diagam, let A represent the centre of the top shot, and B, C, D the centres of the shot on the second course from the top of the pile. Conceive a plane to pass through these centres, and to become the base of a regular tetrahedron whose vertex is A. For the edges we have, suppose, $AB = AC = AD = BC = BD = CD = 2r$.

The angles BAC, CAD, DAB are evidently equal; hence their respective cosines are equal, and each equal to $\frac{1}{2}$. And from an inspection of the figure, we readily perceive that each solid cuts from its concentric shot, a spherical triangle which forms the base of a spherical pyramid that is common to the tetrahedron and said shot. But the sides of the spherical triangles are obviously the measures of the angles BAC, CAD, DAB, whose respective cosines we have already found to be each $= \frac{1}{2}$.

Let the spherical excess of each spherical triangle be designated by ε, and its area by Σ. By a formula due to *De Gua*, we have

$$\text{vers}\,\varepsilon = \frac{1-\cos^2 A_1-\cos^2 A_2-\cos^2 A_3+2\cos A_1\cos A_2\cos A_3}{(1+\cos A_1)(1+\cos A_2)(1+\cos A_3)}$$
$$= \frac{4}{27}.$$

Whence $\varepsilon = \text{vers}^{-1}\frac{4}{27} = \cos^{-1}\frac{23}{27}$.

From this value of ε we have $\Sigma = r^2\varepsilon\dfrac{3.1416}{180°} = r^2\cos^{-1}\dfrac{23}{27}\,\dfrac{3.1416}{180°}$.

The capacity of each spherical pyramid is $\frac{1}{3}r\Sigma$, and the capacity of the four is $\frac{4}{3}r\Sigma$.

By a formula of Le Gendre, the capacity, k, of the tetrahedron is

$$k = \frac{AB.AC.AD\sqrt{(1-\cos^2 A_1-\cos^2 A_2-\cos^2 A_3+2\cos A_1 \cos A_2 \cos A_3)}}{6}$$

$$= \tfrac{2}{3}r^3\sqrt{2}.$$

Hence the capacity of *one* empty space is

$$\frac{r^3}{3}\left(2\sqrt{2}-4\Sigma\right) = \frac{r^3}{3}\left(2\sqrt{2}-4\cos^{-1}\frac{23}{27}\frac{3.1416}{180^\circ}\right)$$

$$= 0.20772r^3 = \text{capacity of } one \text{ shot} \times .0496 \quad \text{f}$$

$$= \text{capacity of one shot} \div 20.16.$$

Let n = number of shot in one side of the bottom course of the pile, then

$\tfrac{1}{6}n(n+1)(n+2)$ = number of shot in the whole pile, and

$\tfrac{1}{6}n(n-1)(n+1)$ = number of empty spaces in same.

Hence, capac. of whole pile : capac. of empty sp's :: $(20.16)(n+1):(n-1)$.

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393. *By J. M. Boorman, Esq., New York City.*—State the general equation of the 4th degree in terms whose coefficients shall be real and direct functions of its roots and admit a solution showing the root's *real* nature—no root to be directly *expressed* by a letter.

394. *Id.*—Show that the general equation of the 4th degree has its companion biquadrate, and state it and the respective relat'ns of their roots.

ANSWER BY THE PROPOSER.

$$X^4-4dX^3+(6d^2-2a)X^2-(4d^3-4ad-f)X+d^4-2ad^2-fd+a^2-b=0, \quad (A)$$

which, for $d=0$, or by removing the second term, is

$$X^4 - 2aX^2 + fX + a^2 - b = 0; \quad (B)$$

where $X = d \,,\!\pm \tfrac{1}{2}S \pm\sqrt{[a + FS \,,\!\mp F(f \text{ and } S)]}, \quad (C)$

$S = r+t;\ r+l;\ r+e;\ t+e;\ r, t, l, e$ being the 4 roots of (B), and $a = \tfrac{1}{4}(S^2+S_{\backslash}^2+S_{\backslash\backslash}^2)$; $b = \tfrac{1}{4}(S^2S_{\backslash}^2+S^2S_{\backslash\backslash}^2+S_{\backslash}^2S_{\backslash\backslash}^2)$; $f = SS_{\backslash}S_{\backslash\backslash}$; $S, S_{\backslash}, S_{\backslash\backslash}$ being 3 roots of the equation $S^6-4aS^4+4bS^2-f^2=0$.

The inverted prime ($_\backslash$), above, means that $\pm$ signs so marked change only in unison.

[In common with some of our readers, we failed to perceive the exact sig-significance of these questions, but, as the author claims to have made some important discoveries in relation to the roots of such equations, we hoped to be enlightened by his answers. The foregoing answer has been submitted and is accompanied by an Example, and also an answer to 394, but as we fail to understand equation (C) above we withhold the remainder until that equation shall be more explicitly stated.—Ed.]

395. *By C. O. Boije af Gennas, Gothenburg, Sweden.*—"Determine the law of density of a sphere in order that its centre of gravity may be coincident with the centre of gravity of the half sphere cut off from the sphere."

SOLUTION BY GEO. E. CURTIS, NEWHAVEN, CONN.

The problem is essentially indeterminate. A simple law is obtained as follows. The distance of the center of gravity of a sphere from a point on its circumference taken as the pole is, by a formula of polar coordinates,

$$\frac{\int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\theta} D.r^3 \sin\theta\cos\theta\, d\theta\, dr}{\int_0^{\frac{1}{2}\pi}\int_0^{2a\cos\theta} D.r^2 \sin\theta\, d\theta\, dr},$$

where a is the radius and D, the density.

The center of gravity must coincide with the center of gravity of a homogeneous hemisphere of equal radius; hence the integral must equal $\frac{5}{8}a$.

If the law of density is $1\div(r\cos\theta)$, the integral gives the distance $\frac{2}{3}a^3\div a^2$; if the law is $1\div r$ we obtain $\frac{8}{15}a^3\div\frac{2}{3}a^2$. As $1\div(r\cos\theta)$ gives a result slightly too large, subtract from it $1\div r$ multiplied by a constant factor, c, and we shall obtain the required law.

To obtain the constant factor, we have the equation $(\frac{2}{3}-\frac{8}{15}c)\div(1-\frac{2}{3}c)=\frac{5}{8}$. This gives $c=\frac{5}{14}$, and the corrected law of the density is

$$\frac{1}{r\cos\theta}-\frac{5}{14r}.$$

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396. No solution received.

397. *By Prof. J. M. Rice, U. S. Naval Academy.*—"Given

$$\varphi(x^2)\,\varphi(y^2) = \varphi(x'^2)\,\varphi(y'^2)$$

and

$$x^2+y^2 = x'^2+y'^2,$$

to determine the form of the function denoted by φ."

SOLUTION BY PROF. ASAPH HALL.

Put $x^2 = u$, $y^2 = v$, etc., then

$$\varphi(u)\,.\,\varphi(v) = \varphi(u')\,.\,\varphi(v') = f(u,v).$$

Differentiating,

$$\varphi'(u)\,.\,\varphi(v) = f'(u,v): \qquad \text{and } \varphi(u)\,.\,\varphi'(v) = f'(u,v).$$

Hence

$$\varphi'(u)\,.\,\varphi(v) = \varphi(u)\,.\,\varphi'(v), \text{ or}$$

$$\frac{\varphi'(u)}{\varphi(u)} = \frac{\varphi'(v)}{\varphi(v)} = k; \text{ a constant.}$$

Therefore $\log \varphi(u) = C.ku,$

and $\varphi(x^2) = C.e^{kx^2}.$

The second given equation does not seem to be necessary for the solution, but follows as a result.

SOLUTION BY THOMAS SPENCER, SOUTH MERIDEN, CONN.

Take the logarithm of the first equation to any base a, and we have

$$\log_a\varphi(x^2) + \log_a\varphi(y^2) = \log_a\varphi(x'^2) + \log_a\varphi(y'^2).$$

But $x^2+y^2 = x'^2+y'^2$ is of that form; therefore we are at liberty to put $\log_a\varphi(x^2) = bx^2$, $\log_a\varphi(y^2) = by^2$, $\log_a\varphi(x'^2) = bx'^2$ and $\log_a\varphi(y'^2) = by'^2$, where b is any quantity.

Hence we have $\varphi(x^2) = a^{bx^2}$, $\varphi(y^2) = a^{by^2}$, &c., which gives the form of the function.

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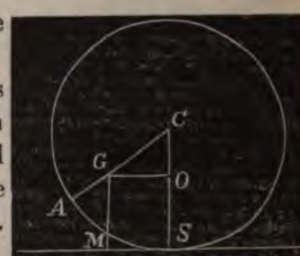
398. *By Wm. Hoover, A. M., Dayton, Ohio*—"An angular velocity having been impressed upon a heterogeneous sphere, about an axis, perp. to the vertical plane which contains its center of gravity G and geometrical center C, and passing through G, it is then placed on a smooth horizontal plane; to find the magnitude of the impressed angular velocity that G may rise into a point in the vertical line SCK through C, and there rest; the angle GCS being a at the beginning of the motion, c, the radius, and φ the req'd angular velocity."

SOLUTION BY THE PROPOSER.



Draw the radius CGA, and from G drop the perpendicular GM upon the plane.

Let $m =$ the mass of the sphere, k, the radius of gyration of the sphere about an axis through G perpendicular to the plane containing C and G, R, the mutual reaction of the sphere and the plane, $SM = x$, $GM = y$, $CS = CA = a$, ang. $AGM =$ angle $ACS = \varphi$ and $CG = c$.

Since there is no friction, we have for the motion of the sphere

$$m\frac{d^2x}{dt^2} = 0; \tag{1}$$

resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg, \tag{2}$$

and taking moments about G,

$$mk^2\frac{d^2\varphi}{dt^2} = -Rc\sin\varphi, \tag{3}$$

φ being the angular velocity of the sphere.

We have from the geometry, $y = ac\cos\varphi$, whence

$$\frac{d^2y}{dt^2} = c\sin\varphi\,\frac{d^2\varphi}{dt^2} + c\cos\varphi\,\frac{d\varphi^2}{dt^2}.$$

Substituting in (2),

$$R = m\left(c\sin\varphi\,\frac{d^2\varphi}{dt^2} + c\cos\varphi\,\frac{d\varphi^2}{dt^2} + g\right).$$

This in (3) gives by reduction

$$(c^2\sin^2\varphi + k^2)\frac{d^2\varphi}{dt^2} + c^2\sin\varphi\cos\varphi\,\frac{d\varphi^2}{dt^2} = -cg\sin\varphi. \tag{4}$$

Integrating,

$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = C + 2cg\cos\varphi. \tag{5}$$

Let $t = 0$, when $\varphi = 0$; $d\varphi \div dt = \omega$, and $C = (c^2\sin^2\varphi + k^2)\omega^2 - 2cg$. Hence (5) becomes

$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = (c^2\sin^2\varphi + k^2)\omega^2 - 2cg(1-\cos\varphi). \tag{6}$$

Now if the initial value of $\varphi = a$, the terminal value $= a+\pi$, when also $\frac{d\varphi}{dt} = 0$; then the left member of (6) becomes 0, and $\omega^2 = \frac{2cg(1-\cos\varphi)}{c^2\sin^2\varphi + k^2}$.

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399. *By E. J. Esselstyn, New Haven, Conn.*—Given two points A and B, and a circle K having its centre at O. Let any circle L be drawn thro' A and B so as to cut the circumf. of the circle K in two variable points m and n. Show that the circle through O, A and B is cut by the variable circle through O, m and n, in a fixed point P.

SOLUTION BY HENRY HEATON.

In the figure the circle through A, B and O intersects the circle K in M and N, and the chords AB and MN intersect in D; $\therefore$ $AD \times DB = MD \times DN$. If we draw MD and prolong it till it cuts the circle K in n' and the circle L in n'', we have $mD \times Dn' = MD \times DN$, and $mD \times Dn'' = AD \times DB$. Hence $Dn' = Dn''$, or n' and n'' coincide in the point n.

If we draw OD and prolong it until it cuts the circle OAB in P' and the circle Omn in P'', we have $OD\times DP' = AD\times DB$ and $OD\times DP'' = mD\times Dn = AD\times DB$. Hence $DP' = DP''$ or P' and P'' coincide in the point P. But AB and MN being fixed lines D is a fixed point, and the line OD cuts the circle OAB in the fixed point P.

SOLUTION BY THOMAS SPENCER.

M, N being the intersections of the circles K and OAB, we see that the straight lines MN, AB, and mn are the radical axes of the three circles K, L and OAB, and D their radical centre, which is a fixed point, because the lines MN and AB are fixed. Also D is the radical centre of the three circles K, Omn and OAB, mn, MN and OP being their radical axes; $\therefore$ OP is a fixed straight line, because the points O and D are fixed points. Hence, because the circle OAB is invariable, the point P is fixed.

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400. *By Prof. Asaph Hall.*—"In a plane passing through the centre of the sun, 12 right lines are drawn from this centre making an angle of 30° with each other. On each of these lines, three homogeneous spherical bodies are placed at distances respectively of 10, 20 and 30 from the centre of the sun; the distance from the earth to the sun being the unit of distance.

The mass of each of these bodies being equal to that of the sun, what will be the velocity of a particle that starts from an infinite distance and moves in a right line towards the centre of the sun, and perpendicular to the plane of the bodies, when the particle is at a distance of 0.01 from the centre of the sun; the law of attraction being that of Newton?"

SOLUTION BY PROFESSOR HALL.

Let m be the mass of the sun, x the distance of the particle from the centre of the sun at the time t; then the equation of motion is

$$\frac{d^2x}{dt^2}+m.\left\{\frac{1}{x^2}+\frac{12x}{(x^2+100)^{\frac{3}{2}}}+\frac{12x}{(x^2+400)^{\frac{3}{2}}}+\frac{12x}{(x^2+900)^{\frac{3}{2}}}\right\}=0.$$

Integrating,

$$\frac{dx}{dt}=v=(2m)^{\frac{1}{2}}\left\{\frac{1}{x}+\frac{12}{(x^2+100)^{\frac{1}{2}}}+\frac{12}{(x^2+400)^{\frac{1}{2}}}+\frac{12}{(x^2+900)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

Let v' be the mean velocity of the earth in its orbit at the distance unity, then $v'^2=m$: and $v'=18.4166$ miles per second. Hence $v=263.3$ m's per second. If there were no body but the sun, the velocity at the same distance would be 260.5 miles per second. The addition of the other bodies therefore produces only a small change in the velocity. An increase in the mass of the sun would be much more effective.

Note by Prof. Johnson.—The following formula in Spherical Trigonometry is not to be found in any treatise I have seen.

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c} = \frac{1+\cos A\cos B\cos C}{1-\cos a\cos b\cos c}.$$

The proof is very simple; from the usual formulæ we have

$$\cos^2 a = \cos a\cos b\cos c + \sin b\sin c\cos A\cos a;$$

adding $\sin^2 a$ and transposing,

$$1 - \cos a\cos b\cos c = \sin^2 a + \sin b\sin c\cos A\cos a; \qquad (1)$$

also $\cos^2 A = -\cos A\cos B\cos C + \sin B\sin C\cos a\cos A,$

whence $1 + \cos A\cos B\cos C = \sin^2 A + \sin B\sin C\cos a\cos A. \qquad (2)$

Dividing (2) by (1), we have the formula.

PROBLEMS.

401. *By M. Updegraff, Madison, Wis.*—If two triang's are so situated that the three lines drawn thro' their corresponding vertices meet in a point, then will the corresponding sides produced meet in three points which lie on the same straight line.

402. *By Prof. W. P. Casey.*—Upon two sides of a triangle, describe equilateral triangles, and upon the same two sides, but in the opposite direction, describe two others, and let O, O_1 be the centres of the inscribed circles in the first pair and P, P_1 those of the second pair. It is required to prove, geometrically, that the sum of the squares of the sides of the triangle $= 3(OO_1)^2 + 3(PP_1)^2$.

403. *By Prof. W. W. Johnson.*—If, from the centre C of an equilat'l hyperbola, CA be drawn bisecting the angle between the axis and an asymptote, and the chord AB be drawn perpendicular to CA; then $AB = 2CA$.

404. *By Prof. M. L. Comstock.*—A heavy triangle ABC is suspended from a point by three strings, mutually at right angles, attached to the angular points of the triangle; if θ be the inclination of the triangle to the horizon in its position of equilibrium, then

$$\cos\theta = \frac{3}{\sqrt{(1+\sec A\sec B\sec C)}}.$$

(Todhunter's Analytical Statics, page 81.)

405. *By Prof. C. A. Van Velzer.*—Prove that a determinant which vanishes may be transformed into one having, at the same time, two identical rows *and* two identical columns.

406. *By William Hoover, A. M. Dayton, Ohio.*—Find x from the eq'n

$$\cot 2^{x-1} a - \cot 2^x a = \operatorname{cosec} 3a.$$

407. *By Henry Heaton, Lewis, Iowa.*—Evaluate

$$\int_0^{\frac{\pi}{2}} \left(1 + \cos^4\theta\right)^{\frac{1}{2}} d\theta.$$

408. *By W. E. Heal.*—Two points, one on each of two confocal ellipsoids, are said to correspond if

$$\frac{x}{a} = \frac{X}{A}, \quad \frac{y}{b} = \frac{Y}{B}, \quad \frac{z}{c} = \frac{Z}{C}.$$

Prove that the distance between two points, one on each of two confocal ellipsoids is equal to the distance bet. the corresp. points. (Ivory's Th.)

PUBLICATIONS RECEIVED.

Logarithms. By H. N. WHEELER. Used at Harvard College in connect'n with Wheeler's Trigonometry and Peirce's Logarithm Tables. 43 pages. Cambridge: 1882.

The Multisector and Polyode. (Pamphlet.) By J. W. NICHOLSON, A. M., Baton Rouge, La.

The Multisector is an instrument devised for drawing a curve, the "Polyode", by which an angle is not only trisected but may be divided into any number of equal parts.

New, simple and Complete Demonstration of the Binomial Theorem and Logarithmic Series. By J. W. NICHOLSON, M. A.

Newcomb's Mathematical Course:

Elements of Geometry; 399 pages. New York: Henry Holt & Co. 1881:

A School Algebra; 279 pages:

Algebra for Schools and Colleges; 474 pages. New York: Henry Holt & Co. By PROFESSOR SIMON NEWCOMB, U. S. Navy.

Professor Newcomb is so well, and favorably, known as a writer, that any commendation of these books is unnecessary. It is sufficient to say that the Public expect from this author nothing below *first class* productions, and that they will not be disappointed in these books.

ERRATA.

On page 65, line 12, for "weights" read masses.

" " 91, " 10, 11 and 14, change last sign from $-$ to $+$.

" " 93, for "S" and "P" on line AB of Fig., read F and F', and for "x", at foot of perp. from D, read S.

" " 94, line 19 from bottom, for "division", read divisors.

" " 110, " 17, for "equation of motion", read equations, &c.

" " " " 20, for "*Celestium*" read *Cœlestium*.

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THE DENSITY OF THE EARTH.

BY PROFESSOR ASAPH HALL.

IN a paper recently published by the U. S. Coast and Geodetic Survey, Mr. C. A. Schott gives the results of a comparison of the geodetic and astronomical positions of sixty places on our eastern coast; these places extending from Calais, Maine, to Atlanta, Georgia. This comparison brings to light some curious differences in latitude and longitude; and if we consider the accuracy with which this survey is conducted, it is probable that the most of these differences are real, and are caused by local attractions and variations in the earth's crust. Thus at Washington, the plumb line is drawn 6″ towards the west. Mr. Schott's interesting paper has recalled to mind an idea that occurred to me some time ago of determining the density of the earth by means of the tides in the Bay of Fundy. No doubt the same idea has occurred to others, and by a reference of Professor Wright, of Yale College, I find in Thomson and Tait's Natural Philosophy, Art. 818, an approximate formula for the effect of such a mass of water on the plumb line. The advantage of this method lies in the fact that the attracting mass is homogeneous and of known density, while in the case of a mountain, the density of the attracting mass is uncertain. On the other hand, the attraction of a mountain will generally be greater than can be got from the shifting of the tides, and the difficulty of measuring small angular quantities with the necessary accuracy is a serious hindrance in the application of the tidal method. This difficulty may be overcome perhaps by the employment of more refined apparatus, such as the horizontal pendulum. Another difficulty would be the determination of the form of the water, but probably this could be found from surveys and soundings. I confess, however, that it seems to me doubtful if any of these methods can ever give results as accurate as those found by Baily from his experiments made in such

a complete manner with the Coulomb torsion balance. Still, the results found by independent methods have a value, and these methods ought not to be neglected.

In order to see how great would be the effect of a rectangular mass of water on a plumb line, let us take the axis of x so that it bisects the upper surface of the water, and is at right angles to the edge. Let the origin be in the axis of x at a distance δ from the edge; let k be a constant depending on the unit and density of mass, and denote by A the component of attraction in the axis of x. Then if a is the distance from the origin to the farther side of the water, 2β the width of the rectangular mass, and γ its depth, we shall have

$$A = k.\int_{\delta}^{a}\int_{-\beta}^{+\beta}\int_{0}^{\gamma}\frac{x.dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

This is an exact differential with respect to x, and after two integrations we have,

$$A = -k.\int_0^{\gamma} dz.\log[\beta+\surd(a^2+\beta^2+z^2)]+k.\int_0^{\gamma} dz.\log[-\beta+\surd(a^2+\beta^2+z^2)]$$
$$+k.\int_0^{\gamma} dz.\log[\beta+\surd(\delta^2+\beta^2+z^2)]-k.\int_0^{\gamma} dz.\log[-\beta+\surd(\delta^2+\beta^2+z^2)]. \quad \ldots (1)$$

These integrals are all of the same form, and an integration by parts gives,

$$\int dz.\log[\beta+\surd(n+z^2)] = z.\log[\beta+\surd(n+z^2)]-\int\frac{z^2.dz}{\beta\surd(n+z^2)+n+z^2}.$$

Adding $ndz—ndz$ to the numerator of the last term, and putting for the moment $n+z^2 = x^2$, we have the two forms

$$\int\frac{n.dx}{(x^2-n)^{\frac{1}{2}}.(\beta+x)},\quad\text{and}\quad\int\frac{x^2.dx}{(x^2-n)^{\frac{1}{2}}.(\beta+x)}.$$

The first of these reduces to a known integral if we put $\beta+x = 1\div y$; and if we add to the numerator of the second $\beta^2dx-\beta^2dx$ we have,

$$\int\frac{x^2.dx}{(x^2-n)^{\frac{1}{2}}(\beta+x)} = \int\frac{xdx}{(x^2-n)^{\frac{1}{2}}}-\beta.\int\frac{dx}{(x^2-n)^{\frac{1}{2}}}+\beta^2.\int\frac{dx}{(x^2-n)^{\frac{1}{2}}(\beta+x)}.$$

These integrals are also known, and we have

$$\int dz.\log[\beta+\surd(a^2+\beta^2+z^2)] = z.\log[\beta+\surd(a^2+\beta^2+z^2)]$$
$$+a.\text{arc tan.}\frac{-a^2-\beta^2-\beta\surd(a^2+\beta^2+z^2)}{az}-z+\beta.\log[z+\surd(a^2+\beta^2+z^2)]$$
$$+\text{ a constant.}$$

In the case therefore of a homogeneous rectangular mass, the attraction can be found exactly; and by uniting terms having the same factor we have,

$$A=2k\beta.\log\frac{[\gamma+\surd(\delta^2+\beta^2+\gamma^2)].\surd(\alpha^2+\beta^2)}{[\gamma+\surd(\alpha^2+\beta^2+\gamma^2)].\surd(\delta^2+\beta^2)}$$

$$+k\gamma.\log\frac{[\beta+\surd(\delta^2+\beta^2+\gamma^2)].[-\beta+\surd(\alpha^2+\beta^2+\gamma^2)]}{[\beta+\surd(\alpha^2+\beta^2+\gamma^2)].[-\beta+\surd(\delta^2+\beta^2+\gamma^2)]}$$

$$+k\alpha.\text{arc tan}\,\frac{2\alpha\,\beta\,\gamma\surd(\alpha^2+\beta^2+\gamma^2)}{\alpha^2(\alpha^2+\beta^2+\gamma^2)-\beta^2\gamma^2}-k\delta.\text{arc tan}\,\frac{2\delta\,\beta\,\gamma\surd(\delta^2+\beta^2+\gamma^2)}{\delta^2(\delta^2+\beta^2+\gamma^2)-\beta^2\gamma^2}.$$

If now we rationalize the second logarithmic term, put

$$s=\surd(\alpha^2+\beta^2+\delta^2),\qquad s_1=\surd(\delta^2+\beta^2+\gamma^2),$$

and change the arcs to half arcs, we have finally,

$$A=2k\beta.\log\frac{(\gamma+s_1).(\alpha^2+\beta^2)^{\frac{1}{2}}}{(\gamma+s\ ).(\delta^2+\beta^2)^{\frac{1}{2}}}+2k\gamma.\log\frac{(\beta+s_1).(\alpha^2+\gamma^2)^{\frac{1}{2}}}{(\beta+s\ ).(\delta^2+\gamma^2)^{\frac{1}{2}}}$$

$$+2k\alpha.\text{arc tan}\,\frac{\beta\gamma}{\alpha s}-2k\delta.\text{arc tan}\,\frac{\beta\gamma}{\delta s}. \tag{2}$$

Since z is a small quantity in nearly all practical cases, if we neglect z^2 in equation (1) and then integrate we have

$$A=k\gamma.\log\frac{[\beta+\surd(\delta^2+\beta^2)].[-\beta+\surd(\alpha^2+\beta^2)]}{[\beta+\surd(\alpha^2+\beta^2)].[-\beta+\surd(\delta^2+\beta^2)]}. \tag{3}$$

This expression is similar to that given by Thomson and Tait, and also by Pratt in his book on the Figure of the Earth. This formula gives the principal part of the attraction, and it may be written

$$A=2k\gamma.\log\frac{\alpha\beta+\alpha\surd(\delta^2+\beta^2)}{\delta\beta+\delta\surd(\alpha^2+\beta^2)}.$$

The logarithms in the preceding expressions are Napierian; and if we use the common tables we must multiply their logarithms by 2.302585.

In order to apply these formulæ to a case that will be nearly represented by the tides in the Bay of Fundy, I take one mile for the unit of length, and put

$$\alpha=30;\ \beta=15;\ \gamma=\delta=0.01=53\text{ feet nearly.}$$

The terms of formula (2) give

first term	=	+ 0.011122.k
second "	=	+ 0.143571 "
third "	=	+ 0.008944 "
fourth "	=	— 0.008411 "
sum	=	+ 0.155226.k

If we denote the force of gravity of the earth by g, we have by Clairaut's formula,

$$g = G.[1 + (\tfrac{5}{2}m - \varepsilon)\sin\varphi^2].$$

G is the force of gravity at the Equator; m is the ratio of the centrifugal force at the Equator to gravity; ε is the flattening of the Earth, and φ is the latitude of the place. Denoting by k' the constant of the Earth's mass, depending on the unit of mass and its density, we have,

$$G = \frac{4\pi}{3}k'.(a^2b)^{⅓}: \qquad m = \frac{1}{289}.$$

$$\varepsilon = \frac{a - b}{a} = \frac{13.5}{3963.3},$$

and the coefficient of sine φ^2 is 0.005244. If we put $\varphi = 45°$,

$$g = 16626.04.k'.$$

The mean density of the earth according to Baily is 5.6747; and that of sea water being 1.026,

$$k' = \frac{5.6747}{1.026}.k = 5.5309\,.k.$$

If v be the variation of the plumb line produced by the tide, then

$$v = \frac{0.155226}{16626.04 \times 5.5309} = 0''.348.$$

It would be difficult to determine this small arc with the necessary accuracy.

Baily's value of the mean density was found from a long and very careful series of experiments made in London in 1841 and 1842. The two attracting masses were balls of lead 12 inches in diameter, and weighing 389 pounds each. The small balls placed at the ends of the torsion rods were 2 inches in diameter, and made of different substances, platinum, lead, zinc, glass, ivory and brass. From all his experiments Baily found

$$\text{mean density} = 5.6747 \pm 0.0038;$$

the unit of density being that of distilled water. The computed probable error of this result is very small, being only $\frac{7}{10000}$ of the whole value.

It would hardly be possible to determine this quantity from the attraction of a mountain or mass of water to such a degree of accuracy. On the other hand, it must be remembered that the theory of probable errors does not recognize the existence of constant errors in the experiments, and that conclusions drawn from the values of probable errors are frequently erroneous. It would be interesting to have this determination repeated under different circumstances.

MR. GLAISHER'S ENUMERATION OF PRIMES FOR THE FIRST NINE MILLIONS.

BY PROFESSOR W. W. JOHNSON.

THE gap between Burckhardt's factor table for the first three millions and Dase's tables for the seventh, eighth and ninth millions has now been filled by the completion of Mr. James Glaisher's tables for the fourth, fifth and sixth millions, and the Report of the British Association for 1881 contains the final results of Mr. J. W. L. Glaisher's Enumeration of Primes alluded to at p. 7 Vol. V and p. 118 Vol. VII of the ANALYST.

The following tables, taken from the Report, give the number of primes in each group of 100, 000.

The numbers for the complete millions are:—

	Number of Primes	Differences
First million	78,499*	
Second "	70,433	—8,066
Third "	67,885	—2,548
Fourth "	66,329	—1,556
Fifth "	65,369	— 960
Sixth "	64,336	—1,033
Seventh "	63,799	— 537
Eighth "	63,158	— 641
Ninth "	62,760	— 398

and the total number of primes in the nine millions is 602,568.

The irregular character of the distribution is strikingly exhibited by the columns of differences. Notice in the following table the occurrence of the difference + 119 between — 102 and — 90; and in the above table the sudden drop from 1,033 to 537 in the value of the difference, preceded and followed by a considerable rise in value.

Both Legendre's formula for the number of primes inferior to x, and Tchebycheff's logarithmic integral $li\,x$, where

$$li\,x = \int_0^x \frac{dx}{\log x},$$

have been shown to be hopelessly wrong. It is to be hoped that comparison will eventually be made with Riemann's formula,

$$li\,x - \tfrac{1}{2}li\,x^{\frac{1}{2}} - \tfrac{1}{3}li\,x^{\frac{1}{3}} - \tfrac{1}{5}li\,x^{\frac{1}{5}} + \tfrac{1}{6}li\,x^{\frac{1}{6}} - \text{etc.,}\dagger$$

which, "it seems clear, is the only one having a sound theoretical basis".

*1 and 2 are counted as primes.

†The coefficients and indices are of the form $1 \div \alpha\beta\gamma\ldots$. where α, β, γ etc., are different primes and the sign is — when there is one prime, + when there are two, – when three etc.

Number of Primes in each Group of 100,000 from 0 to 9,000,000.

		No. of Primes	Differ'e			No. of Primes	Differ'e
0 —	100 000	9 593		4 500 000 —	4 600 000	6 493	— 120
100 000	200 000	8 392	—1201	4 600 000	4 700 000	6 523	+ 30
200 000	300 000	8 013	— 379	4 700 000	4 800 000	6 475	— 48
300 000	400 000	7 863	— 150	4 800 000	4 900 000	6 554	+ 79
400 000	500 000	7 678	— 185	4 900 000	5 000 000	6 522	— 32
500 000	600 000	7 560	— 118	5 000 000	5 100 000	6 458	— 64
600 000	700 000	7 445	— 115	5 100 000	5 200 000	6 436	— 22
700 000	800 000	7 408	— 37	5 200 000	5 300 000	6 493	+ 57
800 000	900 000	7 323	— 85	5 300 000	5 400 000	6 462	— 31
900 000	1 000 000	7 224	— 99	5 400 000	5 500 000	6 438	— 24
1 000 000	1 100 000	7 216	— 8	5 500 000	5 600 000	6 402	— 36
1 100 000	1 200 000	7 225	+ 9	5 600 000	5 700 000	6 404	+ 2
1 200 000	1 300 000	7 081	— 144	5 700 000	5 800 000	6 387	— 17
1 300 000	1 400 000	7 103	+ 22	5 800 000	5 900 000	6 436	+ 49
1 400 000	1 500 000	7 028	— 75	5 900 000	6 000 000	6 420	— 16
1 500 000	1 600 000	6 973	— 55	6 000 000	6 100 000	6 397	— 23
1 600 000	1 700 000	7 015	+ 42	6 100 000	6 200 000	6 402	+ 5
1 700 000	1 800 000	6 932	— 83	6 200 000	6 300 000	6 425	+ 23
1 800 000	1 900 000	6 957	+ 25	6 300 000	6 400 000	6 337	— 88
1 900 000	2 000 000	6 903	— 54	6 400 000	6 500 000	6 347	+ 10
2 000 000	2 100 000	6 874	— 29	6 500 000	6 600 000	6 402	+ 55
2 100 000	2 200 000	6 857	— 17	6 600 000	6 700 000	6 338	— 64
2 200 000	2 300 000	6 849	— 8	6 700 000	6 800 000	6 375	+ 37
2 300 000	2 400 000	6 791	— 58	6 800 000	6 900 000	6 411	+ 36
2 400 000	2 500 000	6 770	— 21	6 900 000	7 000 000	6 365	— 46
2 500 000	2 600 000	6 809	+ 39	7 000 000	7 100 000	6 369	+ 4
2 600 000	2 700 000	6 765	— 44	7 100 000	7 200 000	6 306	— 63
2 700 000	2 800 000	6 716	— 49	7 200 000	7 300 000	6 348	+ 42
2 800 000	2 900 000	6 746	+ 30	7 300 000	7 400 000	6 299	— 49
2 900 000	3 000 000	6 708	— 38	7 400 000	7 500 000	6 301	+ 2
3 000 000	3 100 000	6 676	— 32	7 500 000	7 600 000	6 305	+ 4
3 100 000	3 200 000	6 717	+ 41	7 600 000	7 700 000	6 347	+ 42
3 200 000	3 300 000	6 691	— 26	7 700 000	7 800 000	6 245	— 102
3 300 000	3 400 000	6 639	— 52	7 800 000	7 900 000	6 364	+ 119
3 400 000	3 500 000	6 611	— 28	7 900 000	8 000 000	6 274	— 90
3 500 000	3 600 000	6 575	— 36	8 000 000	8 100 000	6 250	— 24
3 600 000	3 700 000	6 671	+ 96	8 100 000	8 200 000	6 301	+ 51
3 700 000	3 800 000	6 590	— 81	8 200 000	8 300 000	6 283	— 18
3 800 000	3 900 000	6 624	+ 34	8 300 000	8 400 000	6 285	+ 2
3 900 000	4 000 000	6 535	— 89	8 400 000	8 500 000	6 245	— 40
4 000 000	4 100 000	6 628	+ 93	8 500 000	8 600 000	6 3.6	+ 81
4 100 000	4 200 000	6 540	— 88	8 600 000	8 700 000	6 281	— 45
4 200 000	4 300 000	6 510	— 30	8 700 000	8 800 000	6 299	+ 18
4 300 000	4 400 000	6 511	+ 1	8 800 000	8 900 000	6 220	— 79
4 400 000 —	4 500 000	6 613	+ 102	8 900 000 —	9 000 000	6 270	+ 50

ON AN UNSYMMETRICAL PROBABILITY CURVE.

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WHEN repeated observations of a quantity are made, and are liable to error through accidental or unknown causes, it is sometimes found that the facility of error is greater on one side of the arithmetical mean than on the other, so that the limits of possible error are different, and + and — errors of equal amount do not occur with equal frequency. Cases of this kind have been noticed by various writers; see for instance Quetelet's *Lettres sur la Théorie des Probabilités*, pp. 180 and 410. He remarked a similarity between the apparent form of the curve of facility, and that of the series of terms in the expansion of a binomial $p+q$ to a high but finite power, when p and q are very unequal. I am not aware that any writer has attempted to give the analytical equation of such a curve, and it is the object of the present paper to obtain one. We shall do this, however, in the most general way, regarding the desired curve of facility as a limiting form of the series of coefficients in the expansion of a polynomial, which may or may not be a binomial. The limit of a polynomial having none but positive coefficients has been investigated in a peculiar manner, by Laplace and subsequent writers, and found to be the common probability curve. See, for instance, Meyer, *Wahrscheinlichkeitsrechnung*, Leipsic 1879, pp. 141, 350, 407, 412.* A simpler and very different way of obtaining such results was given by me in the ANALYST, Sept. 1879 and Sept. 1881, and I have extended it to the cases of polynomials of two and three variables. Although the general form of the ultimate limiting curve for a polynomial of one variable is represented by the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \tag{1}$$

yet it can be shown that the actual form of an expansion to a high power approaches still more closely to another and more complex curve, of which (1) is only a special case. To discover the nature of this curve, we must take precautions to insure the most accurate approximation.

First, we shall use the method of symmetrical differences. Having an unlimited series of equidistant terms

$$\ldots\ldots l_{i-2},\ l_{i-1},\ l_i,\ l_{i+1},\ l_{i+2},\ \ldots\ldots \tag{2}$$

we take their differences so as to keep l_i always in the middle. The differences of even order are

*Meyer's work is a valuable compendium of the science of probability, brought down to recent date.

$$\Delta_2 = -2l_i + l_{i+1} + l_{i-1}, \quad \Delta_4 = 6l_i - 4(l_{i+1} + l_{i-1}) + l_{i+2} + l_{i-2},$$
$$\Delta_6 = -20l_i + 15(l_{i+1} + l_{i-1}) - 6(l_{i+2} + l_{i-2}) + l_{i+3} + l_{i-3}, \text{ \&c., \&c.}$$

Those of odd order would in the usual form be

$$\Delta_1 = l_{i+\frac{1}{2}} - l_{i-\frac{1}{2}}, \qquad \Delta_3 = l_{i+1\frac{1}{2}} - 3l_{i+\frac{1}{2}} + 3l_{i-\frac{1}{2}} - l_{i-1\frac{1}{2}}, \text{ \&c.}$$

But since only the terms in (2) are supposed to be given, we make the hypotheses

$$l_{i-\frac{1}{2}} = \tfrac{1}{2}(l_{i-1} + l_i), \quad l_{i+\frac{1}{2}} = \tfrac{1}{2}(l_i + l_{i+1}), \quad l_{i+1\frac{1}{2}} = \tfrac{1}{2}(l_{i+1} + l_{i+2}), \text{ \&c.,}$$

and by substitution get expressions which we agree to represent by Δ_1, Δ_3, &c., thus

$$\left.\begin{aligned} \Delta_1 &= \tfrac{1}{2}(l_{i+1} - l_{i-1}), \quad \Delta_3 = \tfrac{1}{2}[l_{i+2} - l_{i-2} - 2(l_{i+1} - l_{i-1})], \\ \Delta_5 &= \tfrac{1}{2}[l_{i+3} - l_{i-3} - 4(l_{i+2} - l_{i-2}) + 5(l_{i+1} - l_{i-1})], \text{ \&c.} \end{aligned}\right\} \quad (3)$$

It will be found that each difference of odd order thus formed is half the sum of the two nearest differences of the same order formed in the usual way, one on each side of the centre or place of l_i. Thus for example

$$\Delta_3 = \tfrac{1}{2}[(l_{i+2} - 3l_{i+1} + 3l_i - l_{i-1}) + (l_{i+1} - 3l_i + 3l_{i-1} - l_{i-2})].$$

From the expressions for Δ_1, Δ_2, &c. we get by successive eliminations

$$l_{i\pm1} = l_i \pm \Delta_1 + \tfrac{1}{2}\Delta_2, \qquad l_{i\pm2} = l_i \pm 2\Delta_1 + 2\Delta_2 \pm \Delta_3 + \tfrac{1}{2}\Delta_4,$$

and in general

$$l_{i+n} = l_i + \frac{n}{1}\Delta_1 + \frac{n^2}{1.2}\Delta_2 + \frac{n(n^2-1^2)}{1.2.3}\Delta_3 + \frac{n^2(n^2-1^2)}{1.2.3.4}\Delta_4 + \frac{n(n^2-1^2)(n^2-2^2)}{1.2.3.4.5}\Delta_5 + \text{\&c.,} \quad (4)$$

where n may be either positive or negative. This formula, which may be used for making interpolations, is given by Lacroix, *Calcul Diff. et Int.*, Paris 1819, Vol. III. pp. 26–28.

It was shown in my Analyst articles above cited, that if any polynomial

$$\lambda_{-m}z^{-m} + \ldots. + \lambda_{-1}z^{-1} + \lambda_0 + \lambda_1 z + \ldots. + \lambda_m z^m, \quad (5)$$

whose coefficients λ may be either + or —, is expanded to the k power, and the expansion is written

$$l_{-km}z^{-km} + \ldots + l_{-1}z^{-1} + l_0 + l_1 z + \ldots + l_{km}z^{km}, \quad (6)$$

then any coefficient l_i in the expansion, and the $2m$ coefficients nearest to it, will be connected by the relation

$$\frac{(\lambda_1 l_{i-1} - \lambda_{-1}l_{i+1}) + 2(\lambda_2 l_{i-2} - \lambda_{-2}l_{i+2}) + \ldots + m(\lambda_m l_{i-m} - \lambda_{-m}l_{i+m})}{\lambda_0 l_i + (\lambda_1 l_{i-1} + \lambda_{-1}l_{i+1}) + (\lambda_2 l_{i-2} + \lambda_{-2}l_{i+2} + \ldots + (\lambda_m l_{i-m} + \lambda_{-m}l_{i+m})} = \frac{i}{k+1}. \quad (7)$$

Let l_{i+1}, l_{i-1} &c, be expressed in terms of l_i and the differences as in (4), and write also

$$\left.\begin{aligned} b_0 &= \lambda_0 + (\lambda_1 + \lambda_{-1}) + (\lambda_2 + \lambda_{-2}) + \ldots. + (\lambda_m + \lambda_{-m}) \\ b_1 &= 1(\lambda_1 - \lambda_{-1}) + 2(\lambda_2 - \lambda_{-2}) + \ldots. + m(\lambda_m - \lambda_{-m}) \\ b_2 &= 1^2(\lambda_1 + \lambda_{-1}) + 2^2(\lambda_2 + \lambda_{-2}) + \ldots. + m^2(\lambda_m + \lambda_{-m}) \\ b_3 &= 1^3(\lambda_1 - \lambda_{-1}) + 2^3(\lambda_2 - \lambda_{-2}) + \ldots. + m^3(\lambda_m - \lambda_{-m}) \\ &\qquad\text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \end{aligned}\right\} \quad (8)$$

Denoting the numerator and denominator in the first member of (7) by N and D, and writing $1.2.3 \dots n = n\,!$, we get

$$\left.\begin{aligned}
N &= b_1 l_i - b_2 \varDelta_1 + \tfrac{1}{2} b_3 \varDelta_2 - \frac{1}{3\,!}(b_4 - b_2)\varDelta_3 + \frac{1}{4\,!}(b_5 - b_3)\varDelta_4 - \frac{1}{5\,!} \\
&\qquad \times (b_6 - 5b_4 + 4b_2)\varDelta_5 \\
&\quad + \frac{1}{6\,!}(b_7 - 5b_5 + 4b_3)\varDelta_6 - \frac{1}{7\,!}(b_8 - 8b_6 + 19b_4 - 12b_2)\varDelta_7 + \&c. \\
D &= b_0 l_i - b_1 \varDelta_1 + \tfrac{1}{2} b_2 \varDelta_2 - \frac{1}{3\,!}(b_3 - b_1)\varDelta_3 + \frac{1}{4\,!}(b_4 - b_2)\varDelta_4 - \frac{1}{5\,!} \\
&\qquad \times (b_5 - 5b_3 + 4b_1)\varDelta_5 \\
&\quad + \frac{1}{6\,!}(b_6 - 5b_4 + 4b_2)\varDelta_6 - \frac{1}{7\,!}(b_7 - 8b_5 + 19b_3 - 12b_1)\varDelta_7 + \&c, \\
\frac{N}{D} &= \frac{i}{k+1}.
\end{aligned}\right\} \qquad (9)$$

It will be seen that in the coefficient of $\varDelta_{2m+1}$ or $\varDelta_{2m+2}$, the numerical coefficients of the b within the parentheses are those of the powers of n in the product of the factors

$$(n^2 - 1^2)\,(n^2 - 2^2) \dots\dots (n^2 - m^2).$$

When k becomes an infinity of the second order, that is, of a magnitude comparable with the quotient of a finite area by $(dx)^2$, and the successive values of l are regarded as consecutive ordinates y to a limiting curve which extends to an infinite distance over the axis of X, we have

$$l_i = y, \qquad \varDelta_1 = dy, \qquad \varDelta_2 = d^2y, \qquad \varDelta_3 = d^3y, \ \&c.$$

The common interval $\varDelta x$ between ordinates becomes dx when they are set close together, and the abscissa corresponding to any y is $x = idx$. Thus (9) becomes the differential equation of the curve, and b_0, b_1, &c. are constants. Any given polynomial may be reduced to one in which $\Sigma(\lambda) = 1$, by dividing it throughout by the sum of its coefficients. We have then $b_0 = 1$. If a constant number is added to or subtracted from all the exponents of z in (5), it will not alter the values of l in (6). Hence, as shown in my former article, b_1 may be reduced to zero by transferring the origin or place of z^0 to the centre of parallel forces of the coefficients λ in (5), when these are regarded as forces ranged equidistantly along the imponderable axis of X and acting upon it at right angles in the plane XY. Then any constant b_n in (8) will denote the sum of the products formed by multiplying each λ into the nth power of its abscissa reckoned from the new origin, if the common interval $\varDelta x$ between the coefficients or forces λ is regarded as unity. But if any other unit of abscissas is employed, the sum of the products will be $b_n(\varDelta x)^n$, or $b_n(dx)^n$ when the coefficients are set close together so as to be consecutive. We may now write (9) as follows.

$$\frac{b_2dy-\frac{1}{2}b_3d^2y+\frac{1}{6}(b_4-b_2)d^3y-\&c}{y+\frac{1}{2}b_2d^2y-\frac{1}{6}b_3d^3y+\&c.}=\frac{-x}{(k+1)dx}. \tag{10}$$

In the numerator of the first member let d^3y, d^4y &c. be neglected in comparison with dy and d^2y, and in the denominator neglect d^2y, d^3y &c. in comparison with y. Since k is infinitely large, we may write k instead of $k+1$. Therefore

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y}=\frac{-x}{kb_2dx}.$$

Invert both members of this equation, subtract $\frac{1}{2}(b_3\div b_2)$ from each, and invert them both back again. This gives

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y-\frac{1}{2}(b_3\div b_2)dy+\frac{1}{4}(b_3\div b_2)^2d^2y}=\frac{-x}{kb_2dx+\frac{1}{2}(b_3\div b_2)x}. \tag{11}$$

In the denominator of the first member, let d^2y be neglected in comparison with y and dy. From the properties of the lever arms of the coeffic'nts in polynomials and their products, as shown by me in the ANALYST articles cited (see also March and Nov. 1880), it follows that since the origin or place of z^0 in the given polynomial (5) is transferred to the centre of forces for the coefficients λ, the origin or place of z^0 in the expansion to the k power will be located at the centre of forces for the coefficients l in (6), which become the ordinates y to the limiting curve.

When the λ's are all positive, as they must be if they represent probabilities, the y's will be all positive, and their centre of forces is the same as their centre of gravity, if the ordinates are regarded as the masses of material points ranged along the X axis at intervals equal to dx. Now in (11) let the origin be transferred from the centre of gravity to another convenient point by putting

$$x-\frac{2kb_2^2\,dx}{b^3} \tag{12}$$

in place of x. This gives

$$\frac{dy-\frac{1}{2}(b_3\div b_2)d^2y}{y-\frac{1}{2}(b_3\div b_2)dy}=\frac{4kb_2dx-2(b_3\div b_2)x}{(b_3\div b_2)^2x}. \tag{13}$$

In the first member, the numerator is the differential of the denominator. Without any further change of origin, we can write approximately

$$y+\tfrac{1}{2}(b_3\div b_2)dy \qquad \text{and} \qquad x+\tfrac{1}{2}(b_3\div b_2)dx$$

in place of y and x respectively, neglecting d^3y in the numerator and d^2y in the denominator, and so get

$$\frac{dy}{y}=\frac{4kb_2dx-(b_3\div b_2)^2dx-2(b_3\div b_2)x}{(b_3\div b_2)^2[x+\frac{1}{2}(b_3\div b_2)dx]}.$$

Since the denominator y in the first member is supposed to be infinitely greater than the numerator dy, the denominator in the second member must

be infinitely greater than its numerator, so that in the denominator we may neglect dx in comparison with x. Also let the constants be expressed by means of two new constants

$$a = \frac{2b_2(dx)^2}{b_3(dx)^3}, \qquad b = kb_2(dx)^2. \tag{14}$$

Since k is supposed to be an infinity of the second order, b represents a finite area. The equation will now stand

$$\frac{dy}{y} = \frac{dx}{x}(a^2b-1)-adx, \tag{15}$$

and integration gives

$$\log' y = (a^2b-1)\log' x - ax + \log' C,$$
$$\therefore y = Cx^{a^2b-1}e^{-ax}. \tag{16}$$

This is a more accurate relation between y and x than we could have obtained without the use of symmetrical differences. In (15) dy is not the difference of the ordinates at x and $x+dx$, but of those at $x-\frac{1}{2}dx$ and $x+\frac{1}{2}dx$.

The finite difference of $\log' y$ is usually considered to be

$$\left.\begin{aligned} \Delta\log' y &= \log'(y+\Delta y)-\log' y = \log'\left(1+\frac{\Delta y}{y}\right) \\ &= \frac{\Delta y}{y}-\tfrac{1}{2}\left(\frac{\Delta y}{y}\right)^2+\tfrac{1}{3}\left(\frac{\Delta y}{y}\right)^3-\&c. \end{aligned}\right\} \tag{17}$$

But we have taken it to be

$$\left.\begin{aligned} \Delta\log' y &= \log'(y+\tfrac{1}{2}\Delta y)-\log'(y-\tfrac{1}{2}\Delta y) = \log'\left(1+\frac{\Delta y}{2y}\right)-\log'\left(1-\frac{\Delta y}{2y}\right) \\ &= \frac{\Delta y}{y}+\frac{1}{3.2^2}\left(\frac{\Delta y}{y}\right)^3+\frac{1}{5.2^4}\left(\frac{\Delta y}{y}\right)^5+\&c. \end{aligned}\right\} \tag{18}$$

These expressions show that when $dy \div y$ is regarded as the differential of $\log' y$, the magnitude of the error committed is either

$$\frac{1}{2}\left(\frac{dy}{y}\right)^2 \qquad \text{or} \qquad \frac{1}{12}\left(\frac{dy}{y}\right)^3, \tag{19}$$

according as we consider dy to be $y_{i+1}-y_i$ or $y_{i+\frac{1}{2}}-y_{i-\frac{1}{2}}$. The second error is of a lower order of magnitude than the first. Conversely, if we take $\log' y$ to be the integral of $dy \div y$, the accuracy of the result is increased when dy or Δy is understood in the sense here adopted.

When we write

$$ax = v, \qquad a^2b = n, \tag{20}$$

the ordinate y in (16) is seen to be proportional to an expression having the known form

$$v^{n-1}e^{-v},$$

the element-function of the Gamma integral. The limiting form (16) of the expansion of a polynomial, therefore, is a curve shaped like the one whose area, between $x = 0$ and $x = \infty$, is $\Gamma(n)$. For convenience, we may call it the gamma curve. It makes $y = 0$ for $x = 0$ and for $x = \infty$. (See Price's *Calculus*, I. p. 208.) We also have

$$\left.\begin{aligned} \frac{dy}{dx} &= Cx^{a^2b-2}\, e^{-ax}(a^2b-1-ax), \\ \frac{d^2y}{dx^2} &= Cx^{a^2b-3}\, e^{-ax}\left[(ax)^2-2(a^2b-1)ax+(a^2b-1)(a^2b-2)\right], \end{aligned}\right\} \quad (21)$$

whence it appears that the X axis is tangent to the curve at $x = 0$ and asymptote to it at $x = \infty$, and that y is a maximum at

$$x = ab - \frac{1}{a}. \quad (22)$$

There are two points of inflexion, equidistant from the maximum, at

$$x = ab - \frac{1}{a} \pm \frac{1}{a}\sqrt{(a^2b-1)}. \quad (23)$$

Since the first $n-2$ differential coefficients will contain both x and e^{-x} as factors, it appears that the X axis has a contact of an order as high as the $n-2$, with the curve at $x = 0$ and $x = \infty$. Hence the curve anywhere near those points is almost a straight line coinciding with the axis, and values of y in the vicinity of either will scarcely differ from zero. This agrees with what we know of the extreme smallness of the coefficients at or near the two ends of the expansion of a binomial or polynomial to a high power.

Since $\Sigma(\lambda) = 1$ in the given polynomial and $\Sigma(l) = 1$ in its expansion, we shall have $\Sigma(y) = 1$ in the curve (16), so that

$$\frac{1}{dx}\int_0^\infty y\,dx = 1, \qquad \therefore \frac{C}{a^{a^2b}\,dx}\int_0^\infty (ax)^{a^2b-1}\, e^{-ax} d(ax) = 1, \quad (24)$$

which gives the value of C, and we get

$$y = \frac{a\,dx}{\Gamma(a^2b)}(ax)^{a^2b-1}\, e^{-ax}, \quad (25)$$

the complete equation of the curve sought. According to the view stated in my former articles, by which the limiting form of the expansion of a polynomial expresses the most plausible law of facility of error, y will represent the probability that any error which occurs will fall within an arbitrary but very small interval dx, the abscissa of whose middle point is x. The constants a and b depend on the nature of the observations. The quantity $b_2(dx)^2$ is the square of what I have called the "radius of gyration" of the coefficients λ in the polynomial (5), about their centre of gravity; but it might better, perhaps, be called the *quadratic radius.*

I have shown (ANALYST, Jan. and May 1880), that the squared radius for the expansion to the k power is k times as great as for the first power. It follows that b in (14) is the square of the quadratic mean error, that is, the mean of the squares of the deviations of the observations from their arithmetical mean. The expression for a in (14) has for its denominator the quantity $b_3(dx)^3$, which is of the same nature as $b_2(dx)^2$, except that the cubes of the deviations are used instead of the squares, deviations on one side being regarded as $+$ and on the other side as $-$. We will now prove by a precisely analogous method, that this quantity also is k times greater in the expansion to the k power, than it is in the first power.

In my article last cited, any two polynomial factors were denoted by

$$\left.\begin{array}{l} a_0+a_1z+a_2z^2+\ldots\ldots+a_mz^m, \\ c_0+c_1z+c_2z^2+\ldots\ldots+c_nz^n. \end{array}\right\} \tag{26}$$

The coefficients a nnd c may be essentially either $+$ or $-$. Their sums were

$$S_1 = a_0+a_1+\ldots\ldots+a_m, \qquad S_2 = c_0+c_1+\ldots\ldots+c_n, \tag{27}$$

and h_1 and h_2 were the lever arms of the coefficients about the place of the first terms, so that

$$S_1h_1=(a_1+2a_2+\ldots+ma_m)\Delta x,\ S_2h_2=(c_1+2c_2+\ldots+nc_n)\Delta x. \tag{28}$$

It was proved that the lever arm of the coefficients in the product of the two factors, about the place of the first term in the product, is

$$H = h_1 + h_2. \tag{29}$$

The quadratic radii of the coefficients in the two factors, about the places of the first terms, were denoted by g_1 and g_2, whence

$$\left.\begin{array}{l} S_1g_1^2 = (1^2a_1+2^2a_2+\ldots+m^2a_m)(\Delta x)^2, \\ S_2g_2^2 = (1^2c_1+2^2c_2+\ldots+n^2c_n)(\Delta x)^2, \end{array}\right\} \tag{30}$$

and putting G for the like radius in the product, it was proved that

$$G^2 = g_1^2 + g_2^2 + 2h_1h_2. \tag{31}$$

Now let u_1^3 and u_2^3 be formed from the two factors in the same way as $b_3(dx)^3$ was. $S_1u_1^3$ and $S_2u_2^3$ will represent the two sums of all the products obtained by multiplying each coefficient by the cube of its $+$ or $-$ deviation from the centre of forces, so that

$$\left.\begin{array}{l} S_1u_1^3 = a_0(-h_1)^3+a_1(\Delta x-h_1)^3+a_2(2\Delta x-h_1)^3\ldots+a_m(m\Delta x-h_1)^3, \\ S_2u_2^3 = c_0(-h_2)^3+c_1(\Delta x-h_2)^3+c_2(2\Delta x-h_2)^3\ldots+c_n(n\Delta x-h_2)^3. \end{array}\right\} \tag{32}$$

For convenience we will call such quantities as u_1 and u_2 *cubic radii* about the point from which the deviations are reckoned. Let l_1 and l_2 denote the cubic radii taken about the first term in each polynomial, then

$$\left.\begin{array}{l} S_1l_1^3 = (1^3a_1+2^3a_2+\ldots+m^3a_m)(\Delta x)^3, \\ S_2l_2^3 = (1^3c_1+2^3c_2+\ldots+n^3c_n)(\Delta x)^3. \end{array}\right\} \tag{33}$$

From (32) we have

$$S_1 u_1^3 = (1^3a_1+2^3a_2+\,..\,+m^3a_m)(\Delta x)^3 - 3h_1(1^2a_1+2^2a_2+\,..\,+m^2a_m)(\Delta x)^2$$
$$+3h_1^2(a_1+2a_2+\,..\,+ma_m)\Delta x - h_1^3(a_0+a_1+\,..\,+a_m),$$

and a similar expression for $S_2u_2^3$. Then by help of (27), (28), (30) and and (33), we get

$$u_1^3 = l_1^3 - 3g_1^2h_1 + 2h_1^3, \qquad u_2^3 = l_2^3 - 3g_2^2h_2 + 2h_2^3, \qquad (34)$$

Likewise denoting by U and L the cubic radii for the product, about its centre of forces and its first term respectively, we shall have

$$U^3 = L^3 - 3G^2H + 2H^3. \qquad (35)$$

The sum of all the coefficients in the product is S_1S_2. Supposing the first polynomial factor to be multiplied successively by the terms of the second one, L^3 will be expressed thus,

$$S_1S_2L^3 = c_0(1^3a_1+2^3a_2+\dots+m^3a_m)(\Delta x)^3$$
$$+\,c_1[1^3a_0+2^3a_1+\dots+(m+1)^3a_m](\Delta x)^3$$
$$+\,c_2[2^3a_0+3^3a_1+\dots+(m+2)^3a_m](\Delta x)^3$$
$$+\dots+c_n[n^3a_0+(n+1)^3a_1+\dots+(n+m)^3a_m](\Delta x)^3.$$

The coefficient of $c_n(\Delta x)^3$ is reducible by means of (27), (28), (30) and (33) to

$$S_1\left\{ n^3+3n^2\left(\frac{h_1}{\Delta x}\right)+3n\left(\frac{g_1}{\Delta x}\right)^2+\left(\frac{l_1}{\Delta x}\right)^3 \right\}.$$

Assigning to n the values 0, 1, 2 &c. in succession, we get expressions for the coefficients of $c_0(\Delta x)^3$, $c_1(\Delta x)^3$, &c., and so find

$$S_1S_2L^3 = c_0S_1l_1^3 + c_1S_1[1^3(\Delta x)^3+3.1^2h_1(\Delta x)^2+3.1g_1^2\Delta x+l_1^3]$$
$$+c_2S_1[2^3(\Delta x)^3+3.2^2h_1(\Delta x)^2+3.2g_1^2\Delta x+l_1^3]$$
$$+\dots+c_nS_1[n^3(\Delta x)^3+3n^2h_1(\Delta x)^2+3ng_1^2\Delta x+l_1^3],$$

which we put in the form

$$S_2L^3 = l_1^3(c_0+c_1+\dots+c_n)+3g_1^2\Delta x(c_1+2c_2+\dots+nc_n)$$
$$+\,3h_1(\Delta x)^2(1^2c_1+2^2c_2+\dots+n^2c_n)+(\Delta x)^3(1^3c_1+2^3c_2+\dots$$
$$\dots+n^3c_n).$$

By help of (27), (28), (30) and (33) this is reduced to

$$L^3 = l_1^3 + l_2^3 + 3g_1^2h_2 + 3g_2^2h_1. \qquad (36)$$

Substituting in (35) the values of H, G and L from (29), (31) and (36), we get

$$U^3 = l_1^3+l_2^3 - 3(g_1^2h_1+g_2^2h_2)+2(h_1^3+h_2^3),$$

and by (34) we have finally

$$U^3 = u_1^3 + u_2^3. \qquad (37)$$

[To be continued.]

ON THE ACTUAL AND PROBABLE ERRORS OF INTERPOLATED VALUES DERIVED FROM NUMERICAL TABLES BY MEANS OF FIRST DIFFERENCES.

BY R. S. WOODWARD, C. E.

§ 1. ALL numerical calculations dependent on tabulated values of logarithms of numbers, sines, tangents etc., or on natural trigonometric functions are subject to certain errors arising from unavoidable inaccuracies in the tabular values themselves. The inaccuracies in the tabular values result from the neglect of figures beyond the tabular number of places, the tabular value being given always to the nearest unit in the last place. The actual error which may exist in the final result of any computation will depend on the form of the computation, or the manner in which the possible actual errors of the tabular values used are combined. The investigation of this class of errors becomes a matter of importance as well as interest whenever the highest degree of precision is sought in computations dependent on values derived from a given table. Thus, the practical question may arise, "What are the limits of accuracy attainable with a 5-place table of logarithms?" A complete discussion of the various problems that may arise in this connection would form an important branch of the theory of errors, and might well form the subject matter of a special treatise. In the present paper, however, it is proposed to consider only the most elementary of the problems, viz., that relative to the error which may exist in an interpolated value derived by means of first differences from consecutive tabular values. In other words, it is proposed to give some answer to the practical question, "What are the possible actual errors and what are the corresponding probable errors to which values interpolated from the ordinary 5-place or 7-place tables of logarithms are subject?"

Before taking up this problem, some remarks on the actual and probable errors of tabular values are required.

§ 2. The possible actual errors of tabular values are confined between the limits $+\frac{1}{2}$ and $-\frac{1}{2}$ of a unit in the last place of a tabular value, and all errors between these lemits are equally likely to occur. These facts may be taken as self-evident and as a necessary result of the method of constructing tables. The probable error, being defined as that error which is as likely to be exceeded as not, will be, in such a system of errors, one-half the maximum error, or $\pm\frac{1}{4}$ of a unit in the last place of a tabular value.

These statements may be readily verified in any special case. For example, if a large number of actual errors of tabular values of 5-place logari'ms

be derived by means of a 7- or higher place table, it will be found that

1st, the numbers of positive and negative errors are sensibly equal;

2nd, the numbers of errors, without regard to sign, lying between equidistant limits are sensibly equal; and hence

3rd, one-half the errors will be greater and one-half less than $\frac{1}{4}$.

That the probable error in a system of errors of constant probability between given limits is one-half the max. or limiting error may also be proved in a more general manner, and some useful principles will result therefrom.

For this purpose, let ε be the general symbol for an error lying between the limits $\pm a$. The facility with which ε can occur may be denoted by $\varphi(\varepsilon)$ which must be in this case a constant, since by hypothesis all values of ε occur with the same facility. Hence we may write

$$\varphi(\varepsilon) = c, \text{ a constant.}$$

Supposing ε to vary continuously between the given limits we may make

$$\int_{-a}^{+a} \varphi(\varepsilon)d\varepsilon = c\int_{-a}^{+a} d\varepsilon = 1.$$

This is the analytical expression of the fact that the sum of all the probabilities of all possible errors within the given limits is unity. The value of the constant c results at once by integration. Thus

$$c\int_{-a}^{+a} d\varepsilon = 2ac = 1, \text{ whence}$$

$$c = \frac{1}{2a}. \tag{1}$$

Whatever form $\varphi(\varepsilon)$ may have, as long as $\varphi(+\varepsilon) = \varphi(-\varepsilon)$, or equal positive and negative errors are equally likely to occur, the probable error is the limit r in the definite integral

$$\int_{-a}^{-r} \varphi(\varepsilon)d\varepsilon = \int_{-r}^{0} \varphi(\varepsilon)d\varepsilon = \int_{0}^{+r} \varphi(\varepsilon)d\varepsilon = \int_{+r}^{+a} \varphi(\varepsilon)d\varepsilon = \tfrac{1}{4}\int_{-a}^{+a} \varphi(\varepsilon)d\varepsilon.$$

In the present case $\frac{1}{4}\int_{-a}^{+a} \varphi(\varepsilon)d\varepsilon = \frac{1}{4}$, and $\varphi(\varepsilon) = \frac{1}{2a}$. Hence

$$\int_{\pm r}^{0} \varphi(\varepsilon)d\varepsilon = \int_{\pm r}^{0} \frac{d\varepsilon}{2a} = \tfrac{1}{4}, \text{ whence } r = \frac{a}{2}.$$

§ 3. The errors to which interpolated values are subject may now be considered.

Let v_1 and v_2 be two consecutive tabular values and v' an interpolated value t tenths from v_1. Then the value of v' actually computed is

$$v' = v_1 + t(v_2 - v_1) = (1 - t)v_1 + tv_2. \tag{2}$$

Denoting the actual errors of v_1, v_2 and v' by ε_1, ε_2 and ε' respectively, we have

$$\varepsilon' = (1 - t)\varepsilon_1 + t\varepsilon_2. \tag{3}$$

The meaning of these equations should be clearly understood. Equation (2) means precisely what the symbols indicate, and does not imply any neglect of decimals beyond the tabular number. However, so far as errors here considered are concerned, it is practically sufficient to retain one figure in addition to the tabular number, retaining for example the 6th figure in interpolated values from a 5-place table and the 8th figure in interpolated values from a 7-place table. It will be noticed that $(v_2 - v_1)$ is the *tabular* difference. Equation (3) shows that the error ε' of v' cannot exceed $\pm \frac{1}{2}$, since ε_1 and ε_2 cannot surpass $\pm \frac{1}{2}$. An important property of ε_1 and ε_2 is their mutual independence; i. e., one of them cannot be expressed in terms of the other. They are also independent of t and continuous between the limits $\pm \frac{1}{2}$. Therefore ε' may have any value between the limits $+ \frac{1}{2}$ and $- \frac{1}{2}$ for any value of t.

The species of interpolated value just defined is that which, as will presently appear, should be used always if the highest precision with a given table is desired. The species of interpolated value more frequently used, however, is that formed by abridging the product $t(v_2 - v_1)$ to the nearest unit in the last place of the tabular value. This neglect of decimals is a source of error additional to those arising from the errors of the tabular values, and may be denoted by ε_3. Calling the actual error of the interpolated value in this case ε'', the meaning of ε_1 and ε_2 being the same as defined above,

$$\varepsilon'' = (1 - t)\varepsilon_1 + t\varepsilon_2 + \varepsilon_3. \qquad (4)$$

It will be observed that ε_3 in this equat'n differs from ε_1 and ε_2 in two important respects; viz., 1st, ε_3 is not independ't of t; 2nd, ε_3 is *discontinuous* for any value of t. Thus, suppose $t = \frac{1}{2}$; then the only possible values of ε_3 are 0, $+ \frac{1}{2}$ and $- \frac{1}{2}$. Again, suppose $t = \frac{1}{3}$; then the only possible values of ε_3 are 0, $+ \frac{1}{3}$ and $- \frac{1}{3}$. In a similar manner the possible values of ε_3 may be assigned for any value of t. The maximum possible value of ε'' varies, therefore, with t. A few cases may be written down.

For $t = \frac{1}{5}$ or $\frac{4}{5}$, the max. of $\varepsilon'' = \frac{4}{5}\times\frac{1}{2} + \frac{1}{5}\times\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$.
" $t = \frac{1}{4}$ or $\frac{3}{4}$, " " " $\varepsilon'' = \frac{3}{4}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2} + \frac{1}{2} = 1$.
" $t = \frac{1}{3}$ or $\frac{2}{3}$, " " " $\varepsilon'' = \frac{2}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.
" $t = \frac{1}{2}$ " " " $\varepsilon'' = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{2} = 1$.

It will also be noticed that for certain values of ε_3, ε'' will be necessarily positive or negative whatever values ε_1 and ε_2 may have. Thus for $t = \frac{1}{2}$, if the tabular difference $(v_2 - v_1)$ be odd, ε_3 must be either $+\frac{1}{2}$ or $-\frac{1}{2}$, and according as ε_3 is taken as $+\frac{1}{2}$ or $-\frac{1}{2}$ so will ε'' be either positive or negative, being confined between the limits 0 and + 1 in the one case and 0

and — 1 in the other. On the other hand, the value of ε_3 may be such as to permit both positive and negative values of ε'' but with unequal limits for the two classes of errors. Thus for $t = \frac{1}{3}$, ε_3 must be either 0, $+\frac{1}{3}$ or $-\frac{1}{3}$. If $\varepsilon_3 = +\frac{1}{3}$, the possible negative values of ε'' must lie between 0 and $-\frac{1}{6}$, while the positive values of ε'' must lie between 0 and $+\frac{5}{6}$.

Since in a given numerical table the differences must be limited, the product $t(v_2 - v_1)$ must vanish for $t = 0$; and hence $\varepsilon_3 = 0$ for $t = 0$.

Besides the two species of interpolated values already defined there may be two more, both of little practical importance, but of some theoretical interest from the fact that their definition gives some light on the real nature of the first two species. It is plain that instead of the tabular difference $(v_2 - v_1)$ there might be used the *true* differences, such as would be obtained practically by inserting in a given table differences derived from a higher place table and carried some decimals beyond the last place of values in the given table. Would there be any advantage or disadvantage in the use of such differences? There may be two cases. Let Δ be the true difference, v_1 the adjacent tabular value and t the interpolating factor. Then, supposing no figures omitted, the interpolated value becomes

$$v_1 + t\Delta. \tag{5}$$

The actual error of this value must be the same as the actual error of v_1, since there is no error in $t\Delta$. Denoting for the sake of distinctness the actual error of $v_1 + t\Delta$ by ε''', we have

$$\varepsilon''' = \varepsilon_1.$$

If, however, the product $t\Delta$ be abridged to the nearest unit of the last tabular place, another source of error will be introduced. Calling the actual error from this source ε_4 and the actual error of the interpolated value ε^{iv}, there results

$$\varepsilon^{\text{iv}} = \varepsilon_1 + \varepsilon_4. \tag{6}$$

In this equation ε_4 may have any value between the limits $+\frac{1}{2}$ and $-\frac{1}{2}$ for any value of t greater than the reciprocal of twice the greatest difference in the table used.

For values of t greater than this limit ε^{iv} may have any value between the limits $+1$ and -1. For values of t less than the above limit, ε_4 must be less than $+\frac{1}{2}$, and hence ε^{iv} must be less than ± 1. When $t = 0$, $t\Delta$ vanishes and hence $\varepsilon_4 = 0$.

The characteristics of the four different species of interpolated values and the expressions for their actual errors may now be collected in tabular form for the purpose of ready comparison.

No. of species.	Obtained by means of	Expression for actual error.	Remarks.
1	Tabular diff.	$\varepsilon' = (1-t)\varepsilon_1 + t\varepsilon_2$	ε_1 and ε_2 are indep. of t and of each other & continu's between $\pm\frac{1}{2}$. ε_3 is dependent on t & discontinu's. ε_4 is con. bet. limits depen't on t.
2	" "	$\varepsilon'' = (1-t)\varepsilon_1 + t\varepsilon_2 + \varepsilon_3$	
3	True differ'e.	$\varepsilon''' = \varepsilon_1$	
4	" "	$\varepsilon^{iv} = \varepsilon_2 + \varepsilon_4$	

§ 4. Thus far the actual errors only of interpolated values have been considered. These errors may indeed be ascertained for any special case by a suitable computation, but such computation is generally inexpedient and of little interest. The probable error of each species of interpolated value is of interest and importance, however, since it affords a measure of the precision of such value. The probable error is, as already defined, that error which is as likely as not to be exceeded. It applies therefore to the aggregate of all possible errors in a system of errors, and may be taken as the representative error of the system. It applies in general only to continuous errors and is itself always a possible error of the system of errors.

This premised, let r be the general symbol for the probable error in a system of errors of which any one is ε, and let $\varphi(\varepsilon)$ express the law of facility of these errors. Then, supposing equal positive and negative errors of equal facility, or $\varphi(+\varepsilon) = \varphi(-\varepsilon)$, r is defined by the definite integral

$$\int_{-l}^{-r}\varphi(\varepsilon)d\varepsilon = \int_{-r}^{0}\varphi(\varepsilon)d\varepsilon = \int_{0}^{+r}\varphi(\varepsilon)d\varepsilon = \int_{+r}^{+l}\varphi(\varepsilon)d\varepsilon = \tfrac{1}{4}\int_{-l}^{+l}\varphi(\varepsilon)d\varepsilon,^{*} \quad (7)$$

in which $\pm l$ is the limiting value of ε. The determination of the probable error corresponding to ε', ε'', ε''' or ε^{iv}, requires therefore a knowledge of $\varphi(\varepsilon)$.

It will now be shown what forms $\varphi(\varepsilon)$ has for the system of errors represented by

$$\varepsilon' = (1-t)\varepsilon_1 + t\varepsilon_2.$$

Put $(1-t)\varepsilon_1 = x$ and $t\varepsilon_2 = y$. Then $\varepsilon' = x + y$, in which x may have any value between the limits $\pm(1-t)\frac{1}{2} = a$, say, and y may have any value between the limits $\pm t\frac{1}{2} = b$, say. All values of x and all values of y between these limits are equally likely to occur, which facts, as shown in § 2, are expressed analytically by

*This permits $\varphi(\varepsilon)$ to be of such a form that $\phi(\varepsilon)d\varepsilon$ will represent a *relative* probability. If $\phi(\varepsilon)d\varepsilon$ represent an *absolute* probability then must

$$\int_{-l}^{+l}\phi(\varepsilon)d\varepsilon = 1.$$

$$\varphi(x) = \frac{1}{2a} \quad \text{and} \quad \varphi(y) = \frac{1}{2b}.$$

The probability of the occurrence of any particular x is

$$\varphi(x)dx = \frac{dx}{2a}.$$

Likewise the probability of the occurrence of any particular y is

$$\varphi(y)dy = \frac{dy}{2b}.$$

Since x and y are independent, the probability of their concurrence is

$$\varphi(x)\varphi(y)dx\,dy = \frac{dx\,dy}{4ab} = \frac{d\varepsilon\,dx}{4ab}, \text{ since } \varepsilon = x+y.$$

But since there must be in general an indefinite number of pairs of values of x and $y=\varepsilon-x$, which will produce ε, $\varphi(\varepsilon)d\varepsilon$ or the probability of ε must be equal to $dx d\varepsilon \div 4ab$ taken as many times as there are units in the range through which x may vary. In other words,

$$\varphi(\varepsilon)d\varepsilon = d\varepsilon\int\frac{dx}{4ab}.$$

In evaluating this integral, x must not surpass $\pm a$ and $\varepsilon - x = y$ must not surpass $\pm b$. For any value of ε lying between $-(a+b)$ and $-(a-b)$, assuming $a > b$, the limits of the integral are $-a$ and $(\varepsilon + b)$. This fact is rendered plain by a numerical example. Thus, suppose $a = \pm 5$ and $b = \pm 3$. Then $-(a+b) = -8$ and $-(a-b) = -2$. Suppose $\varepsilon = -6$ a number intermediate to -8 and -2. Then the following are the possible integer values of x and y which will produce $\varepsilon = -6$.

ε'		x y	Limits of x.
-6	$=$	$-5-1$	$-5 = -a$ and
	$=$	$-4-2$	$-3 = \varepsilon'+b$
	$=$	$-3-3$	

Again, suppose ε' is -3, a number between the limits -8 and -2. Then

ε'		x y	Limits of x.
-3	$=$	$-5+2$	$-5 = -a$.
	$=$	$-4+1$	
	$=$	$-3+0$	
	$=$	$-2-1$	
	$=$	$-1-2$	
	$=$	$0-3$	$0 = \varepsilon'+b$.

Therefore, when ε lies between $-(a+b)$ and $-(a-b)$

$$\varphi(\varepsilon)d\varepsilon = d\varepsilon\int_{-a}^{\varepsilon+b}\frac{dx}{4ab} = \frac{\varepsilon+a+b}{4ab}d\varepsilon, \text{ or } \varphi(\varepsilon) = \frac{\varepsilon+a+b}{4ab}. \qquad (8)$$

Similarly, when ε lies between $-(a-b)$ and $+(a-b)$, the limiting values of x are $\varepsilon-b$ and $\varepsilon+b$. Therefore

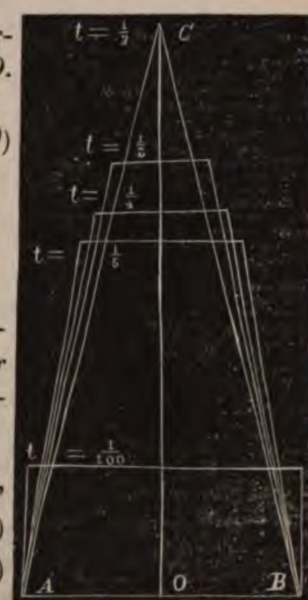
$$\varphi(\varepsilon)d\varepsilon = d\varepsilon\int_{\varepsilon-b}^{\varepsilon+b}\frac{dx}{4ab} = \frac{2b}{4ab}d\varepsilon, \quad \text{or } \varphi(\varepsilon) = \frac{2b}{4ab}. \qquad (9)$$

Finally, when ε lies between $(a-b)$ and $(a+b)$ the limits of x are $\varepsilon-b$ and $+a$, and hence

$$\varphi(\varepsilon) = \frac{-\varepsilon+a+b}{4ab}. \qquad (10)$$

The "probability curve" represented by equations (8), (9), (10), is made up of three straight lines. Equation (8) represents the line joining the p'ts whose coordinates are $-(a+b)$, 0 and $-(a-b)$, $(1\div 2a)$. Equation (9) represents the line joining the points whose coordinates are $-(a-b)$, $(1\div 2a)$ and $+(a-b)$, $(1\div 2a)$. Equation (10) represents the line joining the p'ts $+(a-b)$, $(1\div 2a)$ $+(a+b)$, 0. In the special case under consideration, $a=(1-t)\frac{1}{2}$ and $b=t.\frac{1}{2}$. Therefore

$$a+b = \tfrac{1}{2},$$
$$a-b = \tfrac{1}{2}(1-2t),$$
$$\frac{1}{2a} = \frac{1}{1-t}.$$



The figure shows the "probability curves" corresp'ing to several values of t. The origin is at O. AOB is the axis of ε and OC the axis of $\varphi(\varepsilon)$.

It may now be observed that eq's (8), (9), (10) give

$$\int_{-(a+b)}^{-(a-b)}\frac{\varepsilon+a+b}{4ab}d\varepsilon+\int_{-(a-b)}^{+(a-b)}\frac{2b}{4ab}d\varepsilon$$
$$+\int_{+(a-b)}^{+(a+b)}\frac{-\varepsilon+a+b}{4ab}d\varepsilon = 1.$$

Therefore, by (7) the probable error in the system of errors defined by (8) - (10) is the limit r which will satisfy one of the two following conditions:

$$\int_{-(a+b)}^{-r}\frac{\varepsilon+a+b}{4ab}d\varepsilon = \int_{+r}^{+(a+b)}\frac{-\varepsilon+a+b}{4ab}d\varepsilon = \tfrac{1}{4}, \quad \ldots\ldots (\alpha)$$

$$\int_{-r}^{0}\frac{2b}{4ab}d\varepsilon = \int_{0}^{+r}\frac{2b}{4ab}d\varepsilon = \tfrac{1}{4}. \qquad (\beta)$$

[To be continued.]

A SINGULAR VALUE OF Π.

BY PROF. J. W. NICHOLSON, LOUISIANA STATE UNIV., BATON ROUGE, LA.

On page 291 of Ray's Calculus may be seen a demonstration of the following well known theorem of Wallis:

$$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8.\dots..}{1.3.3.5.5.7.7.9.\dots..} \qquad (1)$$

By the binomial formula

$$(1-1)^n=1-n+\frac{n(n-1)}{2}-\frac{n(n-1)(n-2)}{2.3}+\frac{n(n-1)(n-2)(n-3)}{2.3.4}\dots \qquad (2)$$

Factoring,

$$(1-1)^n=\frac{(1-n)\,(2-n)\,(3-n)\,(4-n)}{1\;.\;2\;.\;3\;.\;4}\dots \qquad (3)$$

Substituting $-n$ for n,

$$(1-1)^{-n}=\frac{(1+n)\,(2+n)\,(3+n)\,(4+n)}{1\;.\;2\;.\;3\;.\;4}\dots \qquad (4)$$

Multiplying (3) by (4),

$$(1+1)^n(1-1)^{-n}=\frac{(1-n^2)(4-n^2)(9-n^2)(16-n^2)}{1\;.\;2.2\;.\;3.3\;.\;4.4}\dots \qquad (5)$$

Substituting $\frac{1}{2}$ for n, and reducing,

$$(1-1)^{\frac{1}{2}}(1-1)^{-\frac{1}{2}}=\frac{1.3.3.5.5.7.7.9\dots}{2.2.4.4.6.6.8.8\dots} \qquad (6)$$

Combining (1) and (6),

$$\pi=\frac{2}{(1-1)^{\frac{1}{2}}(1-1)^{-\frac{1}{2}}}.$$

ANSWER TO PROF. SCHEFFER'S QUER (P. 31, VOL. VIII.)

BY C. B. SEYMOUR, ATTORNEY AT LAW, LOUISVILLE, KY.

Query.—"If of any curve we find the evolute, and of the latter the evolute, and so on ad infin., the ultimate evolute is a cycloid. How is this proved?"

Answer.—The proposition stated is not correct.

Let s_0 be the length of the given curve, measured from the origin to any point (the origin being a point on the curve). Let β_0 be the inclination of the tangent at that point to the axis of abscissas, and let R_0 be the radius of curvature at that point. Let s_n, β_n, R_n be like quantities for the corres-

ponding points on the nth evolute of the curve, and s_{-n}, β_{-n}, R_{-n} like quantities for the corresponding point on its nth involute. In this notation I regard the centre of curvature at any point as corresponding to that point.

Then

$$R_0 = \frac{ds_0}{d\beta_0};\ s_1 = C_0 + R_0 = C_0 + \frac{ds_0}{d\beta_0}.$$

But since the directions of a curve and its evolute at the corresponding points are perpendicular, we have

$$\beta_n = \beta_{n-1} - \tfrac{1}{2}\pi,$$

and by differentiating,

$$d\beta_n = d\beta_{n-1} = d\beta_0.$$

By the principles of the foregoing argument

$$s_2 = C_1 + \frac{ds_1}{d\beta_0} = C_1 + \frac{d^2 s_0}{d\beta_0^2},$$

and generally

$$s_n = C^{n-1} + \frac{d^n s_0}{d\beta_0^n},$$

C with its various subscripts signifying constants.

It is then plain that if s_0 be given as a function of β_0, s_n can be at once deduced as a function of β_0, and of course as a function of β_n; and this function depends on the form of the given function, as the arbitrary constant introduced does not affect the form of the evolute. Thus if $s_0 = \beta_0^i$ (i signifying an integer), successive differentiations will at last bring the eq. $s_n = C_{n-1}\,\beta_n$, which is the equation of a circle.

A cycloid does not in general result from taking successive evolutes. If however successive *involutes* be taken, the arbitrary constant introduced by one integration becomes a coefficient of β_0 in the next integration; so that the form of the ultimate involute depends on the arbitrary constants.

If in determining the arbitrary constants we make β_0 successively zero and $\frac{1}{2}\pi$ the successive integrations will bring at last an equation indefinitely approximating

$$C + s_{-4n} = C_{-4n}\cos\beta_0 = -C_{-4n}\cos\beta_{-4n}.$$

This is the equation of a cycloid, and no doubt the proposition intended was this:—

If of any curve we find the *involute*, taking the extreme radii of curvature perpendicular to each other, and of the latter the involute in like manner, and so on ad infinitum, the ultimate involute is a cycloid.

CORRESPONDENCE.

EDITOR ANALYST:—

Since you decline to publish my review of Prof. Newcomb's article on Limits on the ground that it is a repetition of arguments already gone over and hence may not be interesting to your readers; I desire to say simply, in regard to the criticism upon myself, that Prof. Newcomb's objection to my definition of a limit is not valid, since, according to accepted definitions including his own, *it is true* that any value of the sine less than unity is the limit of a series of sines subjected to such a law as that there shall be an *indefinite* approach to that value. Also that he has not shown why the syllogism, to which reference was made, is not as applicable to a divided time, as to a divided debt, or to a divided space. It is not, by any means, necessary to assume a case of uniform motion in order to illustrate the *reductio ad absurdum* to which it leads.

DE VOLSON WOOD.

Hoboken, N. J., Aug. 1, 1882.

SOLUTION OF PROB. 397 BY PROF. J. M. RICE.—Problem 397 will be found in the new edition of Thomson and Tait's Natural Philosophy, page 349. The following is an algebraic solution.

Putting $y' = 0$ we have $x'^2 = x^2 + y^2$, and from the first equation,

$$\varphi(x^2)\,\varphi(y^2) = \varphi(x^2 + y^2)\,\varphi(0) \tag{a}$$

Again, putting $y^2 = x^2$, $y^2 = 2x^2$, etc., and denoting $\varphi(0)$ by c, we have

$$[\varphi(x^2)]^2 = c.\varphi(2x^2), \tag{b}$$

and $$[\varphi(x^2)]_3 = \varphi(3x^2)c^2, \text{ etc.,}$$

finally $$[\varphi(x^2)]^n = \varphi(nx^2)c^{n-1}.$$

We now substitute z^2 for nx^2 and elminate n, whence

$$[\varphi(x^2)]^{1\div x^2} = [\varphi(z^2)^{1\div z^2}C]^{1\div x^2 - 1\div z^2},$$

or $$\left[\frac{\varphi(x^2)}{c}\right]^{1\div x^2} = \left[\frac{\varphi(z^2)}{c}\right]^{1\div z^2} = k \text{ (a constant)};$$

$$\therefore \varphi(x^2) = ck^{x^2} = ce^{x^2\div h^2}.$$

In Professor Hall's solution of this problem on p. 120, it is assumed that the partial derivatives $df\div du$ and $df\div dv$ are equal [f denoting $f(u, v)$]. I do not see that this assumption is admissible except when φ denotes an exponential function.

396. *Selected by Prof. H. T. Eddy.*—"A smooth horizontal disk revol's with the angular velocity $\sqrt{\mu}$ about a vertical axis at which is placed a material particle attracted to a certain point of the disk by a force whose acceleration is $\mu \times$ distance; prove that the path on the disk will be a cycloid. (Routh's Rigid Dynamics, p. 163.)"

Solution by Prof. Asaph Hall.—Let a and b be the coordin's of the attracting point, the origin being at the centre of the disk; and x and y the coordinates of the particle at the time t. The attracting force being $[(a-x)^2+(b-y)^2]^{\frac{1}{2}}\times\mu$, the parts of this force resolved along the axes are $(a-x)\mu$ and $(b-y)\mu$. If we consider the axis of x as a radius vector the accelerations along this axis and perpendicular to it are,

$$\frac{d^2x}{dt^2}-x\mu, \qquad \text{and} \qquad 2\sqrt{\mu}\,.\frac{dx}{dt};$$

with similar expressions for the axis of y. Hence we have the two equations of motion,

$$\frac{d^2x}{dt^2}-x\mu-2\sqrt{\mu}\,.\frac{dy}{dt}=(a-x)\mu,$$

$$\frac{d^2y}{dt^2}-y\mu+2\sqrt{\mu}\,.\frac{dx}{dt}=(b-y)\mu.$$

These give,

$$\frac{d^3x}{dt^3}+4\mu\,.\frac{dx}{dt}-2b\mu^{\frac{3}{2}}=0,$$

$$\frac{d^3y}{dt^3}+4\mu\,.\frac{dy}{dt}+2a\mu^{\frac{3}{2}}=0.$$

If we differentiate these equations in order to remove the constants we shall have two linear differential equations of the fourth order, the solution of which will introduce eight arbitrary constants. Four of these will be determined by the differential equations, and two more by the condition that when $t=0$, $x=y=0$. Putting $2\sqrt{\mu}\,.t=\theta$, the solution gives

$$x=\quad c_1-c_1\cos\theta+c_2\sin\theta+\tfrac{1}{4}b\theta,$$

$$y=-c_2+c_2\cos\theta+c_1\sin\theta-\tfrac{1}{4}a\theta.$$

These are the equations of a cycloid.

[C. B Seymour, Esq., has also sent a solution of this problem. He finds the equation $-y=\frac{1}{4}\text{ver}\sin^{-1}4x-\sqrt{(\frac{1}{2}x-x^2)}$, and remarks that "This is the equ'n of the path described on the disk by the material particle. It is, as will be seen, a cycloid whose base is the axis of y, and whose generating circle has a diameter of one-half; the cycloid lies on the positive side of the axis of ordinates, and for all positive values of $t\sqrt{\mu}$, y is negative."]

SOLUTION OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 406; Marcus Baker, 403; George Eastwood, 404; W. E. Heal, 401, 404, 406, 407, 408; William Hoover, 405; Prof. P. H. Philbrick, 401, 402; P. Richardson, 402; Prof. J. Scheffer, 401, 402, 403, 404, 406, 407; Prof. E. B. Seitz, 402, 406; M. Updegraff, 401.

401. *By M. Updegraff, Madison, Wis.*—"If two triang's are so situated that the three lines drawn thro' their corresponding vertices meet in a point, then will the corresponding sides produced meet in three points which lie on the same straight line."

SOLUTION BY THE PROPOSER.

As the lines passing through corresponding vertices of the triangles meet in a point, the triangles may be considered as projections of sections of a triangular pyramid whose edges are the three straight lines passing through the corresponding vertices. Now if a pyramid is cut by two planes these planes will intersect in a straight line, and the intersections of these two planes by the faces of the pyramid will be the bounding lines of the triangular sections of the pyramid. But if the two cutting planes are cut by a third the two traces of this third plane must meet on the line of intersection of the first two planes. Therefore the sides of the two triangular sections of the pyramid will meet in three points which lie on the line of intersection of the two cutting planes.

SOLUTION BY PROFESSORS PHILBRICK AND SCHEFFER.

Let ABC and $A'B'C'$ represent the triangles, the lines AA' BB' and CC' meeting in P. The corresponding sides meet in D, E and F.

Since $A'D$, $A'E$ and $B'F$ are respectively transversals to the triangles PAB, PAC and PBC, we have:

$$AD\times BB'\times A'P = BD\times B'P\times AA', \tag{1}$$

$$CE\times C'P\times AA' = AE\times CC'\times A'P, \tag{2}$$

$$BF\times CC'\times B'P = CF\times C'P\times BB'. \tag{3}$$

The product of (1), (2) and (3) gives:

$$AD\times CE\times BF = BD\times AE\times CF. \tag{4}$$

Hence DF is a transversal to the triangle ABC, and therefore D, E and F are in the same straight line.

402. *By Prof. W. P. Casey.*—"Upon two sides of a triangle, describe equilateral triangles, and upon the same two sides, but in the opposite direction, describe two others, and let O, O_1 be the centres of the inscribed circles in the first pair and P, P_1 those of the second pair. It is required to prove, geometrically, that the sum of the squares of the sides of the triangle $= 3(OO_1)^2 + 3(PP_1)^2$."

SOLUTION BY PROF. E. B. SEITZ.

Let ABC be any triangle, ACD, BCE and ACF, BCG the two pairs of equilateral triangles described on the sides AC and BC. Join CO, CO_1, CP, CP_1, OP, O_1P_1, OP_1, O_1P.

Let H, K, M, N, R, S, be the middle points of AC, BC, OP_1, O_1P, OO_1, PP_1, respectively. Join HK, MR, RN, NS, SM, RS, MN.

Let $BC = a$, $AC = b$, $AB = c$. Then $CO = CP = OP = \frac{1}{3}b\sqrt{3}$, $CO_1 = CP_1 = O_1P_1 = \frac{1}{3}a\sqrt{3}$. The angles ACB and OCP_1 are equal; for $\angle ACB = ACP_1 + BCP_1$ and $\angle OCP = ACP_1 + ACO$. But BCP_1 $ACO = 30°$. We also have $CA : CB :: CO : CP_1$; $\therefore$ the triangles ACB and OCP_1 are similar, and we have $AC : OC :: AB : OP_1$, whence $OP_1 = \frac{1}{3}c\sqrt{3}$. In the same way we can prove that $O_1P = \frac{1}{3}c\sqrt{3}$,

We have from similarity of triangles $MR = NS = \frac{1}{2}O_1P_1 = \frac{1}{6}a\sqrt{3}$, $RN = SM = \frac{1}{2}OP = \frac{1}{6}b\sqrt{3}$, and $HK = \frac{1}{2}c$. Since the opposite sides of $RNSM$ are equal, it is a parallelogram, and we have

$$RS^2 + MN^2 = MR^2 + RN^2 + NS^2 + SM^2 = \tfrac{1}{6}a^2 + \tfrac{1}{6}b^2 \qquad (1)$$

Since HK and RS join the middle points of the opposite sides of the q'drilateral OO_1P_1P, we have $RS^2 + HK^2 = \frac{1}{2}OP_1^2 + \frac{1}{2}O_1P^2$, whence $RS^2 = \frac{1}{12}c^2$. Substituting this value in (1), we find

$$MN^2 = \tfrac{1}{6}a^2 + \tfrac{1}{6}b^2 - \tfrac{1}{12}c^2.$$

In the quadrilateral OO_1P_1P we have

$$OO_1^2 + PP_1^2 + OP^2 + O_1P_1^2 = OP_1^2 + O_1P^2 + 4MN^2.$$

Substituting and reducing we find

$$3(OO_1)^2 + 3(PP_1)^2 = a^2 + b^2 + c^2.$$

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403. *By Prof. W. W. Johnson.*—"If, from the centre C of an equilateral hyperbola, CA be drawn bisecting the angle between the axis and an asymptote, and the chord AB be drawn perpendicular to CA; then $AB = 2CA$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURV., LOS ANG'S, CAL.

In the annexed figure CX is the axis and CM and CN the asymptotes. CA and CQ are the bisectors of MCX and NCX. Therefore $MCA=ACX=XCQ=QCN=QLC=22\frac{1}{2}°$;

$$AQC=ACQ=45°;$$

and $$QKC=QCK=67\tfrac{1}{2}°.$$

Hence the three triangles QCL, QCK and QCA are isosceles and $QL=QC=QK$ and $QA=CA$. But $AK=BL$ (property of the hyperbola) and therefore $QA=QB=AC$; $\therefore\ AB=2AC$.

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404. *By Prof. M. L. Comstock.*—"A heavy triangle ABC is suspended from a point by three strings, mutually at right angles, attached to the angular points of the triangle; if θ be the inclination of the triangle to the horizon in its position of equilibrium, then

$$\cos\theta=\frac{3}{\sqrt{(1+\sec A\sec B\sec C)}}.$$

(Todhunter's Analytical Statics, page 81.)"

SOLUTION BY PROF. J. SCHEFFER, MC SHERRYSTOWN, PA.

Let ABC represent the triangle, $DA=l$, $DB=l'$, $DC=l''$, the three strings and F, the centre of gravity of the triangle ABC.

Equilibrium will exist if F is vertically below the point D, that is, if DF is perpendicular to the horizon.

Draw DG perpendicular to the plane of ABC, then will $\angle FDG,=\theta$, be the angle which the $\triangle\ ABC$ makes with the horizon.

Since l, l', l'' are perpendicular to each other, we have $l^2+l'^2=c^2$, $l^2+l''^2=b^2$, $l'^2+l''^2=a^2$; whence

$$\left.\begin{aligned} l^2 &= \tfrac{1}{2}(b^2+c^2-a^2) = bc\cos A\\ l'^2 &= \tfrac{1}{2}(a^2+c^2-b^2) = ac\cos B\\ l''^2 &= \tfrac{1}{2}(a^2+b^2-c^2) = ab\cos C\end{aligned}\right\}\qquad(1)$$

In the triangle ABC we have $b^2+c^2=2.\frac{1}{4}a^2+2(AE)^2$, whence

$$AE^2=\tfrac{1}{4}[2(b^2+c^2)-a^2]=m^2\text{ say.}\qquad(2)$$

Since $BE=CE$, and l', l'' are perpend. to each other, we find $DE=\frac{1}{2}a$; also $$EF=\tfrac{1}{3}AE=\tfrac{1}{3}m.\qquad(3)$$

In the triangle DEA, we have

$$\cos DFA = \frac{DE^2+AE^2-AD^2}{2DE.AE} = \frac{\frac{1}{4}a^2+m^2-l^2}{am},$$

whence we get, after substituting for m and l,

$$\cos DEA = \tfrac{1}{2}a^2. \qquad (4)$$

In the triangle DEF, we have $DF^2 = DE^2+EF^2-2DE.EF.\cos DEF$. Substituting from (3) and (4), we obtain

$$DF^2 = \tfrac{1}{18}(a^2+b^2+c^2). \qquad (5)$$

The volume of the tetrahedron $ABCD$ is $\frac{1}{6}ll'l''$. Denoting the area of the triangle ABC by $\triangle$, we have $\frac{1}{6}ll'l'' = \frac{1}{3}\triangle.DG$, whence

$$DG = ll'l''\div 2\triangle; \qquad (6)$$

$$\therefore \cos\theta = \frac{DG}{DF} = \frac{3ll'l''}{\triangle\sqrt{2}.\sqrt{(a^2+b^2+c^2)}}. \qquad (7)$$

From (1), $ll'l'' = abc\sqrt{(\cos A\cos B\cos C)}$, and since $bc = 2\triangle\div\sin A$, $ac = 2\triangle\div\sin B$, $ab = 2\triangle\div\sin C$;

$$ll'l'' = \sqrt{(8\triangle^3)}.\sqrt{(\cot A\cot B\cot C)}. \qquad (8)$$

Adding the equations in (1), we get $a^2+b^2+c^2 = 2bc\cos A + 2ac\cos B + 2ab\cos C = 4\triangle(\cot A + \cot B + \cot C) = 4\triangle(\cot A\cot B\cot C + \operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C)$. Substituting in (7) and cancelling $\sqrt{(8\triangle^3)}$,

$$\cos\theta = \frac{3\sqrt{(\cot A\cot B\cot C)}}{\sqrt{(\cot A\cot B\cot C+\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C)}}.$$

Dividing numerator and denominator by $\sqrt{(\cot A\cot B\cot C)}$, we get

$$\cos\theta = \frac{3}{\sqrt{(1+\sec A+\sec B+\operatorname{cec} C)}}.$$

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405. No solution received.

406. *By William Hoover, A. M., Dayton, Ohio.*—"Find x from the eq.

$$\cot 2^{x-1}a - \cot 2^x a = \operatorname{cosec} 3a."$$

SOLUTION BY THE PROPOSER.

The given equation may be written

$$\cot\tfrac{1}{2}.2^x a - \cot 2^x a = \operatorname{cosec} 3a.$$

Put $2^x a = y$; then $\cot\frac{1}{2}y - \cot y = \operatorname{cosec} 3a$, or

$$\frac{1+\cos y}{\sin y} - \frac{\cos y}{\sin y} = \frac{1}{\sin 3a};\ \therefore \sin y = \sin 3a, \text{ or}$$

$$2^x a = 3a,\ x\log 2 = \log 3,\ x = \log 3\div\log 2.$$

407. *By Henry Heaton, Lewis, Iowa.*—"Evaluate

$$\int_0^{\frac{\pi}{2}}\left(1+\cos^4\theta\right)^{\frac{1}{2}}d\theta."$$

SOLUTION BY W. E. HEAL, MARION, INDIANA.

Let $x=\cos\theta$; the limits of x are 0 and 1.

$$\int_0^{\frac{1}{2}\pi}(1+\cos^4\theta)^{\frac{1}{2}}d\theta = \int_0^1\frac{(1+x^4)dx}{\sqrt{(1-x^2)}}.$$

$$\int\frac{(1+x^4)dx}{\sqrt{(1-x^2)}} = \int\frac{dx}{\sqrt{(1-x^2)}}+\int\frac{x^4dx}{\sqrt{(1-x^2)}} = \sin^{-1}x+\int\frac{x^4dx}{\sqrt{(1-x^2)}}$$

$$= \tfrac{1}{8}[11\sin^{-1}x-(2x^3+3x)(1-x^2)^{\frac{1}{2}}].$$

$$\therefore \int_0^1\frac{(1+x^4)dx}{\sqrt{(1-x^2)}} = \frac{1}{8}\Big[11\sin^{-1}x-(2x^3+3x)(1-x^2)^{\frac{1}{2}}\Big]_0^1 = \frac{11\pi}{16}.$$

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408. *By W. E. Heal.*—Two points, one on each of two confocal ellipsoids, are said to correspond if

$$\frac{x}{a}=\frac{X}{A},\quad \frac{y}{b}=\frac{Y}{B},\quad \frac{z}{c}=\frac{Z}{C}.$$

Prove that the distance between two points, one on each of two confocal ellipsoids is equal to the distance bet. the corresp. points. (Ivory's Th.)

SOLUTION BY THE PROPOSER.

Let the equations of the ellipsoids be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}+\frac{Z^2}{C^2}=1. \tag{2}$$

Since the ellipsoids are confocal

$$A^2=a^2+k^2\ldots(3);\ B^2=b^2+k^2\ldots(4);\ C^2=c^2+k^2\ldots \quad (5)$$

Let the first pair of points be

$$(x,y,z);\quad (X,Y,Z).$$

The corresponding points are

$$\left(\frac{aX}{A},\frac{bY}{B},\frac{cZ}{C}\right);\ \left(\frac{Ax}{a},\frac{By}{b},\frac{Cz}{c}\right).$$

Let $D=$ the distance between the first pair of points, and $d=$ the dist. between corresponding points;

$$D^2=(x-X)^2+(y-Y)^2+(z-Z)^2=(x^2+y^2+z^2)$$
$$-2(xX+yY+zZ)+(X^2+Y^2+Z^2). \quad (6)$$

$$d^2 = \left(\frac{Ax}{a} - \frac{aX}{A}\right)^2 + \left(\frac{By}{b} - \frac{bY}{B}\right)^2 + \left(\frac{Cz}{c} - \frac{cZ}{C}\right)^2$$

$$= \frac{A^2x^2}{a^2} + \frac{B^2y^2}{b^2} + \frac{C^2z^2}{c^2} - 2(xX+yY+zZ) + \frac{a^2X^2}{A^2} + \frac{b^2Y^2}{B^2} + \frac{c^2Z^2}{C^2}. \quad (7)$$

From (1) and (2)

$$k^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - k^2\left(\frac{X^2}{A^2} + \frac{Y^2}{B^2} + \frac{Z^2}{C^2}\right) = 0. \quad (8)$$

Adding (7) and (8), regarding (3), (4) and (5),

$$d^2 = (x^2+y^2+z^2) - 2(xX+yY+zZ) + (X^2+Y^2+Z^2). \quad (9$$

$$\therefore d = D.$$

PROBLEMS.

409. *By David Trowbridge, A. M., Waterburgh, N. Y.*—If in any triangle ABC, squares be described on the three sides, and the vertices of the squares be joined by the three straight lines a, b, c; show that

$$a^2 + b^2 + c^2 = 3(AB^2 + BC^2 + CA^2).$$

410. *By Prof. J. Scheffer.*—A cone with circular base is cut by a parabolic plane which passes through the centre of the base; to find the position of the centre of gravity of both portions of the cone.

411. *By Alex. S. Christie, U. S. Coast Survey.*—Sum the series

$$1 - \frac{n}{1}\,\frac{1}{3} + \frac{n(n-1)}{2!}\,\frac{1}{5} - \frac{n(n-1)(n-2)}{3!}\,\frac{1}{7} + \&c.,$$

for positive values of n.

412. *By Prof. L. G. Barbour.*—Show that in any hexaedron bounded by quadrilaterals, the three lines respectively connecting the mean points of opposite (non-contiguous) faces, mutually bisect each other.

413. *By William Hoover, A. M.*—A rod rests with one extremity in a smooth plane and the other against a smooth vertical wall at an inclination a to the horizon. If it then slips down, show that it will leave the wall when its inclination is $\sin^{-1}(\frac{2}{3}\sin a)$.

414. Sum the series, $\sec\theta + \sec\frac{1}{2}\theta + \sec\frac{1}{4}\theta + \sec\frac{1}{8}\theta + \ldots + \sec\frac{1}{2^n}\theta$.

415. Evaluate $$\int \frac{dx}{x - dx}.$$

416. *By Prof. W. P. Casey.*—Given the base AB and the angle A of a triangle ABC; find the locus of the foot of the perpendicular CF drawn from C to the side of the inscribed square.

417. *By George Eastwood.*—Two boys, A and B, play a game at marbles. A deposits 50 white marbles in a bag; B deposits 50 yellow marb's of the same size and value as A's, in another bag, and the game is this:—

The bags being placed in convenient positions, B, at a given signal, takes a marble out of A's bag and drops it into his own bag; then A takes a marble out of B's bag and drops it into his own bag. Next B takes another marble out of A's bag and transfers it to his own, and A then transf's one of B's marbles to his own bag, and so on. After 50 transfers from each bag to the other, in the above order, what is the probable number of white marbles found in B's bag, and how many transfers will B have to make to gain 25 of A's marbles?

PUBLICATIONS RECEIVED.

Linear Associative Algebra. By BENJAMIN PEIRCE, LL. D. New Edition with Addenda and Notes, by C. S. PEIRCE, Son of the Author. 133 pp., 4to. New York: D. Van Nostrand, Publisher. 1882.

Brief Description of the Algebra of Relatives. By C. S. PEIRCE. 4to. 1882.

The Reduction of Air-pressure to Sea-level at Elevated Stations West of the Mississippi River. By HENRY A. HAZEN, A. M. 4to. 42 pp., with 20 isobarometric maps. Washington: 1882.

Note on Hanson's General Formulae for Perturbation. By G. W. Hill. [Reprinted from the American Journal of Mathematics, Vol. IV, No. 3.]

Notes, Queries, and Answers. Vol. I. No. 1. July, 1882. 16 pp., 8vo. Monthly. S. C. & L. M. Gould, publishers, Manchester, N. H.

ERRATA.

On page 143, line 10, for value, read values.
" " " " 21, last word in line, for the, read these.
" " " " 7, from bottom, for "lemits", read limits.

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ON AN UNSYMMETRICAL PROBABILITY CURVE.

BY E. L. DE FOREST.

[Continued from page 142.]

Thus it is proved that when any two entire polynomials are multiplied together, the cube of the cubic radius about the centre of forces in the product is equal to the sum of the cubes of the like radii in the two factors. Hence, if any number of such polynomials are multiplied together, the cube of the radius in the final product is equal to the sum of the cubes of the radii in all the factors. The cube of the radius in the k power of a polynomial is k times the cube of the radius for the polynomial itself. These propositions evidently hold true also for polynomials like (5), in which the place of z^0 is not at the first or left hand term. The coefficients in a product are not altered when all the exponents in either factor are increased or diminished by a constant quantity.

Applying the above to the expression for a in (14), we write

$$a = 2\left(\frac{b_2(dx)^2}{b_3(dx)^3}\right) = 2\left(\frac{kb_2(dx)^2}{kb_3(dx)^3}\right), \tag{38}$$

showing that the part within the parentheses may be regarded as the square of the quadratic radius divided by the cube of the cubic radius, either in the first power of the polynomial or in its expansion to the k power, as may be most convenient. The values of a and b may thus be expressed either by means of the coefficients λ in the given polynomial, or by means of the ordinates y to the limiting curve. When the λ's and y's are all positive, and represent probabilities, $kb_2(dx)^2$ is the square of the *quadratic mean error* ε, and $kb_3(dx)^3$ is the cube of what we will call the *cubic mean inequality*, which we denote by ζ. The constants in (14) will then be

$$a = 2\varepsilon^2 \div \zeta^3, \qquad b = \varepsilon^2. \tag{39}$$

The square of the q. m. error is of course positive, but the cube of the c. m. inequality is either + or —, according as the + or — errors preponderate in forming it. If the facility of error is the same on both sides of the arith. mean or centre of gravity, so that + and — errors of equal am't are equally probable, ζ will be zero, and a becomes infinite. If ζ is negative, a is also negative, and the position of the limiting curve is reversed, so that it lies on the left of the origin, or rather, the origin is now at the other side of the expanded polynomial. Since y in (25) is a function of ax, it will have the same value when a and x are both —, as when they are both +.

The notation we adopted in (14) is such that in the expansion of the polynomial (5) to the infinite k power we have

$$(\text{quadratic rad.})^2 = b, \qquad (\text{cubic rad.})^3 = 2b \div a. \qquad (40)$$

We will now show that these values of the radii are deducible from the equation (25) of the gamma curve, and thus verify the proposition that the curve is a true limiting form of the expansion of the polynomial, since it possesses properties which are known to characterize that expansion. The distance from the origin to the centre of parallel forces, or centre of gravity of the masses y, will be

$$\frac{1}{dx}\int_0^\infty xy\,dx \div \left(\frac{1}{dx}\int_0^\infty y\,dx\right).$$

The divisor here is unity, so that the distance sought is

$$\frac{1}{dx}\int_0^\infty xy\,dx = \frac{1}{a\Gamma(a^2b)}\int_0^\infty (ax)^{a^2b}\, e^{-ax} d(ax) = \frac{\Gamma(a^2b+1)}{a\Gamma(a^2b)} = ab. \quad (41)$$

This agrees with (12), for by (14) we have

$$ab = 2kb_2^2 dx \div b_3.$$

The squared quadratic radius of the masses y about the centre of grav. is

$$\frac{1}{dx}\int_0^\infty (x-ab)^2 y\,dx \div \left(\frac{1}{dx}\int_0^\infty y\,dx\right),$$

that is, the divisor being unity as before,

$$\frac{1}{dx}\int_0^\infty (x-ab)^2 y\,dx = \frac{1}{\Gamma(a^2b)}\left\{\frac{1}{a^2}\int_0^\infty (ax)^{a^2b+1} e^{-ax} d(ax) - 2b\int_0^\infty (ax)^{a^2b} e^{-ax} d(ax) \right.$$
$$\left. + a^2b^2\int_0^\infty (ax)^{a^2b-1}\, e^{-ax} d(ax)\right\}$$
$$= \frac{1}{\Gamma(a^2b)}\left\{\frac{\Gamma(a^2b+2)}{a^2} - 2b\Gamma(a^2b+1) + a^2b^2\Gamma(a^2b)\right\}$$
$$= b(a^2b+1) - 2a^2b^2 + a^2b^2 = b, \qquad (42)$$

a result which agrees with (40). Likewise for the cube of the cubic radius, omitting the divisor unity, we have

$$\frac{1}{dx}\int_0^\infty (x-ab)^3 y\,dx = \frac{1}{\Gamma(a^2b)}\left\{\frac{\Gamma(a^2b+3)}{a^3} - \frac{3b\Gamma(a^2b+2)}{a} \begin{array}{l} + 3ab^2\Gamma(a^2b+1) \\ - a^3b^3\Gamma(a^2b) \end{array}\right\}$$

$$= \left(\frac{b}{a}\right)(a^2b+1)(a^2b+2) - 3ab^2(a^2b+1) + 3a^3b^3 - a^3b^3 = \frac{2b}{a}. \quad (43)$$

This also agrees with (40), so that the curve (25) does exactly represent the form of the series of coefficients in the expansion of the polynomial (5) to an infinitely high power, so far as the quadratic and cubic radii about the centre of forces are concerned.

As shown in my former articles, a binomial $p+q$ or $p+qz$, in which $p+q=1$ and the coefficients p and q are separated by the interval dx, has its centre of gravity at the distance qdx from the first term p, and the sq'd quadratic radius about that centre is

$$(qp^2+pq^2)(dx)^2 = pq(dx)^2. \quad (44)$$

If the binomial is raised to the mth power, the centre of gravity in the expansion will be at the distance $qmdx$ from the first term, and the squared quadratic radius about that centre is

$$\epsilon^2 = pqm(dx)^2. \quad (45)$$

The cube of the cubic radius in the first power is

$$(qp^3-pq^3)(dx)^3 = pq(p-q)(dx)^3, \quad (46)$$

and in the mth power, as we have here shown, it is m times as great, or

$$\zeta^3 = pqm(p-q)(dx)^3. \quad (47)$$

Hence, when m becomes infinite, the constants in the limiting curve, according to (39), will be

$$a = \frac{2}{(p-q)dx}, \qquad b = pqm(dx)^2. \quad (48)$$

We found in (22) that y is a maximum when $x = ab - 1 \div a$, so that the vertex of the gamma curve is at the distance $-1 \div a$ from the centre of gravity, and by (48)

$$-\frac{1}{a} = -\tfrac{1}{2}(p-q)dx. \quad (49)$$

The agreement between this result and that which I found by different means in ANALYST, Vol. VII, p. 3, shows that the vertex of the curve (25) accurately represents the position of the vertex in the expanded binomial, with reference to the ordinate through the centre of gravity.

It will often be convenient to have the origin of coordinates transferred to the centre of gravity. Putting $x+ab$ in place of x in (25), we have

$$y = \frac{dx}{ab\Gamma(a^2b)}\left(\frac{a^2b}{e}\right)^{a^2b}\left(1+\frac{x}{ab}\right)^{a^2b-1}\epsilon^{-ax}. \quad (50)$$

A known formula for $\Gamma(n)$ is

$$\Gamma(n) = \left(\frac{n}{e}\right)^n \sqrt{\left(\frac{2\pi}{n}\right)}\left(1+\frac{1}{12n}+\frac{1}{288n^2} - \&c.\right), \quad (51)$$

and by means of this (50) is reduced to

$$\left.\begin{aligned} K &= 1+\frac{1}{12a^2b}+\frac{1}{288(a^2\,b)^2}-\&c., \\ y &= \frac{dx}{K\sqrt{(2\pi b)}}\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax}. \end{aligned}\right\} \qquad (52)$$

This is the equation of the gamma curve referred to the centre of gravity of the masses y as an origin. To illustrate the uses of the curve, we will now employ it in computing the principal terms in the expansion of a binomial to a high power.

Mortality tables show that among persons aged 40, about one per cent may be expected to die within a year, so that the probability of dying within a year is .01, and that of surviving a year is .99. Suppose we wish to find the probabilities that out of 1000 persons aged 40, the number of deaths within a year will be 0, 1, 2, 3 &c. These probabilities are the 1st, 2nd &c. terms in the expansion of the binomial

$$(p+q)^m = (.99+.01)^{1000}. \qquad (53)$$

The common interval between consecutive terms in the expansion being dx, we will take this as the unit of abscissas. By (48) we have

$$a = \frac{2}{.99-.01} = \frac{100}{49}, \quad b = .99\times.01\times1000 = \tfrac{99}{10},$$

and by (52)

$$\log y = \bar{1}.1022148+40.23282\log(1+.04949495x)-.8863153x. \quad (54)$$

Since the distance $qmdx = 10$ of the centre of gravity of the expanded series from its first term is in this instance a whole number, it follows that one term of the expansion stands exactly at that centre, where $x = 0$, and the whole series of terms will be found by putting x successively equal to

$$\ldots\ldots -2, -1, 0, 1, 2, 3 \ldots\ldots$$

The resulting values of y are given in column (2) of Table I. To show the degree of accuracy attained, the true values of the terms in the expansion have been computed and set in column (1), and the differences (2)—(1) are also given, in units of the fifth decimal place. The computed curve intersects the true one at four points. The agreement between them is pretty close, and would be closer if the exp'nt m were a greater number than it is.

Reverting now to the more general significance of the gamma curve, as representing the expansion of a polynomial, we will inquire what simpler form of curve it approximates to. Let (52) be written

$$y = c\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax},$$

TABLE I.

x	(1)	(2)	(2)–(1)	(3)	(3)–(1)	x	(1)	(2)	(2)–(1)	(3)	(3)-(1)
0	.12574	.12654	80	.12679	105	1	.11431	.11482	51	.12055	624
- 1	.12562	.12635	73	.12055	—507	2	.09516	.09525	9	.10360	844
- 2	.11283	.11311	28	.10360	—923	3	.07305	.07282	—23	.08048	743
- 3	.08999	.08966	—33	.08048	—951	4	.05202	.05167	—35	.05651	449
- 4	.06274	.06203	—71	.05651	—623	5	.03454	03422	—32	.03587	133
- 5	.03746	.03680	—66	.03587	—159	6	.02148	.02127	—21	.02058	— 90
- 6	01861	.01834	—27	.02058	197	7	.01256	.01247	— 9	.01067	—189
- 7	.00739	.00748	9	.01067	328	8	.00693	.00692	— 1	.00500	—193
- 8	.00220	.00242	22	.00500	280	9	.00362	.00365	3	.00212	—150
- 9	.00044	.00060	16	.00212	168	10	.00179	.00184	5	.00081	— 98
-10	.00004	.00011	7	.00081	77	11	.00084	.00089	5	.00028	— 56
-11		.00002		.00028		12	.00038	.00041	3	.00009	— 29
-12				.00009		13	.00016	.00018	2	.00002	— 14
-13				.00002		14	.00007	.00008	1	.00001	— 6
-14				.00001		15	.00003	00003	0		— 3
						16	.00001	.00001	0		— 1
						17		.00001	1		

$$\therefore \log'\left(\frac{y}{c}\right) = (a^2b-1)\log'\left(1+\frac{x}{ab}\right)-ax$$

$$= (a^2b-1)\left\{\frac{x}{ab}-\frac{1}{2}\left(\frac{x}{ab}\right)^2+\frac{1}{3}\left(\frac{x}{ab}\right)^3-\&c.\right\}-ax$$

$$=-\frac{x^2}{2b}+\left(\frac{x^2}{3b}-\frac{1}{1}\right)\frac{x}{ab}-\left(\frac{x^2}{4b}-\frac{1}{2}\right)\left(\frac{x}{ab}\right)^2+\left(\frac{x^2}{5b}-\frac{1}{3}\right)\left(\frac{x}{ab}\right)^3\ldots \quad (55)$$

Since b is a finite area, $x^2\div b$ is in general a finite number. By (14) we have

$$\frac{x}{ab} = \left(\frac{b_3}{b_2}\right)\frac{xdx}{2b}, \quad (56)$$

which is in general an infinitesimal. Neglecting all terms in which $x\div ab$ is a factor, (55) is reduced to

$$\log'\left(\frac{y}{c}\right) = -\frac{x^2}{2b}, \qquad \therefore y = ce^{-x^2\div 2b}.$$

Restoring the value of c from (52), noticing that when k is really infinite we have

$$a^2b = 4kb_2^3\div b_3^2 = \infty, \qquad \therefore K=1,$$

we get finally

$$y = \frac{dx}{\sqrt{(2\pi b)}}e^{-x^2\div 2b}, \quad (57)$$

the equation of a common probability curve like (1). This is the same result we would have obtained in the first place, if we had negl'd d^2y in (10).

If instead of retaining only dy and d^2y, we should also retain d^3y, the resulting equation, if we could integrate it, would doubtless give a limiting curve of still more general form, of which the gamma curve is but a particular case. Under this view, the probability curve (57) and the gamma curve (52) are only first and second approximations to the actual form of an expansion to a high power.

It must be observed that since dx represents the common interval between consecutive coefficients y in the expanded series, the abscissas x corresponding to ordinates at and near the origin will have the values

$$\ldots\ldots -dx,\ 0,\ dx,\ 2dx,\ \ldots\ldots,$$

so that $x^2 \div b$ and $x \div ab$ in (55) will there be of the same order of magnitude, and the latter cannot be neglected in comparison with the former. The curve is thus rendered unsymmetrical in the immediate vicinity of the origin, and the maximum or vertex is thrown a very little to one side of the centre of gravity. But if the given polynomial (5) is such as to make $b_3=0$, then (56) makes $x \div ab$ absolutely null, and the terms in which it is a factor disappear altogether from (55), leaving (57) as the exact result, and showing that the probability curve is a special case of the gamma curve, occurr'g when $1 \div a = 0$. This case arises at the moment when the gamma curve passes from the direct to the reversed position, as noticed in connection with (39). The X axis then becomes an asymptote to the curve, not on one side only, but on both sides. Again, if b_3, though not absolutely zero, is quite small in comparison with b_2, making ζ^3 quite small in comparison with ε^2, a thing which often occurs, then a is a very large number, and terms containing $x \div ab$ may be dropped from (55) as before. In other words, if the asymmetry, as measured by the c. m. inequality ζ, is small, the gamma curve does not differ materially from the common probability curve, and the latter may be preferred to it for practical use, as being more simple.

But when ζ is so large that a is a small number, the probability curve will not represent the true form of the expansion of a polynomial to a finite power with sufficient accuracy. Take for instance that of the binomial (53), for which, with $dx = 1$ as before, and $b = 9.9$, we get by (57)

$$\log y = \bar{1}.1030924 - .02193407x^2. \tag{58}$$

The terms of the series computed by this are entered in column (3) of Table I. Their sum is unity as it should be. The differences (3) — (1) between the computed values and the true ones are also shown. On an average they are more than 12 times as great as the differences (2)–(1) afforded by the gamma curve.

That the expansion of a polynomial to a high power, taken as a whole, tends to become more and more symmetrical in form the higher the power

is, may be inferred from the properties of the quadratic and cubic radii, without regard to any precise analytical expression for the limiting curve. While the whole length of the expanded series increases in proportion to the exponent k of the power, the quadratic radius of the coefficients, about their centre of forces, increases only as the square root of k, while the cubic radius increases still more slowly, being proportional to the cube root of k, as seen in connection with (37).

Suppose that the coefficients λ in the polynomial (5) represent the probabilities of the occurrence of the various possible true errors of an observed quantity, these errors being multiples of the unit of measure $\varDelta x$, which may be taken as small as we please, while m is a whole number so large that the greatest error will not exceed $\pm\, m\varDelta x$. In any term $\lambda_i z^i$ the coefficient λ_i is the probability that, in a single observation, the error which occurs will be $x = i\varDelta x$. The centre of gravity of the coeffic's, regarded as the masses of material p'ts ranged along the imponderable axis of X, may or may not coincide with the place of λ_0, but wherever it is, its abscissa x_1, or lever arm about the place of λ_0, is the arithmetical mean of all the possible true errors of a single observation, each error being taken with a weight proportional to the probability of its occurrence. Let ε and ζ denote the quadratic and cubic radii of the masses λ, about the centre of gravity. These are the same as the q. m. error and c. m. inequality of a single observation, if what we call errors are not necessarily true errors, but only deviations from the centre of gravity or *ultimate arith. mean.* (ANALYST, VIII, p. 141.)

If k such observations are taken, the possible true errors of their sum will be the exponents, and their probabilities will be the coefficients l, in the polynomial (6), which is the expansion of (5) to the k power. The centre of gravity of all the coefficients l will be approximately the place of the maximum coefficient, and its abscissa, or lever arm about the place of l_0, will be kx_1. This lever arm is the arithmetic mean of all the possible true errors in the sum of k observations, the errors being weighted for probability of occurrence. The quadratic and cubic radii for the masses l, about the centre of gravity, will be

$$E = \varepsilon\sqrt{k}, \qquad Z = \zeta\sqrt[3]{k}, \tag{59}$$

and these are the q. m. error and c. m. inequality of the sum of k observations.

The probability that the true error of the sum of k observations will be x, is the same as the probability that the true error of their arith. mean will be $x \div k$. If we suppose the coefficients or masses l to be set closer together, so that the common interval between them is reduced from $\varDelta x$ to $\varDelta x \div k$, their distribution along the X axis will represent the law of facility of error in the mean of k observations. The limits of possible error, which were $\pm\, km\varDelta x$ for the sum, will be reduced to $\pm\, m\varDelta x$ for the mean, being the same

as for a single observation. The centre of g. of all the masses l will now, as before, be the approximate place of the maximum, and its abscissa, or lever arm about the place of l_0, will be reduced from kx_1 to x_1, showing that if there is any true error in the ultimate mean, or arith. mean of all the possible values, weighted for probabi'ty of occurrence, it is the same for the arith. mean of k observations, as it is for a single observation.

Since the distance of each of the masses l from the centre of gravity is k times less for the arith. mean than it was for the sum, $1\div k$ becomes a common coefficient of all the distances which go to make up the quadratic and cubic radii, which are consequently k times less for the mean than they were for the sum. Hence by (59), the q. m. error and c. m. inequality for the arith. mean of k observations will be

$$\varepsilon_0 = \varepsilon\div\sqrt{k}, \qquad \zeta_0 = \zeta\div k^{2/3}. \tag{60}$$

When k is increased, the q. m. error of the mean diminishes, being inv'sely as $\sqrt{k}$, while the c. m. inequality diminishes more rapidly, being inversely as $\sqrt[3]{k^2}$. If we take 64 times as many observations, the q. m. error of the mean result will be one eighth as large as before, but the c. m. inequality of the possible errors of the mean will be only one sixteenth as large as before. This goes to show that as k increases, the curve of facility of error in the mean, taken as a whole, becomes more and more symmetrical on either side of the centre of gravity.

It may be remarked here, by the way, that a relation holds for the c. m. inequality, very similar to that which holds for the q. m. error, in a quantity X which is connected with other quantities x_1, x_2 &c. thus,

$$X = a_1x_1+a_2x_2+a_3x_3+\&c., \tag{61}$$

where a_1, a_2 &c. may be essentially either $+$ or $-$. The errors of x_1, x_2 &c. are supposed to be independent, that is, the error of one has no influence on the error of another. If ε_1, ε_2 &c. denote the q. m. errors of x_1, x_2 &c., and ε denotes the q. m. error of X, then as is well known

$$\varepsilon^2 = (a_1\varepsilon_1)^2+(a_2\varepsilon_2)^2+(a_3\varepsilon_3)^2+\&c. \tag{62}$$

(See the method of proof which I gave in ANALYST, VIII, p. 139.) In like manner, it is easily seen that if the errors to which x_1, x_2 &c. are liable are of such nature that $+$ and $-$ errors of equal amount are not equally probable, then denoting the c. m. inequalities of x_1, x_2 &c. by ζ_1, ζ_2 &c., and that of X by ζ, we shall have by virtue of (37)

$$\zeta^3 = (a_1\zeta_1)^3+(a_2\zeta_2)^3+(a_3\zeta_3)^3+\&c. \tag{63}$$

The a_1, a_2 &c. are merely coefficients, so that any actual error of a_1x_1, for instance, is a_1 times the actual error of x_1. Then the c. m. inequality of a_1x_1 is $a_1\zeta_1$, that of a_2x_2 is $a_2\zeta_2$, and so on. The c. m. inequality of $-x_2$ being $-\zeta_2$, that of $(x_1\pm x_2)$ is the cube root of $(\zeta_1^3\pm\zeta_2^3)$.

[To be continued.]

ON THE ACTUAL AND PROBABLE ERRORS OF INTERPOLATED VALUES DERIVED FROM NUMERICAL TABLES BY MEANS OF FIRST DIFFERENCES.

BY R. S. WOODWARD, C. E.

[Continued from page 149.]

Since

$$\int_{-(a+b)}^{-(a-b)} \frac{\varepsilon+a+b}{4ab}\, d\varepsilon = \frac{b}{2a},$$

$(a-b)$ will be the probable error or r when the relation between a and b is such that

$$\frac{b}{2a} = \tfrac{1}{4}, \text{ or when } b = \tfrac{1}{2}a.$$

In this case $r = a-b = a-\frac{1}{2}a = \frac{1}{2}a$. When $b > \frac{1}{2}a$ and $< a$, r will be given by (α); and when $b < \frac{1}{2}a$, r will be given by (β).

Evaluating (α) there results $r = \pm[a+b-\sqrt{(2ab)}]$, while (β) gives $r = \pm\frac{1}{2}a$. Therefore, when b lies between 0 and $\frac{1}{2}a$

$$r = \pm\tfrac{1}{2}a; \tag{11}$$

and when b lies between $\frac{1}{2}a$ and a

$$r = \pm[a+b-\sqrt{(2ab)}]. \tag{12}$$

In the problem of interpolation, $a = (1-t)\frac{1}{2}$ and $b = t\,.\,\frac{1}{2}$. Supposing $1-t > t$,

$$\frac{b}{2a} = \frac{t}{2(1-t)} = \tfrac{1}{4} \text{ for } t = \tfrac{1}{3}.$$

Therefore, according to equations (11) and (12),

$$\left.\begin{array}{llll} \text{when } t \text{ lies between } 0 \text{ and } \frac{1}{3}, & r = \pm\frac{1}{4}(1-t), \\ \text{" } t \text{ " " } \frac{1}{3} \text{ " } \frac{2}{3}, & r = \pm\frac{1}{2}[1-\sqrt{2t(1-t)}], \\ \text{" } t \text{ " " } \frac{2}{3} \text{ " } 1, & r = \pm\frac{1}{4}t. \end{array}\right\} \tag{13}$$

It appears from these formulas that the probable error of an interpolated value of the first species is always less than the probable error of a tabular value. In other words, such interpolated values are more precise than tab. values. The following table gives the values of r corresponding to several values of t.

For $t =$	0	or 1	$r = \pm 0.25$
" $t =$	$\frac{1}{10}$	" $\frac{9}{10}$	$r = \pm 0.23$
" $t =$	$\frac{2}{10}$	" $\frac{8}{10}$	$r = \pm 0.20$
" $t =$	$\frac{3}{10}$	" $\frac{7}{10}$	$r = \pm 0.17$
" $t =$	$\frac{4}{10}$	" $\frac{6}{10}$	$r = \pm 0.16$
" $t =$	$\frac{5}{10}$	"	$r = \pm 0.15$

As a statistical test of the accuracy of the laws of error expressed by eq's (8), (9), (10), 500 actual errors, $\varepsilon' = (1-t)\varepsilon_1 + t\varepsilon_2$, corresponding to interpolated values derived from a 5-place table of log's have been computed by means of a 7-place table. In each case t was taken equal to $\frac{1}{2}$, or the errors of interpolated mid-values were computed. The arguments used were the following numbers: 20005, 20035, 20065, 20105, 20135, 20165, 20205, etc., in the same order to 36635. Now for $t = \frac{1}{2}$, $a = b = \frac{1}{4}$, and hence by (8) and (10)

$$\varphi(\varepsilon) = \quad 4\varepsilon + 2 \text{ for values of } \varepsilon \text{ bet'n } -\tfrac{1}{2} \text{ and } 0,$$
$$\varphi(\varepsilon) = -4\varepsilon + 2 \text{ `` `` `` } \varepsilon \text{ `` } +\tfrac{1}{2} \text{ `` } 0.$$

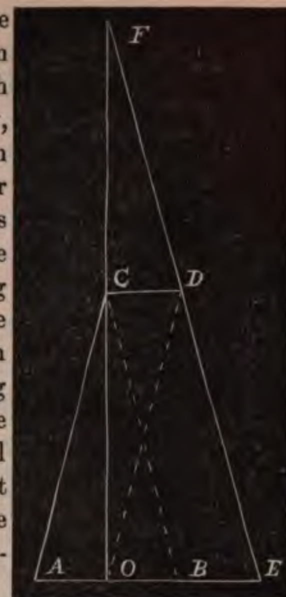
From the integral $\int(2 \pm 4\varepsilon)d\varepsilon$, the percentages of the whole number of errors to be expected between the limits 0 and 0.1, 0.1 and 0.2, etc., up to 0.4 and 0.5 may be computed. The following table gives a comparison of these percentages with those derived from the 500 actual errors.

Between Limits.	Percent of whole number of errors.	
	By Theooy.	By Trial.
0.0 and 0.1	36	33.2
0.1 " 0.2	28	30.2
0.2 " 0.3	20	19.0
0.3 " 0.4	12	13.2
0.4 " 0.5	4	4.4
0.0 " 0.15	50	51.4

The average of the squares of all probable errors given by eq's (13) is

$$(\tfrac{1}{4})^2 \left\{ 2\int_{5/6}^{1} t^2 dt + 4\int_{1/6}^{5/6} [1 - \sqrt{2t(1-t)}]^2 dt \right\} = 0.594(\tfrac{1}{4})^2.$$

2. The consideration of the probable errors of interpolated values of the second species presents some peculiar features, which are, perhaps, more interesting than important. These features arise from the fact already stated that ε_3 in the formula $\varepsilon'' = (1-t)\varepsilon_1 + t\varepsilon_2 + \varepsilon_3$, is discontinuous for any value of t. Thus, when $t = \frac{1}{2}$, ε_3 must be either 0 or $\frac{1}{2}$ according as the tabular difference is odd or even, and these two values of ε_3 are equally likely to occur. For $t = \frac{1}{2}$ therefore we may imagine any large number of resultant errors, ε'', to be made up of two distinct systems of error, namely, 1st that which embraces all cases in which $\varepsilon_3 = 0$, and 2nd that which embraces all cases in which $\varepsilon_3 = \frac{1}{2}$. In the first system the probable error is ± 0.15 as given by the second of (13) for $t = \frac{1}{2}$, since $\varepsilon_3 = 0$. In the second system all values of ε'' will be positive or negative according as ε_3 is taken $= +\frac{1}{2}$ or $= -\frac{1}{2}$; and the probable error will be $+\frac{1}{2}$ or $-\frac{1}{2}$ according as $\varepsilon_3 = +\frac{1}{2}$ or $-\frac{1}{2}$.



On the other hand, we may consider any large number of values of ε'', embracing both classes in which $\varepsilon_3 = 0$ and $\varepsilon_3 = \frac{1}{2}$, as one system. In such a system, supposing the value for ε_3 positive only, the actual errors will vary in magnitude between the limits $-\frac{1}{2}$ and $+1$. The law of facility for the errors in the system in which $\varepsilon_3 = 0$ only, is represented by the isosceles triangle ABC, the base AB being the axis of ε and the altitude OC, being the axis of $\varphi(\varepsilon)$. Likewise, the isosceles triangle ODE represents the law of facility for the system of errors in which $\varepsilon_3 = +\frac{1}{2}$ only, the origin being at O with axes OE and OC respectively. If the element-areas representing the probabilities of eq'l errors in the two systems be summed, the result will be the probability-area corresponding to the combined system of errors. This area is represented by the trapezoid $ACDE$. Therefore

for values of ε between $-\frac{1}{2}$ and 0, $\varphi(\varepsilon) = 2 + 4\varepsilon$,
" " " ε " 0 " $+\frac{1}{2}$, $\varphi(\varepsilon) = 2$,
" " " ε " $+\frac{1}{2}$ " $+1$, $\varphi(\varepsilon) = 4 - 4\varepsilon$.

If we consider all possible errors in this system with reference to magnit'de only, the probability-curve becomes a straight line joining the points whose coordinates are $(0, 4)$ and $(1, 0)$, such line being represented by EDF in the above Figure. The law of facility is

$$\varphi(\varepsilon) = 4(1-\varepsilon).$$

To determine the probable error, r, we have the condition

$$\int_0^r 4(1-\varepsilon)d\varepsilon = \tfrac{1}{2}\int_0^1 4(1-\varepsilon)d\varepsilon = 1, \text{ whence}$$
$$4r - 2r^2 = 1 \text{ and}$$
$$r = 1 - \tfrac{1}{2}\sqrt{2} = +0.29.$$

In order to verify by an actual test this result, which indeed differs from the result reached by a distinguished mathematician, see § 5, the actual errors of 500 interpolated mid-values from a 5-place table of logarithms have been computed by means of a 7-place table. These mid-values correspond to the arguments mentioned under § 4, 1, above; and their actual errors correspond to ε'' in the formula $\varepsilon'' = (1-t)\varepsilon_1 + t\varepsilon_2 + \varepsilon_3$, for $t = \frac{1}{2}$. Now by the law of error just derived, one-half of these actual errors ought to be less than 0.29. Likewise, from the integral $\int 4(1-\varepsilon)d\varepsilon$, the percentages of the

whole number of errors to be expected between the limits 0 and 0.2, 0.2 and 0.4 etc., to 0.8 and 1 can be computed. The following table affords a comparison of the theoretical and observed percentages.

Between Limits.	Per cent by Theo.	Per cent by Obs.
0 and 0.2	36	35.8
0 " 0.29	50	49.8
0.2 " 0.4	28	27.8
0.4 " 0.6	20	18.6
0.6 " 0.8	12	12.2
0.8 " 1.0	4	5.6

Again, suppose $t=\frac{1}{3}$. Then $\varepsilon_3 = 0, +\frac{1}{3}$ or $-\frac{1}{3}$, and these values of ε_3 are equally likely to occur. Any large number of actual errors, ε'', for $t=\frac{1}{3}$ may therefore be considered as a combination of three distinct systems corresponding respectively to the three different values of ε_3. For the system in which $\varepsilon_3 = 0$ only, $\varphi(\varepsilon)$ is given by equations (8), (9), (10) by putting $a = \frac{2}{3}\times\frac{1}{2} = \frac{2}{6}$, and $b = \frac{1}{3}\times\frac{1}{2} = \frac{1}{6}$. For the system in which $\varepsilon_3 = +\frac{1}{3}$ only, $\varphi(\varepsilon)$ will result from (8), (9), (10) by giving to a and b the values just mentioned, and substituting for ε, $\varepsilon + \frac{1}{3}$; or in other words by moving the origin a distance $+\frac{1}{3}$ to the left of its position for the system of errors in which $\varepsilon_3 = 0$ only. Likewise, for the system in which $\varepsilon_3 = -\frac{1}{3}$ only, $\varphi(\varepsilon)$ will result by substituting for ε in $\varphi(\varepsilon)$ for the first system, $\varepsilon - \frac{1}{3}$. See Fig., p. 149, $t=\frac{1}{3}$. If the areas representing the probabilities of equal errors in the three systems be summed, the probability-area for the original or combined system will result. This area will be a trapezoid whose base is $\frac{10}{6}$, whose side parallel to the base is $\frac{2}{6}$, and whose altitude is 3, the other two sides making angles with the base whose tangents are $+\frac{9}{2}$ and $-\frac{9}{2}$ respectively. Therefore

for errors lying between $-\frac{5}{6}$ and $-\frac{1}{6}$, $\varphi(\varepsilon) = \frac{1}{4}(15+18\varepsilon)$,
" " " " $-\frac{1}{6}$ " $+\frac{1}{6}$, $\varphi(\varepsilon) = 3$,
" " " " $+\frac{1}{6}$ " $+\frac{5}{6}$, $\varphi(\varepsilon) = \frac{1}{4}(15-18\varepsilon)$.

Hence the probable error will result from

$$\int_{-\frac{5}{6}}^{-r}\tfrac{1}{4}(15+18\varepsilon)d\varepsilon = \int_{+r}^{+\frac{5}{6}}\tfrac{1}{4}(15-18\varepsilon)d\varepsilon = \tfrac{3}{4},$$

which gives

$$r = \pm\tfrac{1}{6}[5-\sqrt{(12)}] = \pm 0.26.$$

Similarly it may be shown that for $t=\frac{1}{4}$, in which case $\varepsilon_3 = 0, +\frac{1}{4}, \pm\frac{1}{2}$ or $-\frac{1}{4}$, the possible errors being considered with respect to mag. only,

$$\varphi(\varepsilon) = \tfrac{1}{3}(24-16\varepsilon) \text{ for values of } \varepsilon \text{ between } 0 \text{ and } \tfrac{1}{4},$$
$$\varphi(\varepsilon) = \tfrac{1}{3}(28-32\varepsilon) \text{ `` `` `` } \varepsilon \text{ `` } \tfrac{1}{4} \text{ `` } \tfrac{3}{4},$$
$$\varphi(\varepsilon) = \tfrac{1}{3}(16-16\varepsilon) \text{ `` `` `` } \varepsilon \text{ `` } \tfrac{3}{4} \text{ `` } 1,$$

and
$$r = \tfrac{1}{8}[7-\sqrt{(23)}] = 0.28.$$

Likewise for $t = \frac{1}{5}$, in which case $\varepsilon_3 = 0, +\frac{1}{5}, +\frac{2}{5}, -\frac{2}{5}$ or $-\frac{1}{5}$, it may be shown that

$$\varphi(\varepsilon) = \tfrac{25}{4}(\tfrac{9}{10}+\varepsilon) \text{ for values of } \varepsilon \text{ between } -\tfrac{9}{10} \text{ and } -\tfrac{1}{10},$$
$$\varphi(\varepsilon) = 5 \qquad \text{ `` `` `` } \varepsilon \text{ `` } -\tfrac{1}{10} \text{ `` } +\tfrac{1}{10},$$
$$\varphi(\varepsilon) = \tfrac{25}{4}(\tfrac{9}{10}-\varepsilon) \text{ `` `` `` } \varepsilon \text{ `` } +\tfrac{1}{10} \text{ `` } +\tfrac{9}{10},$$

and
$$r = \pm\tfrac{1}{10}[9-\sqrt{(40)}] = \pm 0.27.$$

By extending the above process the probable errors in other and more complex cases might be determined. Without going into these details, however, it is evident that the probable error of an interpolated value of the second species will in general be greater than the probable error of a tabular value. For values of t such that $1 \div t$ is large relatively to the largest tabular difference in the table used, ε_3 may become insignificant in comparison with $(1-t)\varepsilon_1 + t\varepsilon_2$, and the probable error will be given in this case by formulas (13). At the limit, where $t = 0$, ε_3 vanishes, and $r = \pm 0.25$.

It is concluded, therefore, that the use of interpolated values of the sec'd species will not insure the highest precision possible with a given table. In precise computations, indeed, there can be no adequate reason for introducing the error ε_3 since the computer cannot avoid knowing its value.

3. By reference to equation (5) it appears that the probable error of an interpolated value of the third species is the same as that of a tablar value. Hence there must be on the whole a loss of precision in using interpolated values of this species.

4. For interpolated values of the fourth species the probable error will be given by equations (11) and (12) by making $a = \frac{1}{2}$ and assigning to b the limiting value of ε_4, which is subject to the condition mentioned in §3. Whatever this limiting value of ε_4 may be, the probable error of the interpolated value cannot be less than $\pm$ 0.25. When t is such that ε_4 may have any value between $+\frac{1}{2}$ and $-\frac{1}{2}$ the probable error will be

$$r = \pm(1-\tfrac{1}{2}\sqrt{2}) = \pm 0.29.$$

At the limit, where $t = 0$ and $\varepsilon_4 = 0$,

$$r = \pm 0.25.$$

Historical Note.

§ 5. Since at least three distinguished mathematicians have each considered one or more of the problems discussed in this paper, without having

reached conclusions altogether accordant, a brief reference to their investigations may not be uninteresting.

1. Bessel, 1838, in Nos. 358 and 359 of the Astronomische Nachrichten, in an article entitled *Untersuchungen ueber die Wahrscheinlichkeit der Beobachtungsfehler*, has investigated the laws of error discussed in § 4, 1. This article is an admirable one and merits more attention than it seems to have received by writers on the theory of errors. In No. 529 of the Nachrichten, in a review of Captain Shortrede's 7-place tables of logarithms, Bessel has applied the laws of error defined by equations (8), (9), (10), to determine the probable error of an interpolated value of the first species. He observes that the proportional part may be computed by means of the tabular difference or by means of the true difference, giving rise to what we have called the first and third species. He then says

> Man bemerkt leicht, dass die *erste* methode die vortheilhaftere ist, aber ich erinnere mich nicht, dass der *wahrscheinliche* Fehler ihres Resultats methodische untersucht worden wære.

Bessel does not raise the question of the independence of the errors of consecutive tabular values. His analysis leads to equations (13). He does not consider the probable errors of interpolated values of the second species.

2. Bremiker, 1852, in the introduction to the Latin edition of his 6-place tables of logarithms has an important chapter entitled *De Erroribus, Quibus Computationes Logarithmicae Afficiuntur*. This chapt. aims to give a more comprehensive view of this subject than any work with which the writer is acquainted, and its omission in subsequent editions of Bremiker's tables would seem to have relegated it to unmerited obscurity. Among other problems Bremiker considers that of the probable errors of interpolated values of the second species. (See his § 12.) His treatment of this problem is, however, quite erroneous owing to the fact that he has considered ε_3 in $\varepsilon'' = (1-t)\varepsilon_1 + t\varepsilon_2 + \varepsilon_3$ continuous between the limits $+\frac{1}{2}$ and $-\frac{1}{2}$ for any value of t. He says, p. 69, his f_1, f_2, f_3 and ε being the same as our ε_1, ε_2, ε_3 and t, respectively,

> Sumatur, f_1 et f_2 esse errores logarithmorum tabulæ L_1 et L_2, ita intelligendos, ut $L_1 + f_1$ et $L_2 + f_2$ sint exacti logarithmorum valores, error logarithmi $L_1 + \varepsilon\Delta$ erit
>
> $$f_1 + \varepsilon(f_2 - f_1) + f_3$$
>
> si f_3 designat errorem, qui e producto decurtato $\varepsilon\Delta$ proficiscitur, f vero putatur quilibet valorum æque probabilium, qui inter $-\gamma(=\frac{1}{2})$ et $+\gamma$ jacent. . . . . Itaque primum quaeramus, quos valores summa
>
> $$(1-\varepsilon)f_1 + \varepsilon f_2 + f_3$$
>
> accipere possit, si pro illis f omnes valores, que esse possunt, aeque probabiles et intra $-\frac{1}{2}$ et $+\frac{1}{2}$ jacentes ponantur.

He then states that the limits of this sum are -1 and $+1$, and proceeds to determine the probable errors of interpolated values corresponding to several values of $\varepsilon = t$. His results are very different from those reached in §4, 2, inasmuch as they increase gradually from a minimum $\pm$ 0.26 for $t = \frac{1}{2}$, to a maximum $\pm$ 0.29 for $t = 0$. The second of these probable errors is curiously absurd, since the inference from it is that an interpolated value infinitely near a tabular value is less accurate than a tabular value.* It may be remarked that the probable error ± 0.26 for $t = \frac{1}{2}$ does not correspond to the percentages of actual errors given in §4, 2.

3. F. G. W. Struve, 1860, in *Arc Du Me'ridian*, Tome I, p. 94, has considered the problem of the probable error of an interpolated value of the first species. He gives no analysis, but says briefly and somewhat obscurely

> L'erreur probable d'un logarithme donne' imme'diatement dans les tables est 0.25 de la septieme de'cimale; l'erreur probable d'un logarithme de'duit par l' interpolation entre deux chiffres tabulaires n' est que $\sqrt{\frac{2}{3}}$ de l'autre erreur.

This statement may be interpreted thus. If in equation (2), ε_1 and ε_2 be supposed to conform to the ordinary law of error viz., $\varphi(\varepsilon) = (h \div \sqrt{a}) e^{-h^2 \varepsilon^2}$ the probable error corresponding to ε' will be

$$\tfrac{1}{4}[(1-t)^2+t^2]^{\frac{1}{2}} = \tfrac{1}{4}[1-2t+2t^2]^{\frac{1}{2}}.$$

Now the square root of the average of the squares of probable errors given by this formula is

$$\tfrac{1}{4}\left[\int_0^1 (1-2t+2t^2)dt\right]^{\frac{1}{2}} = \tfrac{1}{4}\sqrt{\tfrac{2}{3}} = \pm 0.204.$$

Although ε_1 and ε_2 do not conform to the ordinary law of error, it is interesting to note that the above formula gives results which do not differ widely from the true results given by equations (13). Struve makes no reference to the writings of Bessel or Bremiker on this subject.

*[Because ε_3 must necessarily represent plus or minus 0, 1, 2, 3 or 4 units in the last place figure of the interpolated value

$$v' = (1-t)v_1 + tv_2,$$

and because any one of these digits is equally likely to occur, whatever the value of v' may be, the actual value of ε_3 is therefore independent of the *value* of v', depending only on the terminal digit in the numerical expression for v', and is therefore equally continuous with ε_1 and ε_2. But the probable error of the interpolated value will necessarily be increased by the addition of ε_3, and as its maximum occurs at the greatest distance of the interpolated value from mid-value, the absurdity above alluded to is not apparent.—Ed.]

NOTE ON PROF. NICHOLSON'S SINGULAR VALUE OF Π.

BY PROF. WILLIAM WOOLSEY JOHNSON.

If we regard equation (6), p. 150, as simply equivalent to $\infty.0 = 2 \div \pi$ it presents no difficulty; but if, on the other hand, the symbols $(1—1)^{\frac{1}{2}}$ and $(1—1)^{-\frac{1}{2}}$ be supposed to stand for the limits of $(1—x)^{\frac{1}{2}}$ and $(1—x)^{-\frac{1}{2}}$ when $x = 1$, the result appears paradoxical, since then the product of these quantities and therefore its limit is equal to unity.

The former is in fact correct for although equation (2) is the result of putting $x = 1$ in the expansion of $(1-x)^n$, (3) is not, since the process of transforming the infinite series into an infinite product is applicable only when $x = 1$. Thus equation (3) means nothing more than that the infinite product in the second member has zero for its limit; in like manner equation (4) means only that the infinite product in its second member has no limit.

Moreover, the product obtained by taking an infinite number of factors from each series may have any value we choose, for this value is a function of the ratio of the infinite numbers of the factors taken from the two series. If p factors be taken from (4) and q factors from (3) the value of the product, when p and q are both infinite but $p \div q = a$, is

$$\frac{a^n \sin n\pi}{n\pi};$$

putting $n = \frac{1}{2}$ and assuming $a = 1$, as implied in the manner of writing Wallis' Theorem, the result becomes $2 \div \pi$.

Note on Experimental Confirmation of Theoretical Deduction, by the Editor.—If a plane surface is ruled with parallel and equidistant lines and a slender rod, the length of which equals the perpendicular distance between two consecutive lines, is thrown at hazard upon the plane, the probability that it will fall across a line is $2 \div \pi$. (See *Mathematical Monthly*, Vol. II, p. 236.)

If we denote this probability by P, we shall have

$$P = \frac{2}{\pi} = \frac{2}{3.14159} = .6366.$$

Hence a rod thrown at hazard upon the plane 10,000 times should fall across a line 6366 times.

At the recent Montreal meeting of the American Association for the Advancement of Science, Prof. Mendenhall exhibited before Section A. of the Association the result of 30,000 experiments which he had performed by

casting a rod upon a ruled surface as above described. And as he claimed that he had avoided all conceivable sources of bias in pitching the rod, it was expected that the mean of so great a number of experiments would agree very nearly with the theoretical value of P; but, contrary to expectation, though the mean of the first 3000 experiments gave almost exactly the theoretical value of P, from that point onward to the end of the 30,000 trials the experimental result steadily diverged from the theoretical value of P.

In the discussion which followed the reading of this paper, various reasons were assigned for the discrepancy between the theoretcal and practical results, none of which seemed satisfactory to the Section. Among other possible sources of error in the result of the trials a personal eq'n was suggested by Prof. Mendenhall. The source of the error in this case, however, will, most likely, be found in that physiological characteristic of muscular action in accordance with which any act, frequently repeated, is continued unconciously in the same way. As this peculiarity cannot be eliminated by a personal equation, it follows that, in experiments of this kind, instead of approaching the theoretical value indefinitely by extending the experiments to a very great number, the liability of falling into special habits after a cert'n number of experiments have been performed not only renders further experimentation, by the same individual, useless, but even prejudicial.

INTEGRATION OF SOME GENERAL CLASSES OF TRIGONOMETRIC FUNCTIONS.

BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY, IOWA CITY.

In solving some of the more difficult problems that naturally arise in the higher branches of Applied Mathematics and in connection with a general course in engineering, one frequently has occasion to perform integrations, the formulas for which are not found in the books, at least not in those accessible to the operator.

Some recent experience of this nature led me to integrate some very general classes of Trigonometric functions to which those I incidentally encountered belong, and I offer the results and the methods of arriving at them for such consideration as they deserve. It is perhaps needless to remark that the subsidiary formulas required, are developed and used without special enquiry in regard to their existence elsewhere.

Integration of $\dfrac{dx}{(a+b\sin x)^n}$.

$$\int\frac{dx}{(a+b\sin x)^n}=\int\frac{a\,dx}{(a+b\sin x)^{n+1}}+\int\frac{b\sin x\,dx}{(a+b\sin x)^{n+1}}.$$

Let $u=1\div(a+b\sin x)^{n+1}$, $dv=b\sin x\,dx$; then

$$du=-b(n+1)\frac{\cos x}{(a+b\sin x)^{n+2}},\qquad v=-b\cos x.$$

$$\therefore\int\frac{dx}{(a+b\sin x)^n}=\int\frac{a\,dx}{(a+b\sin x)^{n+1}}-\frac{b\cos x}{(a+b\sin x)^{n+1}}$$
$$-b^2(n+1)\int\frac{\cos^2 x\,dx}{(a+b\sin x)^{n+2}}.$$

But $$\int\frac{b^2\cos^2 x\,dx}{(a+b\sin x)^{n+2}}=\int\frac{(b^2-b^2\sin^2 x)dx}{(a+b\sin x)^{n+2}}=(b^2-a^2)\int\frac{dx}{(a+b\sin x)^{n+2}}$$
$$+2a\int\frac{dx}{(a+b\sin x)^{n+1}}-\int\frac{dx}{(a+b\sin x)^n};$$

$$\therefore\int\frac{dx}{(a+b\sin x)^n}=\int\frac{a\,dx}{(a+b\sin x)^{n+1}}-\frac{b\cos x}{(a+b\sin x)^{n+1}}+(n+1)(a^2-b^2)$$
$$\times\int\frac{dx}{(a+b\sin x)^{n+2}}-2a(n+1)\int\frac{dx}{(a+b\sin x)^{n+1}}+(n+1)\int\frac{dx}{(a+b\sin x)^n}.$$

Transposing, dividing by $(n+1)(a^2-b^2)$ and writing $n-2$ for n, we have finally:

$$\int\frac{dx}{(a+b\sin x)^n}=\frac{b\cos x}{(n-1)(a^2-b^2)(a+b\sin x)^{n-1}}+\frac{(2n-3)a}{(n-1)(a^2-b^2)}$$
$$\times\frac{dx}{(a+b\sin x)^{n-1}}-\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sin x)^{n-2}}.\quad (1$$

In a similar manner we may obtain

$$\int\frac{dx}{(a+b\cos x)^n}=-\frac{b\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-1}}+\frac{(2n-3)a}{(n-1)(a^2-b^2)}$$
$$\times\frac{dx}{(a+b\cos x)^{n-1}}-\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\cos x)^{n-2}}.\quad (2$$

Integration of $\dfrac{dx}{(a+b\tan x)^n}$.

$$\int\frac{dx}{(a+b\tan x)^n}=\int(a\cos x+b\sin x)dx\frac{\sec x}{(a+b\tan x)^{n+1}}.$$

Let $(a\cos x+b\sin x)dx=dv$, $a\sin x-b\cos x=v$; $u=\dfrac{\sec x}{(a+b\tan x)^{n+1}}$,

$$\therefore du=\left[\frac{\sec x\tan x}{(a+b\tan x)^{n+1}}-b(n+1)\frac{\sec^3 x}{(a+b\tan x)^{n+2}}\right]dx;$$

and
$$vdu=\left[\frac{a\tan^2x}{(a+b\tan x)^{n+1}}-ab(n+1)\frac{\tan x(1+\tan^2x)}{(a+b\tan x)^{n+2}}-\frac{b\tan x}{(a+b\tan x)^{n+1}}\right.$$
$$\left.+b^2(n+1)\frac{(1+\tan^2x)}{(a+b\tan x)^{n+2}}\right]dx$$
$$=\left[\frac{b^2(n+1)}{(a+b\tan x)^{n+2}}-\frac{b\tan x}{(a+b\tan x)^{n+1}}-\frac{ab(n+1)\tan x}{(a+b\tan x)^{n+2}}\right.$$
$$\left.+\frac{a\tan^2x}{(a+b\tan x)^{n+1}}+\frac{b^2(n+1)\tan^2x}{(a+b\tan x)^{n+2}}-\frac{ab(n+1)\tan^3x}{(a+b\tan x)^{n+2}}\right]dx.$$

Now
$$\frac{\tan x}{(a+b\tan x)^{n+1}}=\frac{1}{b}\left[\frac{1}{(a+b\tan x)^n}-\frac{a}{(a+b\tan x)^{n+1}}\right],$$
$$\frac{\tan x}{(a+b\tan x)^{n+2}}=\frac{1}{b}\left[\frac{1}{(a+b\tan x)^{n+1}}-\frac{a}{(a+b\tan x)^{n+2}}\right],$$
$$\frac{\tan^2x}{(a+b\tan x)^{n+1}}=\frac{1}{b^2}\left[\frac{1}{(a+b\tan x)^{n-1}}-\frac{2a}{(a+b\tan x)^n}+\frac{a^2}{(a+b\tan x)^{n+1}}\right]$$
$$\frac{\tan^2x}{(a+b\tan x)^{n+2}}=\frac{1}{b^2}\left[\frac{1}{(a+b\tan x)^n}-\frac{2a}{(a+b\tan x)^{n+1}}+\frac{a^2}{(a+b\tan x)^{n+2}}\right]$$
$$\frac{\tan^3x}{(a+b\tan x)^{n+2}}=\frac{1}{b^3}\left[\frac{1}{(a+b\tan x)^{n-1}}-\frac{3a}{(a+b\tan x)^n}+\frac{3a^2}{(a+b\tan x)^{n+1}}\right.$$
$$\left.-\frac{a^3}{(a+b\tan x)^{n+2}}\right].$$

Hence
$$vdu=\frac{b^2(n+1)dx}{(a+b\tan x)^{n+2}}-\frac{dx}{(a+b\tan x)^n}+\frac{adx}{(a+b\tan x)^{n+1}}$$
$$-\frac{a(n+1)dx}{(a+b\tan x)^{n+1}}+\frac{a^2(n+1)dx}{(a+b\tan x)^{n+2}}+\frac{a}{b^2}\frac{dx}{(a+b\tan x)^{n-1}}$$
$$-\frac{2a^2}{b^2}\frac{dx}{(a+b\tan x)^n}+\frac{a^3}{b^2}\frac{dx}{(a+b\tan x)^{n+1}}+\frac{(n+1)dx}{(a+b\tan x)^n}$$
$$-\frac{2a(n+1)dx}{(a+b\tan x)^{n+1}}+\frac{a^2(n+1)dx}{(a+b\tan x)^{n+2}}-\frac{a(n+1)}{b^2}\frac{dx}{(a+b\tan x)^{n-1}}$$
$$+\frac{3a^2(n+1)}{b^2}\frac{dx}{(a+b\tan x)^n}-\frac{3a^3(n+1)}{b^2}\frac{dx}{(a+b\tan x)^{n+1}}$$
$$+\frac{a^4(n+1)}{b^2}\frac{dx}{(a+b\tan x)^{n+2}}.$$

Condensing we find:—
$$\int vdu=\frac{n+1}{b^2}(a^2+b^2)^2\int\frac{dx}{(a+b\tan x)^{n+2}}-\frac{a}{b^2}(3n+2)(a^2+b^2)$$
$$\times\int\frac{dx}{(a+b\tan x)^{n+1}}+\frac{a^2}{b^2}(3n+1)\int\frac{dx}{(a+b\tan x)^n}-\frac{an}{b^2}\int\frac{dx}{(a+b\tan x)^{n-1}}.$$

Hence $\int \frac{dx}{(a+b\tan x)^n} = \int u dv = uv - \int v du$

$$= \int \frac{a\tan x - b}{(a+b\tan x)^{n+1}} - \frac{n+1}{b^2}(a^2+b^2)^2 \int \frac{dx}{(a+b\tan x)^{n+2}} + \frac{a}{b^2}(3n+2)$$

$$\times (a^2+b^2) \int \frac{dx}{(a+b\tan x)^{n+1}} - \frac{a^2}{b^2}(3n+1) \int \frac{dx}{(a+b\tan x)^n} + \frac{an}{b^2} \int \frac{dx}{(a+b\tan x)^{n-1}}$$

Transposing, dividing and writing $n-2$ for n:—

$$\int \frac{dx}{(a+b\tan x)^n} = \frac{b^2}{(n-1)(a^2+b^2)^2} \frac{a\tan x - b}{(a+b\tan x)^{n-1}} + \frac{3n-4}{n-1} \frac{a}{a^2+b^2}$$

$$\times \int \frac{dx}{(a+b\tan x)^{n-1}} - \frac{3a^2(n-2)+(a^2+b^2)}{(n-1)(a^2+b^2)^2} \int \frac{dx}{(a+b\tan x)^{n-2}}$$

$$+ \frac{n-2}{n-1} \frac{a}{(a^2+b^2)^2} \int \frac{dx}{(a+b\tan x)^{n-3}}. \qquad (3)$$

In a similar manner we may deduce:—

$$\int \frac{dx}{(a+b\cot x)^n} = -\frac{b^2}{(n-1)(a^2+b^2)^2} \frac{a\cot x - b}{(a+b\cot x)^{n-1}} + \frac{3n-4}{n-1} \frac{a}{a^2+b^2}$$

$$\times \int \frac{dx}{(a+b\cot x)^{n-1}} - \frac{3a^2(n-2)+(a^2+b^2)}{(n-1)(a^2+b^2)^2} \int \frac{dx}{(a+b\cot x)^{n-2}}$$

$$+ \frac{n-2}{n-1} \frac{a}{(a^2+b^2)^2} \int \frac{dx}{(a+b\cot x)^{n-3}}. \qquad (4)$$

Integration of $\frac{dx}{(a+b\sec x)^n}$.

$$\int \frac{dx}{(a+b\sec x)^n} = \int \frac{adx}{(a+b\sec x)^{n+1}} + b \int \sec x dx \frac{1}{(a+b\sec x)^{n+1}}$$

$$= \int \frac{adx}{(a+b\sec x)^{n+1}} + b \int \operatorname{cosec} x \cot x \, dx \frac{\tan^2 x}{(a+b\sec x)^{n+1}}.$$

Let $\operatorname{cosec} x \cot x \, dx = dv$, $-\operatorname{cosec} x = v$;

$$\frac{\tan^2 x}{(a+b\sec x)^{n+1}} = u, \quad \frac{2\tan x \sec^2 x \, dx}{(a+b\sec x)^{n+1}} - \frac{\tan^2 x (n+1) b \sec x \tan x \, dx}{(a+b\sec x)^{n+2}} = du;$$

$$\therefore \; vdu = -\frac{2\sec^3 x \, dx}{(a+b\sec x)^{n+1}} + (n+1)b \frac{\sec^2 x(\sec^2 x - 1)dx}{(a+b\sec x)^{n+2}}.$$

[To be continued.]

VALUE OF e (NAPIERIAN BASE) TO 137 DECIMALS, COMPUTED BY J. M BOORMAN, ESP.—

$e =$ 2.718281 828459 045235 360287 471352 662497 757247 093699 959574 966967 627724 076630 353547 594571 382178 525166 427427 466391 932003 059921 817413 596629 04357.

ON A NEW CURVE FOR THE TRISECTION OF AN ANGLE.

BY DR. WILLIAM HILLHOUSE.

THE following method for the trisection of an angle from 0 to 360°, is believed to be new. The eq'n to the curve has not been found in any book relating to curves that has come to the notice of the writer, nor has its mechanical description ever been seen. It is now offered to those interested in such matters, as a new contribution to the subject, and if it should excite any interest in them, or in any way prove of value in geometry, the object of the writer will be attained.

ABCD, Fig. 1, represents a jointed parallelogram (*AC*, *FE* and *BD* are supposed to be infinite). Into *CD* is fixed firmly the piece *FE*, whose edge *FE* is at right angles to *CD*. This piece *FE* is about one-third the thickness of *CD* to admit of its passing freely under the bar *AC*, which is elevated on the under side for that purpose. The point *F* is also equidistant from the inner edges of *AC* and *BD*. If the bar *BD* be fixed on a plane, and the bar *AC* be depressed, as indicated by the dotted lines in the figure, *AC* will always remain parallel to *BD*, and the bar *FE* will incline to the left, as *fe*, and intersect the edge of *AC*. The p't of intersection of the inner edges of *AC* and *FE* will trace out (by applying a pencil at the point) a locus, represented in part by the line *bpc*, which will be of the fourth order, having four infinite branches, *AH*, *AG*, *BN*, *BK* (Fig. 2). The inner edge of *AC*, when it is parallel to *EF* will be an asymptote to the curve. The eq'n of the curve may be found as follows, by refer'g to Fig. 2.

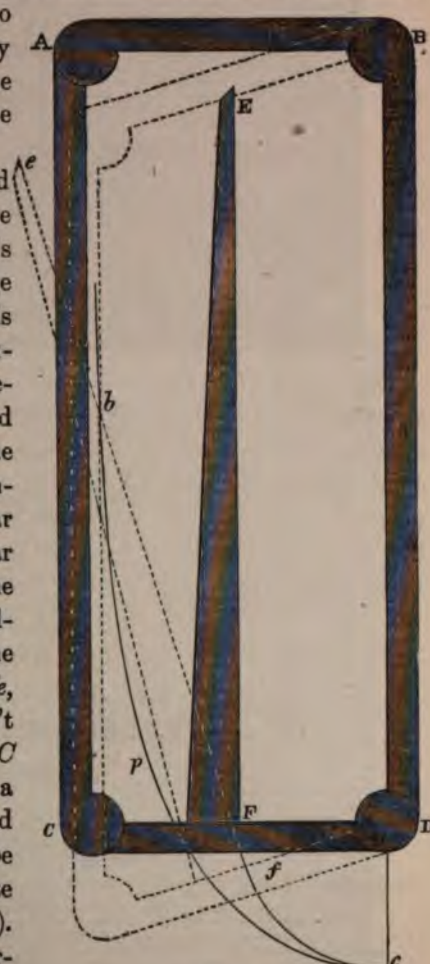


Fig 1.

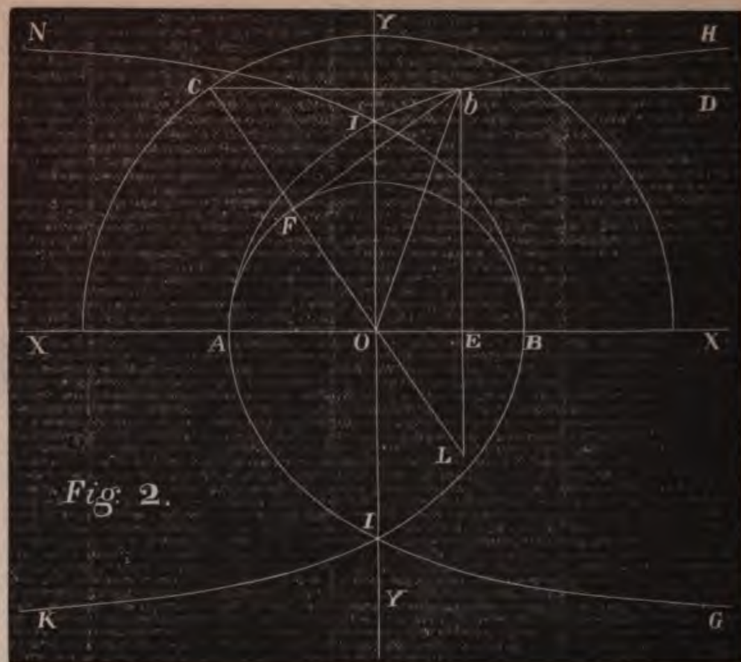


Fig. 2.

Let XX, YY be the rectangular axes; O the origin. With O as centre and radius $OA = a$, describe a semi-circle AFB, and with radius $OC = 2a$ describe a semi-circle XCX. Let b be a point on the curve; through b draw CD parallel to XX, cutting the outer circle in C; from b demit bE perpendicular to XX, and produce it towards L; join CO and produce the line to cut bE in L; now Cb and Ob are constantly equal, and CD is parallel to XX. Then, since the triangles LCb and LOE are similar,

$$Cb : CL :: OE : OL,$$
$$Cb : bL :: OE : EL;$$

or putting $OE = x$, $bE = y$, $OF = OA = a$, $OC = 2a$, $Cb = Ob = r$, $OL = z$, $EL = v$, $CL = 2a + z$, it becomes

$$r : 2a+z :: x : z,$$
$$r : y+v :: x : v;$$

$$\therefore rz = x(2a+z), \qquad z = \frac{2ax}{r-x},$$

$$rv = x(y+v), \qquad v = \frac{xy}{r-x}.$$

But we have $\qquad z^2 = x^2 + v^2, \qquad r^2 = x^2 + y^2;$

$$\therefore z^2 = \frac{4a^2x^2}{(r-x)^2} = x^2 + \frac{x^2y^2}{(r-x)^2},$$

$$\text{or } 4a^2 = (r-x)^2 + y^2. \qquad (1)$$

Substituting for r in (1), and reducing, we have finally

$$x^2 = \frac{(2a^2-y^2)^2}{4a^2-y^2}$$

r the equation of the locus.

To trisect any given angle, draw a straight line (Fig. 3) BOX and a raight line bBD at right angles to it. From B lay off $BO = a$; with O centre and distance BO, describe a circle fBG, and also from the same int a circle with radius $OC = 2a$. Let BOX be the initial line from hich the given angle is measured, passing around to the left; let HOX be e angle to be trisected. From O draw OKN at right angles to HO, and ace the parallelogram so that the inner edge of CD (Fig. 1) is on the line O, and the inner edge of BD (Fig. 1) on OKN; depress the side AC of e parallelogram so that the intersection of the lines AC and FE shall be the line DBb, say the point b; join bO, and from the point b draw bf ngent to the circle BGf; from O through f draw OfC, which will be per- ndicular to bf, and join bC. Now the angle $OKB = CbB = HOX$,

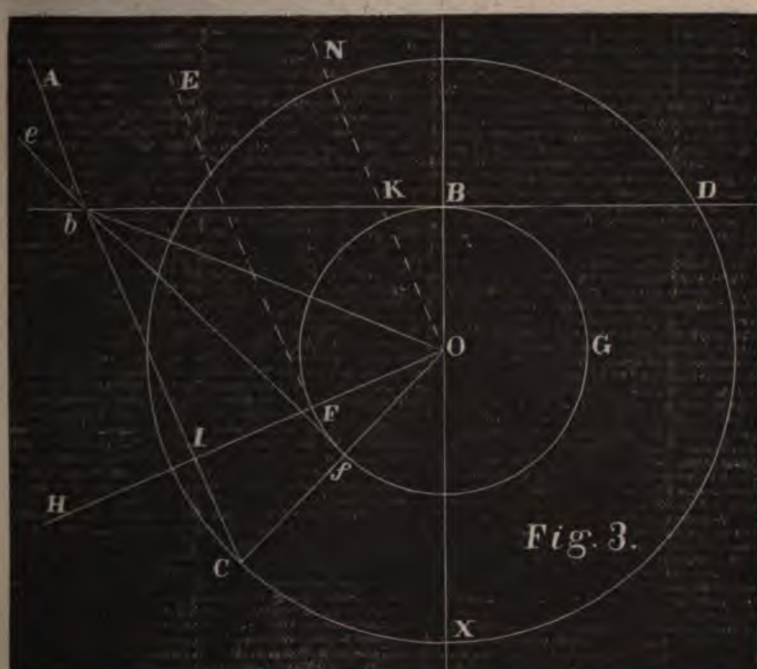


Fig. 3.

and in the right angled triangles bBO, bfO, the side BO = the side Of, since they are radii of the same circle, and the side bO common, therefore the other angles are equal, that is $ObB = Obf$; also in the right triangles bfO, bfC, the side Of = the side fC by construction and the side bf is common, therefore the angle Obf = the angle Cbf. But Obf was shown to be $= ObB$; therefore $Cbf = ObB$, that is, the three angles are equal to each other. But the three angles Cbf, fbO, ObB, make up the angle $CbB = HOX$ the given angle.

Now in the right angled triangles bfC and OIC the angle bCO is common to the two triangles, therefore the remaining angles are equal, that is, the angle COI = the angle Cbf; but Cbf has been shown to be equal to ObB, therefore $COI = ObB$; but ObB is one-third of CbB or HOX, therefore HOC = one-third HOX.

Wm. Hillhouse.

New Haven, Conn., Jan., 1878.

Solution of Prob. 405 by Prof. C. A. Van Velzer.—To fix the idea take the determinant of the fourth order

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

To the second column add l_1 times the first, m_1 times the third, and n_1 times the fourth, where l_1, m_1, n_1 are chosen to satisfy the equations

$$(l_1-1)a_1+m_1c_1+n_1d_1+b_1 = 0,$$
$$(l_1-1)a_2+m_1c_2+n_1d_2+b_2 = 0,$$
$$(l_1-1)a_3+m_1c_3+n_1d_3+b_3 = 0.$$

These three equations are sufficient to determine the values of (l_1-1), m_1, n_1, but if to these we add a fourth

$$(l_1-1)a_4+m_1c_4+n_1d_4+b_4 = 0$$

these *four* equations form a consistent set in (l_1-1), m_1, n_1, since the determinant of the coefficients (viz. the original determinant) vanishes.

We see that by this first transformation the determinant reduces to

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_2 & a_2 & c_2 & d_2 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

Now to the second row of this determinant add l_2 times the first, m_2 times the third and n_2 times the fourth, where l_2, m_2, n_2 satisfy the equ'ns

$$(l_2-1)a_1+m_2a_3+n_2a_4+a_2 = 0,$$
$$(l_2-1)c_1+m_2c_3+n_2c_4+c_2 = 0,$$
$$(l_2-1)d_1+m_2d_3+n_2d_4+d_2 = 0.$$

By this transformation the determinant is changed into

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_1 & a_1 & c_1 & d_1 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

a determinant in which the first two rows are identical and also the first two columns. The same process evidently applies to determinants of any order.

NEW NOTATION FOR ANHARMONIC RATIOS.

BY PROF. WILLIAM WOOLSEY JOHNSON.

1. THE notation here proposed has for its object to express in a symmetrical manner the relation between the six distinct values of the anharmonic ratio of four points and the four modes of writing each which constitute the 24 arrangements of the four letters concerned.

2. The anharmonic ratio of the section of AB by PQ is the ratio

$$\frac{AP}{BP} : \frac{AQ}{BQ};$$

regarding A and B as fixed points of reference, the constituent ratios $AP \div BP$ and $AQ \div BQ$ may be called the *position ratios* of P and Q. A distinction of sign being made between AP and PA, each value of the position ratio determines the position of a point, negative position ratios corresponding to points between A and B; while the position ratio of A is zero and that of B is infinity. The anharmonic ratio is the position ratio of P divided by that of Q, and regarding Q as a third fixed point of reference the value of the anharmonic ratio is a fixed multiple of the position ratio of P, and may be considered a coordinate determining the position of P, in such a manner that the coordinate of Q is unity.

3. Now let this anharmonic ratio be denoted by writing the four letters in a square form thus,

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \frac{AP}{BP} : \frac{AQ}{BQ} = \frac{AP.BQ}{BP.AQ} = x, \tag{1}$$

in which it is to be remembered that the letters occupy the position given to their coordinates in the form

$$\begin{matrix} x & 0 \\ \infty & 1 \end{matrix}.$$

Equation (1) shows that with this notation the anharmonic ratio is *the product of the distances expressed by the rows divided by the product expressed by the columns,* care being taken to read the letters *similarly* in the two cases; that is, if in oppsite directions in the two rows, then also in opposite directions in the two columns, as in equation (1), but if in the same direction in the two rows then in the same direction in the two columns; thus,

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \frac{AP.BQ}{BP.AQ} = \frac{PA.BQ}{PB.AQ}.$$

4. It follows at once, from the rule, that an interchange of A and B or an interchange of P and Q converts the anharmonic ratio x into its reciprocal since it changes rows into columns and columns into rows. That is *the interchange of the letters forming either diagonal produces the reciprocal.* Simultaneous interchanges in both diagonals therefore do not affect the value, and in general, *simultaneous interchanges in two pairs of letters* (that is, in both rows, or in both columns, as well as in both diagonals) *do not alter the value of the ratio,* since the rows thus remain rows and the columns remain columns. Thus the four modes of writieg the same anharmonic ratio are

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \begin{matrix} Q & B \\ A & P \end{matrix} = \begin{matrix} A & P \\ Q & B \end{matrix} = \begin{matrix} B & Q \\ P & A \end{matrix}. \tag{2}$$

5. The six distinct values of the anharmonic ratio of the four points may, as is well known, be obtained one from another by the process of alternately taking the reciprocal, and taking the *complement* (defining as complements quantities whose sum is unity). Thus, starting with x, the six values are

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ x & \frac{1}{x} & \frac{x-1}{x} & \frac{x}{x-1} & \frac{1}{1-x} & 1-x; \end{matrix}$$

the seventh term of the series being identical with the first. There are thus three pairs of reciprocals, or values having the symmetrical relation

$$xy = 1;$$

and also, otherwise grouped, there are three pairs of complements having the symmetrical relation

$$x + z = 1;$$

but furthermore, there are three pairs having the relation borne by the fourth term of the series to the first; viz.,

$$t = \frac{x}{x-1},$$

or

$$xt = x + t,$$

which is also a symmetrical relation. For want of a more distinctive term, I shall call two quantities bearing this relation *conjugates.*

6. If we denote the operation of taking the reciprocal by R, that of taking the complement by C, and that of taking the conjugate by J, we have

$$R.Rx = x \qquad C.Cx = x \qquad J.Jx = x,$$

or symbolically,

$$R^2 = 1 \qquad C^2 = 1 \qquad J^2 = 1. \tag{3}$$

Placing the six anharmonic ratios at the vertices of a hexagon as in the diagram, R is the operation of passing in either direction along one of the full-lined sides, C that of passing along one of the dotted sides, and J that of passing either way along one of the diagonals. It is obvious that we have

$$\left.\begin{aligned} CRC &= RCR = J, \\ JCJ &= CJC = R, \\ RJR &= JRJ = C. \end{aligned}\right\} \tag{4}$$

7. We shall now show that in the proposed notation each of the operations C and J, as well as the operation R, may be effected by a simple interchange. Consider first the effect of interchanging the letters in one of the columns of the square form x, and let

$$z = \begin{matrix} B & A \\ P & Q \end{matrix} = \frac{AB\,.\,PQ}{PB\,.\,AQ}$$

in which the denominator is the negative of that of x in equation (1).

We have then

$$x + z = \frac{AP\,.\,BQ - AB\,.\,PQ}{BP\,.\,AQ};$$

denoting for the moment AB by a, BP by b, and PQ by c,

$$AP = a + b, \qquad BQ = b + c,$$

and

$$AQ = a + b + c,$$

whence substituting we have

$$x + z = \frac{(a+b)(b+c) - ac}{b(a+b+c)} = 1$$

and z is the complement of x. An interchange in the other column has a like effect, since, as shown in 4, the two interchanges reproduce x. Thus *an interchange of the letters in either column produces the complement.*

8. If in the form $\qquad x = \begin{matrix} P & A \\ B & Q \end{matrix}$

we allow one letter to remain stationary, the other three letters form a triangle, and it is readily seen that an interchange along one side of the triangle followed by an interchange along a second side and another interchange

along the first side restores the letter which was at the common vertex of these sides and is therefore equivalent to a single interchange along the third side. Thus allowing Q to remain stationary in the value of x, and making the interchanges represented by R, C and R in this order, we have

$$Rx = \frac{P\ B}{A\ Q}$$

$$CRx = \frac{A\ B}{P\ Q}$$

$$RCRx = \frac{A\ P}{B\ Q}$$

which might have been obtained by interchanging the letters of the upper row. Now it is shown in 6 that $RCRx = Jx$; hence this interchange has the effect of taking the conjugate, and since interchanges in both rows reproduce x, we see that *an interchange of the letters in either row produces the conjugate.*

9. The operations R, C and J respectively convert x into three of the other five values of the anharmonic ratio; the other two values are the complement of the reciprocal

$$CRx = \frac{x-1}{x},$$

and the reciprocal of the complement

$$RCx = \frac{1}{1-x}.$$

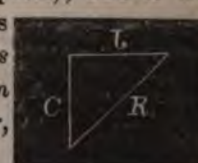
We readily derive from the hexagonal arrangement in 6, or from equations (3) and (4), the identities

$$CR = RJ = JC, \tag{5}$$

$$RC = CJ = JR. \tag{6}$$

Each of these operations is such that thrice repeated it reproduces x, and each is the inverse of the other; (5) is equivalent to passing in the direction of the hands of a watch along one of the sides of the triangle 2 4 6, or in the opposite direction along one of the sides of 1 3 5.

10. Recurring to the square form $x = \frac{P\ A}{B\ Q}$, it will be found that *the effect of two unlike interchanges* as in (5) or (6) *is to produce a cyclic displacement of three of the letters* (the fourth remaining in its place); and if we mark the sides of the triangle formed by these three letters R, C and J, *the direction of the displacement is the same as that in which the letters of the compound symbol occur on this triangle.* For example, if Q is the stationary letter, the triangle is as in the diagram;



the symbols in (5) occur on this triangle in the direction contrary to that of the hands of a watch, and the displacement of the letters takes place in this direction; thus

$$CRx = \begin{matrix} A & B \\ P & Q \end{matrix}.$$

On the other hand

$$RCx = \begin{matrix} B & P \\ A & Q \end{matrix}.$$

11. In comparing any two of the 24 square arrangements of the four letters, one of the letters may by the process indicated in equation (2) be brought into the same position in each form; the forms will then differ either in the position of two of the remaining letters indicating one of the relations denoted by R, C or J or else in the position of all three letters indicating one of the relations denoted by CR or RC, discussed above.

Solution of Prob. 407 by Prof. Edgar Frisby.—I notice that the solution of problem 407 [see p. 158] contains a remarkable mistake in omitting the exponent $\frac{1}{2}$. The expression is an elliptic integral; it can be reduced to a series thus:

$$\int_0^1 \frac{(1+x^4)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\,dx = \int_0^1 \frac{1+\frac{1}{2}x^4-\frac{1}{2}.\frac{1}{4}x^8+\frac{1}{2}.\frac{1}{4}\frac{3}{6}x^{12}-\frac{1}{2}.\frac{1}{4}.\frac{3}{6}\frac{5}{8}x^{16}+-}{\sqrt{(1-x^2)}}\,dx.$$

Integrating each term by parts and only retaining the last term of each series, as the other terms vanish at both limits, we have

$$\int_0^{\frac{\pi}{2}} (1+\cos^4\theta)^{\frac{1}{2}} d\theta = \frac{\pi}{2}\left\{ \frac{1}{2}\frac{1.3}{2.4} - \frac{1.1}{2.4}\frac{1.3.57}{2.4.68} + \frac{1.1.3}{2.4.6}\frac{1.3.5.7.9.11}{2.4.6.8.10.12} \cdots \right\}.$$

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in No. 5 have been received as follows:

From Prof. L. G. Barbour, 412; Alex. S. Christie, 411; Prof, W. P. Casey, 409, 412, 416; Geo. Eastwood, 413, 416, 417; C. E. Everett, 409, 410, 415. 416; Prof. Edgar Frisby, 409, 411, 415; Wm. Hoover, 409, 411, 413 416; L. S. Hulbert, 412; E. H. Moore, Jr., 416; Prof. P. H. Philbrick, 409, 414, 415, 416; A. L. Parman, 409; Prof. E. B. Seitz, 409; Prof. J. Scheffer, 409, 410, 411, 416; Prof. C. M. Woodward, 409 410, 413; R. S. Woodward, 413.

[Solutions of 401, 402, 403 and 406 by Chas. E. Everett and Prof. Casey, respectively, were received too late for acknowledgment in No. 5.]

409. *By David Trowbridge, A. M., Waterburgh, N. Y.*—"If in any triangle ABC, squares be described on the three sides, and the vertices of the squares be joined by the three straight lines a, b, c; show that

$$a^2 + b^2 + c^2 = 3(AB^2 + BC^2 + CA^2)."$$

SOLUTION BY CHARLES E. EVERETT, SPIRIT LAKE, IOWA.

The angle $A'AA''$ is the supplement of A, $B'BB''$ is the supplement of B and $C'CC''$ is the supplement of C; hence

$$a^2 = AB^2 + AC^2 + 2AB.AC\cos A,$$
$$b^2 = AB^2 + BC^2 + 2AB.BC\cos B$$
$$c^2 = BC^2 + AC^2 + 2AC.BC\cos C.$$

Substituting for $\cos A$, $\cos B$ and $\cos C$ their values in terms of the sides of the triang. and adding the eq's together,

$$a^2 + b^2 + c^2 = 3(AB^2 + AC^2 + BC^2).$$

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410. *By Prof. J. Scheffer.*—"A cone with circular base is cut by a parabolic plane which passes through the centre of the base; to find the position of the centre of gravity of both portions of the cone."

SOLUTION BY C. M. WOODWARD, PH. D., WASH. UNIV., ST. LOUIS, MO.

Let the plane of the figure be the meridian plane perpendicular to the given parabolic plane, and CS the trace of the given cutting plane. If $PV = \frac{3}{8}AV$, the straight line BP will pass through the center of gravity of the cone at O (giving $OV = \frac{3}{4}CV$) and through the centers required, since it passes through the centers of gravity of all the parllel parabolic laminæ.

Let MN be the trace of any one of these laminæ; its distance from B, measured on BP, be z; the height of the lamina x; and its limiting ordinate y.

From the figure $x = sz \div l$ (in which s is the slant height of the cone, and l is the length BP), and $y^2 = (zl - z^2)(4r^2 \div l^2)$. Hence the area of the lamina is

$$\tfrac{4}{3}xy = \frac{8}{3}\cdot\frac{rs}{l^2}\cdot\sqrt{(z^3l - z^4)}.$$

Finding the moment of the lamina about an axis at B,

substituting in formula for center of gravity, and cancelling constants, we have the general expression

$$z_0 = \frac{\int (zl - z^2)^{\frac{1}{2}} z^2 dz}{\int (zl - z^2)^{\frac{1}{2}} z \, dz},$$

in which the proper limits are to be inserted. The general value of the first integral is

$$-\tfrac{1}{4}z^{\frac{5}{2}}(l-z)^{\frac{3}{2}} - \tfrac{5}{24}lz^{\frac{3}{2}}(l-z)^{\frac{3}{2}} - \tfrac{5}{32}l^2z^{\frac{1}{2}}(l-z)^{\frac{3}{2}} + \tfrac{5}{64}l^3z^{\frac{1}{2}}(l-z)^{\frac{1}{2}}$$
$$+ \tfrac{5}{128}l^4 \text{ arc ver. sin } (2z \div l)$$

The general value of the second integral is

$$-\tfrac{1}{3}z^{\frac{3}{2}}(l-z)^{\frac{3}{2}} - \tfrac{1}{4}lz^{\frac{1}{2}}(l-z)^{\frac{3}{2}} + \tfrac{1}{8}l^2z^{\frac{1}{2}}(l-z)^{\frac{1}{2}} + \tfrac{1}{16}l^3 \text{ arc ver. sin } (2z \div l).$$

For the smaller of the two cone segments the limits are 0 and $\frac{1}{2}l$; for the other $\frac{1}{2}l$ and l. Putting in the limits we have

$$z_1 = l\frac{\frac{5}{256}\pi - \frac{5}{192} - \frac{1}{64}}{\frac{1}{32}\pi - \frac{1}{24}} = \tfrac{5}{8}l - \tfrac{3}{2}\cdot\frac{l}{3\pi - 4},$$

$$z_2 = l\frac{\frac{5}{128}\pi - \frac{5}{256}\pi + \frac{5}{192} + \frac{1}{16}}{\frac{1}{16}\pi - \frac{1}{32}\pi + \frac{1}{24}} = \tfrac{5}{8}l + \tfrac{3}{2}\cdot\frac{l}{3\pi + 4};$$

or in general

$$z_0 = \tfrac{5}{8}l \pm \tfrac{3}{2}.l \div (3\pi \pm 4),$$

in which the upper signs refer to the greater, and the lower signs to the smaller segment.

The first term in this result is the distance from B to O, the center of gravity of the entire cone, as is also evident if we take limits 0 and l.

The ratio of the distances of the two centers from the center of the cone is $3\pi - 4 : 3\pi + 4$, hence that is the inverse ratio of their volumes.

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411. *By Alex. S. Christie, U. S. Coast Survey.*—"Sum the series

$$1 - \frac{n}{1}\frac{1}{3} + \frac{n(n-1)}{2!}\frac{1}{5} - \frac{n(n-1)(n-2)}{3!}\frac{1}{7} + \text{\&c.},$$

for positive values of n."

SOLUTION BY THE PROPOSER.

Let $\Sigma = 1 - \frac{n}{1}\frac{1}{3} + \frac{n(n-1)}{2!}\frac{1}{5} - \frac{n(n-1)(n-2)}{3!}\frac{1}{7} + \text{\&c.}$, and

$$S = x - \frac{n}{1}\frac{x^3}{3} + \frac{n(n-1)}{2!}\frac{x^5}{5} - \frac{n(n-1)(n-2)}{3!}\frac{x^7}{7} + \text{\&c.}$$

Then $\frac{dS}{dx} = 1 - \frac{n}{1}x^2 + \frac{n(n-1)}{2!}x^4 - \frac{n(n-1)(n-2)}{3!}x^6 + \text{\&c.} = (1-x^2)^n$

and $\quad S = \int_0^x (1-x^2)^n dx = \frac{1}{2}\int_{-x}^{+x}(1-x^2)dx;$

$$\therefore\ \Sigma = \tfrac{1}{2}\int_{-1}^{+1}(1-x^2)^n dx = \tfrac{1}{2}\int_{+1}^{+1}(1+x)^n(1-x)^n dx.$$

Put $1+x = 2y$, $\therefore\ 1-x = 2(1-y)$, $dx = 2dy$.

Then $\quad \Sigma = 2^{2n}\int_0^1 y^n(1-y)^n dy = 2^{2n}B(n+1,\ n+1)$

$$= 2^{2n}.\frac{\Gamma(n+1)\,\Gamma(n+1)}{\Gamma(2n+2)}.$$

SOLUTION BY WM. HOOVER, A. M., DAYTON, OHIO.

The given series is the value of

$$\int_0^1(1-x^2)^n dx = \frac{2n\,.\,2(n-1)\;2(n-2)\,\ldots\,.\,2}{(2n+1)(2n-1)(2n-3)\ldots n},$$

which is therefore the required sum.

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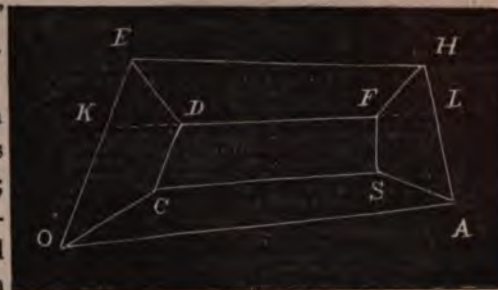
412. *By Prof. L. G. Barbour.*—"Show that in any hexaedron bounded by quadrilaterals, the three lines respectively connecting the mean points of opposite (non-contiguous) faces, mutually bisect each other."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $ABCDRPHG$ be the hexaedron, and Q, E, S, F, K, L, V the middle points of AG, AB, BH, HF, BC, CP, PH. Join QS and EF, SL and KV intersecting in o and n, which are the mean points of the faces AH, HC, and as $So = oQ$ and $Sn = nL$, $\therefore$ on is parallel to QL and equal to one-half of it. And if m, z be the mean points of the faces AR, RC, zm is for a like reason parallel to QL and equal to one-half of it, and so is mn equal and parallel to oz, xz to yn, xn to yz and ov to my (x, y being the mean p'ts of the faces AC, GP); $\therefore$ xy, om, nz are concurrent and bisect each other.

[Prof. Barbour and Mr. Hulbert, each, demonstrates this proposition by quaternions, and Prof. Barbour appends to his demonstration the following corollaries.]

Cor. 1. The face $BCDF$ may approach and finally coincide with the plane of $AOEH$, and the volume degenerate into a plane surface as in the Fig. Thus, the lines joining the mean points of $EDFH$ and $AOCB$



$OCDE$ and $ABFH$, $BCDF$ and $AOEH$ will mutually bisect each other.

Cor. 2. If E should app'ch until it reached K where EO is intersected by FD produced; and H app'ch A until it reaches L where HA is intersected by DF produced, the theorem would still hold good with this modification; viz., the mean point of $EDFH$ would become the middle point of KL, $EOCD$ would become $KOCD$, and $HFBA$, $LFBA$.

Cor. 3. Continuing the process, we should at length have left only $BCDF$. The lines joining the middle points of the opposite (non-contiguons) sides would bisect each other in the mean point, which is thus a special case of the general theorem.

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413. *By William Hoover, A. M.*—"A rod rests with one extremity in a smooth plane and the other against a smooth vertical wall at an inclination α to the horizon. If it then slips down, show that it will leave the wall when its inclination is $\sin^{-1}(\frac{2}{3}\sin\alpha)$."

SOLUTION BY R. S. WOODWARD, C. E., DETROIT, MICH.

Let the mass of the rod be m, its length $2l$, its inclination to the horizon θ and the coordinates of its centre of gravity x, y, the origin being such that for the time, t, considered, $x = l\cos\theta$ and $y = l\sin\theta$. Let the horizontal and vertical reactions at the ends of the rod be H and V respectively. Then the equations of motion are

$$m\frac{d^2y}{dt^2} = ml\frac{d^2(\sin\theta)}{dt^2} = -my + V, \tag{1}$$

$$m\frac{d^2x}{dt^2} = ml\frac{d^2(\cos\theta)}{dt^2} = +H, \tag{2}$$

$$\tfrac{1}{3}ml^2\frac{d^2\theta}{dt^2} = Hl\sin\theta - Vl\cos\theta. \tag{3}$$

Multiply (1) by $\cos\theta$, (2) by $-\sin\theta$ and (3) by $(1\div l)$ and add the products. There results

$$\tfrac{4}{3}ml\frac{d^2\theta}{dt^2} = -mg\cos\theta, \tag{4}$$

whence, since $d\theta\div dt = 0$ and $\theta = \alpha$ when $t = 0$,

$$\frac{d\theta^2}{dt^2} = \frac{3}{2}\frac{g}{l}(\sin\alpha - \sin\theta). \tag{5}$$

The rod will leave the vertical wall when

$$H = -mt\left(\sin\theta\frac{d^2\theta}{dt^2} + \cos\theta\frac{d\theta^2}{dt^2}\right) = 0.$$

Substituting in this the values of $d^2\theta \div dt^2$ and $d\theta^2 \div dt^2$ given by (4) and (5) there results

$$\theta = \sin^{-1}(\tfrac{2}{3}\sin a).$$

---

414. "Sum the series, $\sec\theta + \sec\frac{1}{2}\theta + \sec\frac{1}{4}\theta + \sec\frac{1}{8}\theta + \ldots + \sec\frac{1}{2^n}\theta$."

SOLUTION BY PROF. P. H. PHILBRICK.

$$\text{Sec}\,\theta = 1 + \tfrac{1}{2}\theta^2 + \tfrac{5}{24}\theta^4 + \ldots . = 1 + a\theta^2 + b\theta^4 + \ldots . \text{ say};$$

$$\therefore \sec\tfrac{1}{2}\theta = 1 + (\tfrac{1}{2})^2 a\theta^2 + (\tfrac{1}{2})^4 b\theta^4 + \ldots .$$

$$\sec\tfrac{1}{4}\theta = 1 + (\tfrac{1}{4})^2 a\theta^2 + (\tfrac{1}{4})^4 b\theta^4 + \ldots .$$

. . . .

$$\sec(\tfrac{1}{2})^n\theta = 1 + (\tfrac{1}{2})^{2n} a\theta^2 + (\tfrac{1}{2})^{4n} b\theta^4 + \ldots .$$

Now the corresponding terms of these partial series form geometrical series; hence,

$$\sec\theta + \sec\tfrac{1}{2}\theta + \sec\tfrac{1}{4}\theta + \ldots + \sec(\tfrac{1}{2})^n\theta \text{ (to } n+1 \text{ terms)}$$

$$= n+1+a\theta^2[1+\tfrac{1}{4}+\tfrac{1}{16}+\ldots . + (\tfrac{1}{4})^n]$$

$$+b\theta^4[1+\tfrac{1}{16}+\tfrac{1}{256}+\ldots . + (\tfrac{1}{16})^n]$$

$$= n+1+a\theta^2\left[\frac{1-(\frac{1}{4})^{n+1}}{1-\frac{1}{4}}\right] + b\theta^4\left[\frac{1+(\frac{1}{16})^{n+1}}{1-\frac{1}{16}}\right] + \ldots$$

---

415. "Evaluate $\displaystyle\int\frac{dx}{x-dx}$."

SOLUTION BY PROF. FRISBY.

$$\int_0^x\frac{x}{x-dx} = \int\frac{dx}{x}\left(1-\frac{dx}{x}\right)^{-1} = \int\frac{dx}{x}+\left(\frac{dx}{x}\right)^2+\left(\frac{dx}{x}\right)^3+ \text{\&c.},$$

$$= \log x;$$

all terms after the first being evanescent.

[Prof. Philbrick and Mr. Everett obtain the same result as above. The problem was inserted for the benefit of a correspondent who insists that Prof. Bartlett is in error for treating a similar expression (see *Acoustics and Optics*, p. 29, line 3; edition of 1853) in accordance with the above solution and whose criticism was declined because we believed it to be erroneous, but who, nevertheless, insists on our presenting the subject to the readers of the ANALYLT.—Ed.

416. *By Prof. W. P. Casey.*—"Given the base AB and the angle A of a triangle ABC; find the locus of the foot of the perpendicular CF drawn from C to the side of the inscribed square."

SOLUTION BY GEO. EASTWOOD.

Let $GHIJ$ denote the inscribed square, and let x, y be the coordinates of the point F, A being the origin. Put $AB = a$ and the tangent of the given angle $= t$; then is (K denoting the intersect. of CF produced with AB) $CK = tx$, $CF = tx - y$; and $\triangle ABC : \triangle GHC :: AB^2 : GH^2 (= FK^2)$, which gives $tx : tx - y :: a : y$, or $txy + ay - atx = 0$, which being of the form $Bxy + Dy + Ex = 0$, dentes a hyperbola whose asymptotes are represented by the equations $Bx + D = 0$, $By + E = 0$.

[For want of room, the solution of 417 is deferred to a future number.]

PROBLEMS.

418. *By Levi W. Meech, A. M., Norwich, Conn.*—Required to express Lagrange's Theorem in terms of Finite Differences, as far as practicable, instead of the usual differentials.

419. *By C. E. Everett, Spirit Lake, Iowa.*—Find the locus of a point starting from the centre of a given circle and moving so that the arc included between any two postions of the point shall equal the arc of the circle intercepted by the radii drawn through the same positions.

420. *By Prof. Asaph Hall.*—Transform the definite integral

$$\int_b^a \varphi(x).dx,$$

so that the limits of integration shall be m and n.

421. *By George Eastwood, Saxonville, Mass.*—In a Bicycle exercise on a level, circular course of given radius, what angle ought the plane of the machine to make with the vertical, so that the rider may move on the circumference of a perfect circle?

421. *By W. E. Heal, Marion, Indiana.*—Determine the most general form of two algebraic functions φ and θ such that

$$\varphi(x) + \varphi(y) = \varphi[\theta(x, y)],$$

or prove that there are no such functions.

Announcement of Vol. X.—As this No. completes the ninth annual volume of the Analyst, we again renew our promise to its readers to *continue* the publication, as long as our health will permit and the interest of our subscribers is manifested by appropriate contributions for its pages.

Contributors should not infer that their papers are not acceptable if they should not immediately appear in print, as by limiting each issue to a particular number of pages (32) it is not possible always to insert all the acceptable matter on hand when the No. is issued.

The present number has been enlarged to make room for the interesting paper by Dr. Hillhouse, which has been on hand for some time, but the Dr., besides furnishing the engraved blocks used, has generously paid all the additional expense for the enlargement.

No. 1 of Vol. X will appear about the first of Jan., 1883, and will contain the conclusion of Mr. De Forest's paper and of Prof. Philbrick's paper; also a paper by Mr. Stockwell, and other interesting matter.

J. E. Hendricks.

PUBLICATIONS RECEIVED.

Elements of Plane and Spherical Trigonometry, with Logarithmic and other Mathematical Tables. By Simon Newcomb. 384 pp. 8vo. New York: Henry Holt and Company. 1882.

Instructions for Observing the Transit of Venus, Dec. 6, 1882. 4to. 60 pp., with Maps. Washington: 1882.

Character of Six Hundred Tornadoes. 4to. 19 pp., with Maps. Washington: 1882.

Radiant Heat an Exception to the Second Law of Thermo-Dynamics. By H. T. Eddy, Ph. D. [From the Scientific Proceedings of the Ohio Mechanics' Institute, for July, 1882.]

Alhazen's Problem. Its Bibliography and an Extension of the Problem. By Marcus Baker. [Reprinted from the American Journal of Mathematics, Vol. IV, No. 3.]

ERRATA.

On page 131, line 8, eq. (2), for δs, read δs_1.
" " 132, " 13, from bottom, for 389, read 380.
" " 137, " 5 and 8, for 8, 19 and 12, read 14, 49 and 36.
" " 157, " 4, for $\cos DEA = \frac{1}{2}a^2$, read $\cos DEA = a \div [2(b^2+c^2-a^2]^{\frac{1}{2}}$.
" " " " 20, for $\sqrt{(1+\sec A+\sec B+\sec C)}$,
read $\sqrt{(1+\sec A \sec B \sec C)}$.
" " 175, " 18, for $\sqrt{a}$, read $\sqrt{\pi}$.
" " " " 10 and 11, from bot., for "0, 1, 2, 3 or 4 units",
read 0, 1, 2, 3, 4 or 5 tenths of a unit.
" " 180, " 4, from bottom, for Esp, read Esq.
" " 189, " 17, for $(1-x)$, in denominator, read $(1-x^2)$.
" " 193, for S in the diagram, read B.

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ALPHABETICAL INDEX.

THE ANALYST.

VOL. X. JANUARY, 1883. No. 1.

ON AN UNSYMMETRICAL PROBABILITY CURVE.

BY E. L. DE FOREST.

[Continued from page 168, Vol. IX.]

We will now illustrate the applicability of the gamma curve to represent series which are not expansions of any known polynomial, but are simply the results of repeated observation of some phenomenon or occurrence, in which there is a manifest inequality in the distribution of the errors or deviations on either side of the mean. Take for example the observations given by Quetelet in his *Letters* already cited, of the amplitude of diurnal variation of temperature (centigrade) at Brussels in the month of January, as observed for a period of 10 years, from 1833 to 1842. Column (1) of

TABLE II.

(1) Amp.	(2) Days	(3) g	(4)	(5)	(6) x	(7) gx^2	(8) gx^3	(9) y	(10) $g-y$
0° to 1°	0	.000	0.5	.000	—4.7	.000	— .000	.000	—.000
1 2	8	.026	1.5	.039	—3.7	.356	—1.317	.028	—.002
2 3	31	.100	2.5	.250	—2.7	.729	—1.968	.110	—.010
3 4	61	.197	3.5	.689	—1.7	.569	— .967	.183	+.014
4 5	68	.220	4.5	.990	—0.7	.108	— .075	.200	+.020
5 6	50	.162	5.5	.891	0.3	.015	.004	.171	—.009
6 7	32	.104	6.5	.676	1.3	.176	.229	.124	—.020
7 8	22	.071	7.5	.533	2.3	.376	.864	.081	—.010
8 9	20	.065	8.5	.552	3.3	.708	2.336	.048	+.017
9 10	8	.026	9.5	.247	4.3	.481	2.067	.027	—.001
10 11	4	.013	10.5	.137	5.3	.365	1.935	.014	—.001
11 12	3	.010	11.5	.115	6.3	.397	2.500	.007	+.003
12 13	1	.003	12.5	.037	7.3	.160	1.167	.004	—.001
13 14	1	.003	13.5	.041	8.3	.207	1.715	.002	+.001
14 15	0	.000	14.5	.000	9.3	.000	.000	.001	—.001
	309	1.000		5.197		4.647	8.490	1.000	

When $v>n$, and n is somewhat large, it makes no difference for our purposes whether n is an integer or not, because the series in the numerator will be so convergent that some of its last terms may be neglected, and if this is true for the two nearest integers above and below n, it is also true for n, even though it be fractional. The series does not always converge rapidly, but its terms are easily computed, each from the one that precedes it. To insure accuracy, this part of the work should be carried to two more places of decimals than are required in the sum of the series. To integrate between the limits x_1 and x_2, we take the difference of two integrals from x_1 to ∞ and from x_2 to ∞.

But when $v<n$, or when n is small, we can use by preference another formula, also obtained by integration by parts,

$$\int v^{n-1}\,e^{-v}\,dv = v^n e^{-v}\left\{\frac{1}{n}+\frac{v}{n(n+1)}+\frac{v^2}{n(n+1)\,(n+2)}+\&c.\right\}+C, \qquad (70)$$

where n need not be a whole number. Taking this integral between the limits 0 and v, C disappears, and we get by (66) as before

$$\left.\begin{aligned} n &= a^2b, \qquad v = ax, \\ \int_0^x Ydx &= \frac{v^n e^{-v}}{\Gamma(n+1)}\left\{1+\frac{v}{n+1}+\frac{v^2}{(n+1)\,(n+2)}+\&c.\right\}, \end{aligned}\right\} \qquad (71)$$

or, with the expression for $\Gamma(n+1) = n\Gamma(n)$ from (51),

$$\int_0^x Ydx = \left(\frac{v}{n}\right)^n e^{n-v}\left\{\frac{1+\dfrac{v}{n+1}+\dfrac{v^2}{(n+1)\,(n+2)}+\&c.}{\sqrt{(2\pi n)}\left(1+\dfrac{1}{12n}+\dfrac{1}{288n^2}-\&c.\right)}\right\}. \qquad (72)$$

When either one of the two integrals (69) and (72) is known, the other is known also, because

$$\int_0^x Ydx + \int_x^\infty Ydx = 1. \qquad (73)$$

Now in Table II. we have, by (64),

$$ab = 5.088, \qquad n = 5.572.$$

To find the probability that a single observed amplitude will fall below 3° for instance, the upper limit of integration is

$$x = 5.088-1.7-0.5 = 2.888, \qquad \therefore v = 3.162,$$

and with these values of n and v, (72) gives

$$\int_0^x Ydx = .1413.$$

For the probability that the amplitude will exceed 6°, the lower limit is

$$x = 5.088+0.3+0.5 = 5.888, \therefore v = 6.447,$$

and (69) gives

$$\int_x^\infty Ydx = .3111.$$

The series was carried only so far as the factors $n-1$, $n-2$, &c, were positive, and as none of the terms were small enough to be neglected, it might be doubted whether the result is correct. But when (72) is used, with the same value of v, we get

$$\int_0^x Ydx = .6890,$$

and $.3111+.6890 = 1$ nearly, as it should be, so that the sufficient accuracy of the other result is confirmed. Thus the probabilities that an amplitude will fall below 3°, or between 3° and 6°, or above 6° are as found by integration

.141, .548, .311,

and as found by addition of terms in column (9) of the table,

.138, .554, .308.

The differences existing are due to the fact that the terms y in the table are middle ordinates, while the integration gives areas. The area between two ordinates which are separated by a unit interval will be numerically a little greater or less than the ordinate at the middle of the interval, according as the curve there is convex or concave toward the X axis.

The representation of these observations by the computed gamma curve might have been made a little more accurate if the 309 observed amplitudes had been published and treated separately, instead of being grouped within intervals of 1° each. It is of course only an approximation to the truth when we take the middle of such an interval as the point whose position represents that of all the observations in the group, for the purpose of finding the centre of gravity of the whole series, and the deviations from it by which we estimate the q. m. error ε and the c. m. inequality ζ, and thence get the values of a and b. When the observations are separately given, ε^2 is found just as in constructing a common probability curve, and ζ^3 in like manner, only taking the cubes of the + and — errors instead of their squares. The unit of x may be chosen at pleasure.

We might have made small corrections in a and b on account of the fact that the errors x in our table are residuals and not true errors. The calculation, I think, would be as follows. Any particular true error is the algebraic sum of the residual error and the error of the mean from which the residuals are reckoned. The residual error and the error of the mean may be treated as approximately independent of each other. Denote by ε and ζ the q. m. error and c. m. inequality for a system of true errors. The (q.m. e.)^2 and (c. m. i.)^3 for the residuals are

$$\frac{[gx^2]}{[g]}, \quad \text{and} \quad \frac{[gx^3]}{[g]},$$

where [ ] signifies summation throughout the series. The q. m. error of the mean is nearly $\varepsilon \div \sqrt{m}$, where m denotes 309, the whole number of observations. We have then by (62)

$$\varepsilon^2 = \frac{[gx^2]}{[g]} + \frac{\varepsilon^2}{m}. \tag{74}$$

The approximate c. m. inequality for the mean is $\zeta \div m^{2/3}$ according to (60), and (63) gives

$$\zeta^3 = \frac{[gx^3]}{[g]} + \frac{\zeta^3}{m^2}. \tag{75}$$

From the above we get, since $[g] = 1$,

$$\varepsilon^2 = \left(\frac{m}{m-1}\right)[gx^2], \qquad \zeta^3 = \left(\frac{m^2}{m^2-1}\right)[gx^3]. \tag{76}$$

Now $[gx^2]$ and $[gx^3]$ are the two sums 4.647 and 8.490 of the numbers in columns (7) and (8) of our table, so that for the system of true errors we have

$$\varepsilon^2 = \frac{309}{308} \times 4.647 = 4.662, \qquad \zeta^3 = \frac{95481}{95480} \times 8.490 = 8.490,$$

and the corrected values of a and b are by (64)

$$a = 1.098, \qquad b = 4.662. \tag{77}$$

It will be noticed that while ε^2 is quite perceptibly larger for true errors than for residuals, ζ^3 is hardly increased at all. It seems reasonable that this should be so, for the residuals are reckoned from the place of the arithmetical mean as an origin, and the q. m. error is thereby made a minimum. Any change in the place of the origin must increase ε. But there is no such necessity in the case of ζ. A change of origin may increase its absolute value or may diminish it. According to our formula, the chances are that it will be very slightly increased.

If the observations were separately given, we should find, in like manner, first, that the square of the q. m. error is greater for true errors than for residuals, in the ratio of m to $m-1$; and secondly, that the absolute value of the cube of the c. m. inequality is greater also, in the ratio of m^2 to m^2-1. The first of these is a well known result.

In any given set of observations there will usually be some inequality on the + and — sides of the mean, even when the real law of error is symmetrical on both sides, so that the asymmetry is purely fortuitous. To decide whether ζ^3 as found from the residuals is fortuitous or not, we shall sometimes need to know what its probable value would be, on the assumption that the true errors are represented by x in the symmetrical curve

$$Y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}. \tag{78}$$

The whole number of possible errors, each taken a number of times proportional to the probability of its ocucrrence, is represented by

$$\int_{-\infty}^{\infty} Ydx = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}dx = 1.$$

Hence the mean of the squares, as well as the sum of the squares, of the cubes of all the possible errors is represented by

$$\int_{-\infty}^{\infty} x^6\, Ydx = \frac{1}{h^6\sqrt{\pi}}\int_{-\infty}^{\infty} (hx)^6 e^{-h^2x^2}d(hx),$$

or, putting $hx = t$ and $h^2 = 1 \div 2\varepsilon^2$,

$$\int_{-\infty}^{\infty} x^6\, Ydx = \frac{8\,\varepsilon^6}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^6 e^{-t^2}\, dt = 15\varepsilon^6, \qquad (79)$$

the known value of the last definite integral being $\frac{15}{8}\sqrt{\pi}$. (Sturm, *Cours d' Analyse*, II. p. 19.) The probable value of the cube of a single error is found approximately by taking the square root of the result in (79) and multiplying it by .6745, which gives

$$\pm .6745\varepsilon^3\sqrt{15}.$$

The probable value of the mean of the cubes of m errors is therefore

$$(\zeta^3) = \pm .6745\varepsilon^3\sqrt{(15 \div m)}. \qquad (80)$$

This is a standard which the actual value of ζ^3 ought not very much to exceed, if the law of error is to be considered symmetrical.

For example, in the set of observations at p. 495 of Vol. II. of Chauvenet's *Astronomy*, $m = 40$ and $\varepsilon = .202$, and (80) gives

$$(\zeta^3) = \pm .00340.$$

Actually, the algebraic sum of the cubes of the residuals is $-.1364$, so that

$$\zeta^3 = \frac{-.1364}{40} = -.00341.$$

Of course such a very close agreement between the actual and the prob'le value would not often occur, but in this and other cases, where ζ^3 does not much exceed (ζ^3), we may infer that no real c. m. inequality exists, and that the true law of error is probably symmetrical as in (78).

On the other hand, for the set of observations in our Table II. we have $m = 309$ and $\varepsilon^2 = 4.662$, and (80) gives the probable value

$$(\zeta^3) = \pm 1.496.$$

The actual value is $\zeta^3 = 8.490$, being almost 6 times as great. The chances are something like 10000 to 1 against the fortuitous occurrence of an error 6 times as great as the probable error. We must infer that, as indeed a simple inspection of the observations in this case indicates, the c. m. ineq. here is not only apparent, but real; so that an unsymmetrical curve alone can represent the true law of error with reasonable accuracy.

NOTE ON THE TRANSFORMATION OF A DETERMINANT INTO ANY OTHER EQUIVALENT DETERMINANT.

BY THOMAS MUIR, M. A., F. R. S. E.

PROFESSOR Van Velzer's interesting note on the above subject in ANALYST, Vol. IX, pp. 116–118, has recalled to my mind a theorem to which I was led in dealing with the "Transformations connecting General Determinants with Continuants". (Trans. Roy. Soc. Edinb., XXX, pp. 5–14.) Taking determinants of the fourth order, the theorem is as follows:—

The first three elements of the last column (say) of the determinant $|\, a_1\ b_2\ c_3\ d_4 \,|$ *may be replaced by any three magnitudes whatever,* α, β, γ, *provided the fourth element be changed into*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4-\alpha \\ b_1 & b_2 & b_3 & b_4-\beta \\ c_1 & c_2 & c_3 & c_4-\gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \div |\, a_1\ b_2\ c_3 \,|.$$

For, calling the said fourth element x we are to have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \alpha \\ b_1 & b_2 & b_3 & \beta \\ c_1 & c_2 & c_3 & \gamma \\ d_1 & d_2 & d_3 & x \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & \alpha \\ b_1 & b_2 & b_3 & \beta \\ c_1 & c_2 & c_3 & \gamma \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & x \end{vmatrix}$$

$$\therefore \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4-\alpha \\ b_1 & b_2 & b_3 & b_4-\beta \\ c_1 & c_2 & c_3 & c_4-\gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = x\,|\, a_1\ b_2\ c_3 \,|.$$

$$\text{and } \therefore x = \begin{vmatrix} a_1 & a_2 & a_3 & a_4-\alpha \\ b_1 & b_2 & b_3 & b_4-\beta \\ c_1 & c_2 & c_3 & c_4-\gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \div |\, a_1\ b_2\ c_3 \,|$$

as was to be shown.

The condition for the possibility of effecting the transformation is, as before, indicated by the occurrence of $|\, a_1\ b_2\ c_3 \,|$ as a *divisor* in the value of x.

Applying the theorem to the case of the transformation of

$$\begin{vmatrix} 0 & b & c \\ b & 1 & a \\ c & a & 1 \end{vmatrix} \quad \text{into} \quad \begin{vmatrix} 2abc & b & c \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix}$$

we first change the column $c,\ a,\ 1$ into

$$c,\quad 0,\quad \begin{vmatrix} 0 & b & 0 \\ b & 1 & a \\ c & a & 1 \end{vmatrix} \div(-b^2),$$

i. e., into

$$c,\quad 0,\quad 1-(ac\div b);$$

and so on, exactly as Professor Van Velzer does.

This example fortunately is easy, and the process as applied to it appears to the best advantage. It is desirable however to see the shady side as well, and for this purpose I give the curious identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5-b_2 & a_1-b_3 & a_2-b_4 & a_3-b_5 & a_4-b_1 \\ a_4-b_3 & a_5-b_4 & a_1-b_5 & a_2-b_1 & a_3-b_2 \\ a_3-b_4 & a_4-b_5 & a_5-b_1 & a_1-b_2 & a_2-b_3 \\ a_2-b_5 & a_3-b_1 & a_4-b_2 & a_5-b_3 & a_1-b_4 \end{vmatrix}$$

which possesses considerable interest in the theory of alternants.

Bishopton, Glasgow, Scotland, Oct. 1882.

INTEGRATION OF SOME GENERAL CLASSES OF TRIGONOMETRIC FUNCTIONS.

BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY, IOWA CITY.

[Continued from page 180, Vol. IX.]

$$\therefore \int\frac{dx}{(a+b\sec x)^n}=\int\frac{adx}{(a+b\sec x)^{n+1}}-\frac{\tan x\sec x}{(a+b\sec x)^{n+1}}+(n+1)b$$
$$\times\int\frac{\sec^2 x dx}{(a+b\sec x)^{n+2}}+2\int\frac{\sec^3 x\,dx}{(a+b\sec x)^{n+1}}-(n+1)b\int\frac{\sec^4 x dx}{(a+b\sec x)^{n+2}}.$$

Now

$$\frac{\sec^2 x}{(a+b\sec x)^{n+2}}=\frac{1}{b^2}\left[\frac{1}{(a+b\sec x^n}-\frac{2a}{(a+b\sec x)^{n+1}}+\frac{a^2}{(a+b\sec x)^{n+2}}\right]$$

$$\frac{\sec^3 x}{(a+b\sec x)^{n+1}}=\frac{1}{b^3}\left[\frac{1}{(a+b\sec x)^{n-2}}-\frac{3a}{(a+b\sec x)^{n-1}}+\frac{3a^2}{(a+b\sec x)^{n}}\right.$$
$$\left.-\frac{a^3}{(a+b\sec x)^{n+1}}\right]$$

$$\frac{\sec^4 x}{(a+b\sec x)^{n+2}}=\frac{1}{b^4}\left[\frac{1}{(a+b\sec x)^{n-2}}-\frac{4a}{(a+b\sec x)^{n-1}}+\frac{6a^2}{(a+b\sec x)^{n}}\right.$$
$$\left.-\frac{4a^3}{(a+b\sec x)^{n+1}}+\frac{a4}{(a+b\sec x)^{n+2}}\right].$$

$$\therefore \int\frac{dx}{(a+b\sec x)^n}=\int\frac{adx}{(a+b\sec x)^{n+1}}-\frac{\tan x\sec x}{(a+b\sec x)^{n+1}}+\frac{n+1}{b}\int\frac{dx}{(a+b\sec x)^n}$$
$$-\frac{2a(n+1)}{b}\int\frac{dx}{(a+b\sec x)^{n+1}}+\frac{a^2(n+1)}{b}\int\frac{dx}{(a+b\sec x)^{n+2}}+\frac{2}{b^3}\int\frac{dx}{(a+b\sec x)^{n-2}}$$
$$-\frac{6a}{b^3}\int\frac{dx}{(a+b\sec x)^{n-1}}+\frac{6a^2}{b^3}\int\frac{dx}{(a+b\sec x)^n}-\frac{2a^3}{b^3}\int\frac{dx}{(a+b\sec x)^{n+1}}$$
$$-\frac{n+1}{b^3}\int\frac{dx}{(a+b\sec x)^{n-2}}+\frac{4a(n+1)}{b^3}\int\frac{dx}{(a+b\sec x)^{n-1}}-\frac{6a^2(n+1)}{b^3}\int\frac{dx}{(a+b\sec x)^n}$$
$$+\frac{4a^3(n+1)}{b^3}\int\frac{dx}{(a+b\sec x)^{n+1}}-\frac{a^4(n+1)}{b^3}\int\frac{dx}{(a+b\sec x)^{n+2}}, \text{ or}$$

$$\int\frac{dx}{(a+b\sec x)^n}=-\frac{n-1}{b^3}\int\frac{dx}{(a+b\sec x)^{n-2}}+\frac{2a(2n-1)}{b^3}\int\frac{dx}{(a+b\sec x)^{n-1}}$$
$$+\frac{(n+1)(b^2-6a^2)+6a}{b^3}\int\frac{dx}{(a+b\sec x)^n}+\frac{ab^2(b-2n-2)+2a^3(2n+1)}{b^3}$$
$$\times\int\frac{dx}{(a+b\sec x)^{n+1}}-\frac{a^2(n+1)(a^2-b^2)}{b^3}\int\frac{dx}{(a+b\sec x)^{n+2}}-\frac{\tan x\sec x}{(a+b\sec x)^{n+1}}.$$

Transposing, dividing and writing $n-2$ for n we have :—

$$\int\frac{dx}{(a+b\sec x)^n}=-\frac{n-3}{(n-1)a^2(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-4}}+\frac{2(2n-5)}{a(n-1)(a^2-b^2)}$$
$$\times\int\frac{dx}{(a+b\sec x)^{n-3}}+\frac{(n-1)(b^2-6a^2)+6a-b^3}{a^2(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-2}}$$
$$+\frac{b^2(b-2n+2)+2a^2(2n-3)}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-1}}-\frac{b^3}{a^2(n-1)(a^2-b^2)}\frac{\tan x\sec x}{(a+b\sec x)^{n-1}} \tag{5}$$

In a similar manner we may find :—

$$\int\frac{dx}{(a+b\,\text{co'c}\,x)^n}=-\frac{n-3}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\,\text{co'c}\,x)^{n-4}}+\frac{2(2n-5x)}{a(n-1)(a^2-b^2)}$$
$$\times\int\frac{dx}{(a+b\,\text{cosec}\,x)^{n-3}}+\frac{(n-1)(b^2-6a^2)+6a-b^3}{a^2(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\,\text{cosec}\,x)^{n-2}}$$
$$+\frac{b^2(b-2n+2)+2a^2(2n-3)}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\,\text{co'c}\,x)^{n-1}}+\frac{b^3}{a^2(n-1)(a^2-b^2)}\frac{\text{cotan}\,x\,\text{cosec}\,x}{(a+b\,\text{co'c}\,x)^{n-1}} \tag{6}$$

Now

$$\int\frac{\sin^m x dx}{(a+b\sin x)^n}=\frac{1}{b^m}\left[\int\frac{dx}{(a+b\tan x)^{n-m}}-ma\int\frac{dx}{(a+b\tan x)^{n-m+1}}\right.$$
$$\left.+\frac{m(m-1)}{1.2}a^2\int\frac{dx}{(a+b\tan x)^{n-m+2}}+\dots+a^m\int\frac{dx}{(a+b\tan x)^m}\right], \tag{m}$$

the coefficients of the partial fractions within the brackets *being the successive terms* of $(1-a)^m$. Similarly for other Trigonometric functions. Hence

the preceding equations enable us to integrate the general form

$$\frac{\sin^m x dx}{(a+b\sin x)^n}$$

involving any of the trigonometric functions.

It is important to notice that m may have any value, positive or negative as I will presently show, and hence the equations apply also to

$$\int\frac{dx}{\sin^m x(a+b\sin x)^n}, \text{ etc., constituting a general class.}$$

If in (m), $m>n$ some of the terms are of the form $\int\frac{dx}{(a+b\sin x)^{-p}} = \int(a+b\sin x)^p dx$, and they must be expanded and integrated, the others being integrated by the formulas above.

Let $y = 90° - x$, then $\cos y = \sin x$, $dy = -dx$.

$$\therefore \int\frac{dy}{(a+b\cos y)^n} = -\int\frac{dx}{(a+b\sin x)^n} = -\frac{b\cos x}{(n-1)(a^2-b^2)(a+b\sin x)^{n-1}}$$

$$-\frac{2n-3}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sin x)^{n-1}}+\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sin x)^{n-2}}$$

$$= -\frac{b\sin y}{(n-1)(a^2-b^2)(a+b\cos y)^{n-1}}+\frac{2n-3}{(n-1)(a^2-b^2)}\int\frac{dy}{(a+b\cos y)^{n-1}}$$

$$-\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dy}{(a+b\cos y)^{n-2}}, \qquad (7)$$

which is the same as eq. (2).

Hence the formula for $\int F(\cos x)dx$ may be obt'd from that for $\int F(\sin x)dx$ by replacing $\sin x$ with $\cos x$, etc., and changing the signs of the terms without the sign of integration. The formulas for $\int F(\cot x)dx$ and $\int F(\operatorname{cosec} x)dx$ may be writ'n in a similar manner from those of $\int F(\tan x)dx$ and $\int F(\sec x)dx$.

Again, $\operatorname{cosec} x = (1\div\sin x)$, $\sec x = (1\div\cos x)$ and $\cot x = (1\div\tan)$.

$$\therefore \frac{\operatorname{cosec}^m x\, dx}{(a+b\operatorname{cosec} x)^n} = \frac{\sin^{n-m} x\, dx}{(b+a\sin x)^n}. \qquad (10)$$

$$\frac{\sec^m x\, dx}{(a+b\sec x)^n} = \frac{\cos^{n-m} x\, dx}{(b+a\cos x)^n}. \qquad (11)$$

$$\frac{\cot^m x\, dx}{(a+b\cot x)^n} = \frac{\tan^{n-m} x\, dx}{(b+a\tan x)^n}. \qquad (12)$$

These eq's enable us to integrate the three former forms in terms of the three latter, or *vice versa*, whether m is *positive* or *negative*. See eq. (m) and the remarks following.

Example (1). $\int\frac{dx}{\sec x(a+b\sec x)} = \int\frac{\cos^2 x\, dx}{(b+a\cos x)}$

$$= \frac{1}{a^2}\left[\int(b+a\cos x)dx - 2b\int dx + b^2\int\frac{dx}{(b+a\cos x)}\right] \qquad (13)$$

$$= \frac{1}{a^2}\left[\int(b + a\cos x)dx - 2b\int dx + b^2\int\frac{\sec x\,dx}{(a+b\sec x)}\right] \qquad (14)$$

Eqn. (13) may be used, giving results in terms of the cosine throughout, or (14) may be used. Using (13) we have:

$$\int\frac{dx}{\sec x(a+b\sin x)} = \frac{\sin x}{a} - \frac{bx}{a^2} + \frac{2b^2}{(b^2-a^2)^{½}}\tan^{-1}\left[\left(\frac{b-a}{b+a}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right].$$

Example (2). $\displaystyle\int\frac{dx}{\tan x(a+b\tan x)} = \int\frac{\cot^2 x\,dx}{(b+a\cot x)}$

$$= \frac{1}{a^2}\left[\int(b+a\cot x)dx - 2b\int dx + b^2\int\frac{dx}{(b+a\cot x)}\right] \qquad (16)$$

or
$$= \frac{1}{a^2}\left[\int(b+a\cot x)dx - 2b\int dx + b^2\int\frac{\tan x\,dx}{a+b\tan x)}\right] \qquad (17)$$

which are in known terms and may be treated similarly to the above.

Example (3). $\displaystyle\int\frac{\cos x\,dx}{(a+b\cos x)^3} = \frac{1}{b}\left[\int\frac{dx}{(a+b\cos x)^2} + a\int\frac{dx}{a+b\cos x)^3}\right].$

Now $\displaystyle a\int\frac{dx}{(a+b\cos x)^3} = A - B\int\frac{dx}{(a+b\cos x)^2} + C\int\frac{dx}{(a+b\cos x)};$

$$= A - B\left(D - E\int\frac{dx}{(a+b\cos x)}\right) + C\int\frac{dx}{(a+b\cos x)}$$

$$= A - BD + (BE+C)\int\frac{dx}{(a+b\cos x)}.$$

$$\therefore \int\frac{dx}{(a+b\cos x)^2} - a\int\frac{dx}{(a+b\cos x)^3} = D - E\int\frac{dx}{(a+b\cos x)} - A + BD$$

$$-(BE+C)\int\frac{dx}{(a+b\cos x)} = D + BD - A - (C+E+BE)\int\frac{dx}{(a+b\cos x)}.$$

In the above $A = \dfrac{ab\sin x}{2(b^2-a^2)(a+b\cos x)^2}$, $B = \dfrac{3a^2}{2(b^2-a^2)}$, $C = \dfrac{a}{2(b^2-a^2)}$,

$D = \dfrac{b\sin x}{(b^2-a^2)(a+b\cos x)}$, $E = \dfrac{a}{b^2-a^2}$; $\therefore BD = \dfrac{3a^2 b\sin x}{2(b^2-a^2)^2(a+b\cos x)}$

and $BE = \dfrac{3a^3}{2(b^2-a^2)^2}$. Therefore

$$\int\frac{\cos x\,dx}{(a+b\cos x)^3} = \frac{a\sin x(2a^2+b^2) + b\sin x\cos x(a^2+2b^2)}{2(b^2-a^2)(a+b\cos x)^2}$$

$$- \frac{3ab}{2(b^2-a^2)}\int\frac{dx}{(a+b\cos x)}$$

$$= \frac{a\sin x(2a^2+b^2) + b\sin x\cos x(a^2+2b^2)}{2(b^2-a^2)(a+b\cos x)^2} + \frac{3ab}{(a^2-b^2)^{½}}\tan^{-1}\left[\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right]$$

where $a > b$.

$$\int \frac{\sin^m x \cos^r x\, dx}{(a+b\cos x)^n} = \int \frac{\sin^m x \cos^{r-n} x\, dx}{(b+a\sec x)^n} = \int \frac{(1-\cos^2 x)^{\frac{1}{2}m} \cos^r x\, dx}{(a+b\cos x)^n} \qquad (a)$$

$$= -\int \frac{(1-\cos^2 x)^{\frac{1}{2}(m-1)} \cos^r x\, d\cos x}{(a+b\cos x)^n}. \qquad (b)$$

If m is even the right-hand member of (a) may be developed into a finite number of terms and integrated by the preceding formulas. If m is odd eq. (b) may be developed and will take the form of a rational fraction, with or without a series of mononomial terms, and may be integrated accordingly; or since each term would be of the form: $z^m(a+bz)^{-n}dz$, the exponent within the parentheses being *unity*, the successive terms may be integrated by substitution.

$$\frac{\sin^m x \cos^r x\, dx}{(a+b\sin x)^n} = \frac{\sin^{m-n} x \cos^r x\, dx}{(b+a\operatorname{cosec} x)^n} = \frac{(1-\sin^2 x)^{\frac{1}{2}r} \sin^m x\, dx}{(a+b\sin x)^n}. \qquad (c)$$

$$= \frac{(1-\sin^2 x)^{\frac{1}{2}(r-1)} \sin^m x\, d\sin x}{(a+b\sin x)^n}. \qquad (d)$$

These may be treated the same as (a) and (b). If $n = 0$ in (a) and (b) we have the more special form:

$$\sin^m x \cos^r x\, dx = (1-\cos^2 x)^{\frac{1}{2}m} \cos^r x\, dx, \text{ if } m \text{ is even, or}$$
$$= (1-\cos^2 x)^{\frac{1}{2}(m-1)} \cos^r x\, d\cos x, \text{ if } m \text{ is odd.}$$

This may be developed and at once integrated which process is preferable to that involving the usual formu's for this case, formulas which provide for special sub-cases and involve successive integrals of different orders.

The above includes also the more special case in which $m = 0$ or $r = 0$.

$$\int \frac{\cos^m x\, dx}{(a+b\tan x)^n} = \int \frac{\cos^{m-1} x \cos x\, dx}{(a+b\tan x)^n}. \quad \text{Let } \cos x\, dx = dv,\ \sin x = v,$$

$$\frac{\cos^{m-1} x}{(a+b\tan x)^n} = u, \ -(m-1)\frac{\cos^{m-2} x \sin x\, dx}{(a+b\tan x)^n} - nb\frac{\cos^{m-1} x \sec^2 x\, dx}{(a+b\tan x)^{n+1}} = du.$$

$$\therefore vdu = -(m-1)\frac{\cos^{m-2} x \sin x\, dx}{(a+b\tan x)^n} - nb\frac{\cos^m x \tan x(1+\tan^2 x) dx}{(a+b\tan x)^{n+1}}$$

$$= -nb\frac{\cos^m x \tan x}{(a+b\tan x)^{n+1}} - a(m-1)\frac{\cos^m x \tan^2 x}{(a+b\tan x)^{n+1}}$$

$$-b(m+n-1)\frac{\cos^m x \tan^3 x}{(a+b\tan x)^{n+1}}$$

$$= -\frac{n\cos^m x}{(a+b\tan x)^n} + \frac{an\cos^m x}{(a+b\tan x)^{n+1}} - \frac{a(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$

$$+\frac{2a^2(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^n} - \frac{a^3(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}}$$

$$-\frac{m+n-1}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-2}}+\frac{3a(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$

$$-\frac{3a^2(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n}}+\frac{a^3(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}};$$

$$=-\frac{m+n-1}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-2}}+\frac{a(2m+3n-2)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$

$$-\frac{a^2(m+3n-1)+b^2n}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n}}+\frac{an(a^2+b^2)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}}.$$

$$\therefore\int\frac{\cos^m x dx}{(a+b\tan x)^n}=\frac{\cos^{m-1}x\sin x}{(a+b\tan x)^n}+\frac{m+n-1}{b^2}\int\frac{\cos^m x dx}{(a+b\tan x)^{n+2}}$$

$$-\frac{a(2m+3n-2)}{b^2}\int\frac{\cos^m x dx}{(a+b\tan x)^{n-1}}+\frac{a^2(m+3n-1)+b^2n}{b^2}\int\frac{\cos^m x dx}{(a+b\tan x)^{n}}$$

$$-\frac{an(a^2+b^2)}{b^2}\int\frac{\cos^m x dx}{(a+b\tan x)^{n+1}}.$$

Transposing, dividing by $\frac{an(a^2+b^2)}{b^2}$, and writing $n-1$ for n we finally get:—

$$\int\frac{\cos^m x dx}{(a+b\tan x)^n}=\frac{b^2}{a(n-1)(a^2-b^2)}\frac{\cos^{m-1}x\sin x}{(a+b\tan x)^{n-1}}$$

$$+\frac{a^2(m+3n-4)+b^2(n-2)}{a(n-1)(a^2+b^2)}\int\frac{\cos^m x dx}{(a+b\tan x)^{n-1}}$$

$$-\frac{(2m+3n-5)}{(n-1)(a^2+b^2)}\int\frac{\cos^m x dx}{(a+b\tan x)^{n-2}}+\frac{m+n-2}{a(n-1)(a^2+b^2)}\int\frac{\cos^m x dx}{(a+b\tan x)^{n-3}}. \quad (e)$$

Now

$$\int\frac{\sin^m x dx}{(a+b\tan x)^n}=\int\frac{\cos^m x\tan^m x dx}{(a+b\tan x)^n}.$$

$\frac{\tan^m x}{(a+b\tan x)^n}$ may be separated into a series of partial fractions, each may be multiplied by $\cos^m x dx$ and integrated by eq. (e).

The above includes

$$\frac{\sin^m x\cos^r x}{(a+b\tan x)^n}=\frac{\cos^{m+r}x\tan^m x}{(a+b\tan x)^n}.$$

Again

$$\int\frac{\sin^m x\cos^r x dx}{(a+b\cot x)^n}=\int\frac{\cos^{m+r-n}x\tan^{m+n}x dx}{(b+a\tan x)^n},$$

which may be integrated by the above, and which includes special cases in which $r=0$ or $m=0$.

ON THE LUNAR AND PLANETARY THEORIES.

BY JOHN N. STOCKWELL.

In my Theory of the Moon's Motion, which was published about a year ago, I have called the attention of astronomers to a class of inequalities which appear to have been incorrectly calculated by all my predecessors in this interesting field of inquiry. The inequalities to which I refer, are of great importance in the lunar theory, not only on account of their own magnitude, but also on account of their modifying influence on some of the other inequalities of the moon's motion. They are wholly independent of the coordinates of the disturbing body; and the coefficients of the time in the arguments of the equations differ from unity but by quantities of the order of the disturbing forces. They arise from the integration of equations of the form,

$$\frac{d^2\delta r}{dt^2} + N^2\delta r + m^2\cos(it-\varepsilon) = 0; \qquad (1)$$

in which N and i differ from unity but by quantities of the order m^2, which here represents the disturbing function.

The general integral of equation (1) is

$$\delta r = \frac{m^2}{i^2-N^2}\cos(it-\varepsilon). \qquad (2)$$

For the particular inequality to which I wish to call attention in this paper, equation (1) becomes

$$\frac{d^2\delta\frac{1}{r}}{dt^2} + N^2\delta\frac{1}{r} - \tfrac{15}{4}m^2e\gamma^2\cos(nt+\omega-2\Omega) = 0; \qquad (3)$$

the development being in accordance with Pontécoulant's work on the lunar theory, and the notation being the same as in my own work; namely, r denotes the radius vector; e and γ, the eccentricity and inclination of the moon's orbit; nt, ω and Ω, the mean longitude of the moon, the perigee and the node, respectively.

By reason of the perturbations ω and Ω are variable, and we have

$$\delta\omega = -\delta\Omega = \tfrac{3}{4}m^2nt. \qquad (4)$$

The coefficient of nt in the argument of equation (3) therefore becomes $1+\frac{3}{4}m^2 = i$; while N is equal to uuity minus the motion of the perigee, or $N = 1 - \frac{3}{4}m^2$. Therefore

$$i^2 - N^2 = 6m^2, \qquad (5)$$

and equation (3) gives by integration,

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2\cos(nt + \omega - 2\Omega). \tag{6}$$

The inequality depending on this argument is, therefore, apparently independent of the disturbing force, since the disturbing function becomes a factor of both numerator and denominator of the integral expressed by equation (2). La Place designates such inequalities as *finite.* They are, however, really indeterminate by equation (2), as we shall now proceed to show. It is evident that equation (1) becomes more accurate the smaller the disturbing function m^2 becomes, since it is the first term of a converging series depending on the ascending powers of m^2. Equation (1) would therefore be true if m^2 were an infinitesimal; but equation (2) gives the same value of δr whether m^2 be infinitesimal or finite; a result which is in direct contravention of the general principle of mechanics that the effect is proporportional to the cause.

But at this point we are met by the objection of Mr. G. W. Hill, who remarks, in the March No. of the ANALYST, that the value "$m=0$ implies that we have either an infinitely short month or an infinitely long year"; and is therefore not an admissible hypothesis, since the value of m is really finite.

Now Mr. Hill's remark is correct only in the case in which the sun is the disturbing body; and does not apply to the case of a second satellite or a planet. In this latter case, if we put the constant term of the expression of the inverse cube of the planet's distance from the earth equal to $b \div a''^3$, a'' being the mean distance of the planet from the sun; and also put

$$\mathrm{m}^2 = m''b\frac{a^3}{a''^3}, \tag{7}$$

m'' being the planet's mass; we shall have

$$\delta\omega = -\delta\Omega = \tfrac{3}{4}\mathrm{m}^2nt \qquad \text{and} \qquad i^2 - N^2 = 6\mathrm{m}^2;$$

but epuation (6) would remain unchanged.

In the case of Neptune being the disturbing body, we shall find $\mathrm{m}^2 = \frac{1}{9000000000000}$; and this value of the disturbing force would cause the perigee and node of the moon's orbit to move at the rate of about one second of arc in a period of 7000 years; but the substitution of these quantities in equation (3) gives the same perturbation as the disturbing force of the sun, which is about 500,000,000 times greater. It is therefore evident that eq'n (2) fails to give correct results in the case where i differs from unity but by quantities of the order m^2.

In the lunar theory, however, the disturbing force of the sun is comparatively great in respect to the central force; and as some doubt may exist

as to its competency to completely change the character of the elliptical inequalities due to the central force, we shall now show that the application of the same method to the planetary theories, in which case the disturbing forces are almost infinitesimally small in comparison with the central force, leads to results as extravagant and absurd as the perturbations produced by analysis (though ascribed to gravita'n), in the theories of the moon's motion.

For this purpose we shall consider the motion of Mercury as disturbed by Neptune. If we designate the coordinates of Mercury and Neptune by the same notation which we have employed for the moon and sun, respectively, in the lunar theory, we shall have $\mathrm{m}^2 = m'(a^3 \div a'^3) = 0.0000000000114$ = the disturbing function. And if we neglect the eccentricity and inclination of Neptune's orbit, the perihelion and node of Mercury's orbit would have the following motions on the ecliptic, t being reckoned in Julian years; namely,

$$\delta\omega = -\delta\Omega = +\tfrac{3}{4}\mathrm{m}^2 nt = +0''.0004603\, t.$$

Now according to Mr. Hill's logic, so long as the perihelion and node move at all, equation (2) must give the correct value of the perturbation of the radius vector depending on the argument $nt+\omega-2\Omega$; consequently

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2 \cos(nt+\omega-2\Omega);$$

and this gives for the corresponding perturbation in longitude

$$\delta v = -\tfrac{5}{4}e\gamma^2\,(=805'')\sin(nt+\omega-2\Omega). \qquad (7)$$

Now the pure elliptic motion of Mercury gives rise to the following inequality in the longitude; namely,

$$\delta v = +\tfrac{1}{2}e\gamma^2\,(=322'')\sin(nt+\omega-2\Omega);$$

and the sum of these two values of δv would be the true value of the equation, in order to allow for the perturbation by Neptune.

Now I find that the greatest perturbation of Mercury by Neptune is that depending on the argum't of the evection in the lunar theory, and amounts to only $0''.0125$, while the coefficient of equation (7) amounts to only $-0''.000000193$; which is altogether more probable than the preceding value.

Suppose now that we have a second disturbing planet, and call the disturbing function m'^2, the motion of the perihelion and node arising in consequence would be $\delta\omega = -\delta\Omega = \tfrac{3}{4}\mathrm{m}'^2 nt$; and the integral of (3) would be

$$\delta\frac{1}{r} = -\tfrac{5}{8}e\gamma^2\left[\frac{\mathrm{m}^2+\mathrm{m}'^2}{\mathrm{m}^2+\mathrm{m}'^2}\right]\cos(nt+\omega-2\Omega) = -\tfrac{5}{8}e\gamma^2\cos(nt+\omega-2\Omega),$$

the same as for a single disturbing planet. It is evident that any number of additional disturbing planets might be considered in the same way, and give the total perturbation by all the planets the same as that arising from a

single one. The perturbations depending on the proposed argument are therefore entirely indeterminate by equation (2).

Now suppose we were to force such an inequality as is given by equation (7) into the theory and tables of Mercury; it is evident that it would give rise to several inequalities of sensible magnitude, depending on different arguments; and also that the differences between theory and observation wo'd be very nearly equal to the sum of the inequalities arising from the new equations thus introduced; since the existing tables very closely represent the motion of Mercury. Assuming the theory including these new equat's to be correct, we should endeavor to make the differences between theory and observation disappear by applying corrections to the elements of the orbit. In this way we should be able to make the residuals disappear more or less completely for a time, since the errors of the elements would partially compensate for the errors of the theory and tables. But no amount of tinkering of the elements and theory possessing such an inherent source of error would suffice to produce tables which would permanently represent the motion of Mercury with the precision required by observation.

Now it appears to me that the lunar theory is in just the condition that the theory and tables of Mercury would be in this supposed case. Several inequalities of considerable magnitude which have no existence in nature have been forced in to the theory and tables of the moon; then corrections of the elements have been determined by means of numerous equations of condition, by which means we have neither correct elements nor correct theories; since equations of condition are powerless to correct for constant or systematic sources of error.

It is proper to remark in this connection, that La Place has stated in book II., chapt. V, of the *Mécanique Céleste*, where the integral of equation (1) first appears, that the integral takes a different form from equation (2), in the case of N being equal to i. For this supposed case he gives

$$\delta r = -\frac{m^2}{4N^2}\cos(Nt-\varepsilon)-\frac{m^2t}{2N}\sin(Nt-\varepsilon) \qquad (8)$$

which is easily proved by differentiation to be correct. Now the last term of this equation vanishes at the epoch, when $t = 0$; and we need only consider the second term in this connection. Since $N = 1-\frac{3}{4}m^2$, and $i = 1+\frac{9}{4}m^2$, we shall have $N = 0.9958$ and $i = 1.0126$, in the case of the moon disturbed by the sun. These numbers are not exactly equal, although very nearly so. In the case of Mercury disturbed by Neptune, we have

$$N = 0.99999999999145, \text{ and } i = 1.00000000002565.$$

These numbers approach very nearly to the ratio of equality; but it may very easily be shown that the elements of the orbits should be treated as con-

stant in the differential equations, and we should then have, rigorously, $N = i = 1$; and the integral of equation (1) would become

$$\delta r = -\tfrac{1}{2}m^2\cos(it-\varepsilon).$$

Equation (3) would also become

$$\frac{d^2\delta\frac{1}{r}}{dt^2}+\delta\frac{1}{r}-\tfrac{9}{2}m^2e\gamma^2\cos(nt+\omega-2\Omega) = 0; \tag{9}$$

the integral of which would be

$$\delta\frac{1}{r} = \tfrac{9}{8}m^2e\gamma^2\cos(nt+\omega-2\Omega). \tag{10}$$

This would give for the perturbation of the moon's longitude

$$\delta v = [\tfrac{9}{4}m^2e\gamma^2 = 1''.2]\sin(nt+\omega-2\Omega). \tag{11}$$

The perturbation depending on this argument, therefore, amounts to only $1''.2$ instead of $111''$ as determined by Ponte'coulaut, Plana and others.

From this examination it is apparent that the integral given by equation (2) is not applicable to those equations in which the coefficient of t in the argument is simply the mean motion of the moon; and had La Place and his successors remembered this circumstance in their calculations, the theory of the moon's motion would have been relieved of its most embarrassing features; its development would have been confined within narrower limits, and its permanent improvement been thereby greatly facilitated.

Cleveland, Aug. 25, 1882.

Postscript.—The preceding article was prepared for the November number of the ANALYST; but as it was not transmitted to the editor until the matter for that number had been selected, it was thought best to add the following as a *postscript* to that article, in order that the whole difficulty with which the lunar theory is embarrassed may be clearly and unmistakably traced to its source,

The equation

$$\left.\begin{array}{l}\dfrac{d^2\delta u}{dv^2}+(1-\tfrac{3}{2}\mu)\delta u-\tfrac{21}{4}m^2e\gamma^2\cos(v+\omega-2\Omega)\\ \qquad\qquad +\tfrac{15}{8}m^2\dfrac{a}{a'}e'\cos(v-\omega')\end{array}\right\} = 0, \tag{12}$$

given by Plana on page 72, *tome* II of his *The'orie du Mouvement de la Lune*, is a particular case of the general equation

$$\frac{d^2y}{dt^2}+N^2y+k\cos(it-\varepsilon) = 0, \tag{13}$$

the integral of which is, in the case where $N = i$,

$$y = -\frac{k}{4i^2}\cos(it-\varepsilon)-\frac{kt}{2i}\sin(it-\varepsilon). \tag{14}$$

The integral of equ'n (12), neglecting the last term, should therefore be

$$\delta u = \tfrac{21}{16} m^2 e\gamma^2 \cos(v+\omega-2\Omega) + \tfrac{21}{8} m^2 e\gamma^2\, v \sin(v+\omega-2\Omega), \qquad (15)$$

instead of

$$\delta u = -\tfrac{7}{8} e\gamma^2 \cos(v+\omega-2\Omega), \qquad (16)$$

as given by Plana.

The last term of equation (15) arises from the secular variation of the elements of the moon's orbit; while the first term of the second member arises from their periodic variations. In order to prove this we shall take the values of the secular variation of the elements given by Plana, in *tome* I, pp. 96, 97; namely,

$$\left.\begin{aligned} \delta\gamma &= \tfrac{5}{8} e^2\gamma \cos(2\omega-2\Omega) \\ \gamma\delta\Omega &= \tfrac{5}{8} e^2\gamma \sin(2\omega-2\Omega) \end{aligned}\right\}; \ (17) \qquad \left.\begin{aligned} \delta e &= -\tfrac{7}{8} e\gamma^2 \cos(2\omega-2\Omega) \\ e\delta\omega &= \tfrac{7}{8} e\gamma^2 \sin(2\omega-2\Omega) \end{aligned}\right\}. \ (18)$$

It is evident that the variations of the elements ought to simultaneously vanish at the epoch of the tables; and as equations (17) and (18) do not satisfy that condition, we must add a constant to each of these equations in order to make the variations vanish at a given epoch, when $\omega = \omega_0$, $\Omega = \Omega_0$. Equations (17) and (18) will therefore become

$$\left.\begin{aligned} \delta\gamma &= \tfrac{5}{8} e^2\gamma\,[\cos(2\omega-2\Omega) - \cos(2\omega_0-2\Omega_0)] \\ \gamma\delta\Omega &= \tfrac{5}{8} e^2\gamma\,[\sin(2\omega-2\Omega) - \sin(2\omega_0-2\Omega_0)] \end{aligned}\right\}; \qquad (19)$$

$$\left.\begin{aligned} \delta e &= -\tfrac{7}{8} e\gamma^2\,[\cos(2\omega-2\Omega) - \cos(2\omega_0-2\Omega_0)] \\ e\delta\omega &= \tfrac{7}{8} e\gamma^2\,[\sin(2\omega-2\Omega) - \sin(2\omega_0-2\Omega_0)] \end{aligned}\right\}. \qquad (20$$

Now the principal term in the value of u is

$$u = e\cos(v-\omega_0); \qquad (21)$$

and its variation corresponding to small variations of e and ω_0 will be given by the equation

$$\delta u = \left(\frac{du}{de}\right)\delta e + \left(\frac{du}{d\omega}\right)\delta\omega = \cos(v-\omega_0)\delta e + \sin(v-\omega_0)e\delta\omega. \qquad (22)$$

If we substitute the values of δe and $e\delta\omega$ given by eq'ns (20) we shall find

$$\left.\begin{aligned} \delta u &= -\tfrac{7}{8} e\gamma^2\,[\cos(v-\omega_0+2\omega-2\Omega) - \cos(v+\omega_0-2\Omega_0)] \\ &= \tfrac{7}{4} e\gamma^2 \sin(\omega-\Omega-\omega_0+\Omega_0)\sin(v+\omega-\Omega-\Omega_0) \end{aligned}\right\}. \qquad (23)$$

In the case of constant elements we have $\omega = \omega_0$, $\Omega = \Omega_0$, and δu vanishes. But since ω and Ω are variable, we have, according to Plana,

$$\omega = \omega_0 + \tfrac{3}{4} m^2 v, \qquad \Omega = \Omega_0 - \tfrac{3}{4} m^2 v, \qquad (24)$$

so that $\omega-\Omega-\omega_0+\Omega_0 = \frac{3}{2} m^2 v$; and as we are retaining only the terms of the order m^2, we may put

$$\sin(\omega-\Omega-\omega_0+\Omega_0) = \omega-\Omega-\omega_0+\Omega_0 = \tfrac{3}{2} m^2 v, \qquad (25)$$

and equation (23) will become

$$\delta u = \tfrac{21}{8} m^2 e\gamma^2\, v \sin(v+\omega-\Omega-\Omega_0 = v+\omega-2\Omega \textit{ at the epoch}). \qquad (26)$$

This is the same as the last term of equation (15), and proves the first p't of the proposition. The second part may be proved in a similar manner by taking the variation of the second term of the value of u and substituting the periodic variation of the elements.

We thus see that the variation of the elements leads to the same results as are derived from the variation of the coordinates, when the proper constants are added to the integrals of these variations.

The integral of equation (12) depending on $\cos(v-\omega')$ is

$$\delta u = -\tfrac{15}{32} m^2 (a \div a') e' \cos(v-\omega') - \tfrac{15}{16} m^2 (a \div a') e' v \sin(v-\omega'), \quad (27)$$

instead of $\delta u = \frac{5}{4}(a \div a') e' \cos(v-\omega')$, as given by Plana.

The last term of equation (27) arises from the secular variation of e and ω, depending on the distance between the perigees of the sun and moon. To prove this we shall observe that the differential equations of e and ω contain the terms

$$\left.\begin{aligned} \frac{de}{dv} &= -\tfrac{15}{16} m^2 \frac{a}{a'} e' \sin(\omega-\omega') \\ e\frac{d\omega}{dv} &= -\tfrac{15}{16} m^2 \frac{a}{a'} e' \cos(\omega-\omega') \end{aligned}\right\}. \quad (28)$$

If we integrate these equations in the same way that Plana has done, and also add a constant to the integrals so that δe and $\delta\omega$ may *simultaneously vanish at the epoch*, we shall find,

$$\left.\begin{aligned} \delta e &= \tfrac{5}{4}(a \div a') e' [\cos(\omega-\omega') - \cos(\omega_0-\omega'_0)] \\ e\delta\omega &= -\tfrac{5}{4}(a \div a') e' [\sin(\omega-\omega') - \sin(\omega_0-\omega'_0)] \end{aligned}\right\}. \quad (29)$$

If these values be substituted in equation (22), we shall find, after making the necessary reductions,

$$\delta u = -\tfrac{15}{16} m^2 (a \div a') e' v \sin(v-\omega'), \quad (30)$$

as in the last term of equation (27).

The other term of equation (27) may be found in like manner by substituting the proper periodic variations of e and ω in equation (22).

The two cases which we have examined are the most important ones among the equations of the moon's longitude; but there are two terms of considerable importance in the equations of the moon's latitude, which we shall now examine.

The first of these terms of the latitude is given by Plana in *tome* II, page 118, of his work, as follows:

$$\frac{d^2\delta s}{dv^2} + (1+\tfrac{3}{2}m^2)\delta s - \tfrac{15}{4} m^2 e^2 \gamma \sin(v-2\omega+\Omega) = 0. \quad (31)$$

The integral of this equation is

$$\delta s = \tfrac{15}{16} m^2 e^2 \gamma \sin(v-2\omega+\Omega) - \tfrac{15}{8} m^2 e^2 \gamma v \cos(v-2\omega+\Omega); \quad (32)$$

instead of $\delta s = +\frac{5}{8} e^2 \gamma \sin(v-2\omega+\Omega)$, as given by Plana.

The last term of equation (32) is produced by the secular variation of the node and inclination of the moon's orbit, as may be shown in the following manner:—Since the principal term of the latitude is given by the equation

$$s = \gamma \sin (v - \Omega_0), \tag{33}$$

we shall have

$$\delta s = \sin (v - \Omega_0)\delta\gamma - \cos (v - \Omega_0)\gamma\delta\Omega; \tag{34}$$

and if we substitute the values of $\delta\gamma$ and $\gamma\delta\Omega$, which are given by eq. (19) we shall find

$$\left.\begin{aligned} \delta s &= \tfrac{5}{4}e^2\gamma \sin (\omega_0 - \Omega_0 - \omega + \Omega) \cos (v - \omega_0 - \omega + \Omega) \\ &= -\tfrac{15}{8}m^2e^2\gamma\, v \cos (v - 2\omega + \Omega) \end{aligned}\right\}, \tag{35}$$

the same as the last term of equation (32).

The second term of the latitude which seems to be entirely erroneous is the one arising from the oblateness of the earth. La Place first computed this inequality; and I am especially desirous of calling attention to it, since the error is common to every theory of the moon's motion which I have examined, including my own. In this examination I shall first follow the method given by La Place in the *Mécanique Céleste,* making use of the translation by Bowditch, since it is more convenient for reference than the original work. I shall then show how the same results may be obtained by the method given in my own work.

If, in the function [5347], *Méc. Cél.,* we substitute for $u \div h^2$ its value $1 \div a^2$, and also put for abridgement

$$m = (a\rho - \tfrac{1}{2}a\varphi)\frac{D^2}{a^2}, \tag{36}$$

it will become

$$2m \sin \varepsilon \cos \varepsilon \sin fv + (g^2 - 1)H \sin fv, \tag{37}$$

ε denoting the obliquity of the ecliptic to the equator.

But in [5350] La Place gives

$$H = -\frac{2m}{g^2 - 1} \sin \varepsilon \cos \varepsilon; \tag{38}$$

consequently the function (37) is *identically equal to nothing.*

Now La Place remarks that the oblateness of the earth *adds* the function (37) to the differential equation of the latitude, by which means it becomes

$$\frac{d^2\delta s}{dv^2} + \delta s + [2m \sin \varepsilon \cos \varepsilon + (g^2 - 1)H] \sin fv = 0; \tag{39}$$

the integral of which he gives as

$$\delta s = -\frac{m}{g - 1} \sin \varepsilon \cos \varepsilon \sin fv. \tag{40}$$

In other words, if *nothing* be *added* to the differential equation of the latitude, its integral will be *increased* by the value of δs in equation (40);—a result which is exceedingly interesting as well as mysterious to a person unacquainted with the wonderful powers of the *calculus.*

But the correct value of the integral of equation (39) is

$$\left.\begin{aligned}\delta s = &-\tfrac{1}{4}[2m\sin\varepsilon\cos\varepsilon+(g^2-1)H]\sin fv\\ &+\tfrac{1}{2}[2m\sin\varepsilon\cos\varepsilon+(g^2-1)H]\,v\cos fv\end{aligned}\right\}, \qquad (41)$$

since $f=1$.

The function H is therefore not correctly given by equation (38), and the function (37) is not equal to *zero*.

In equation (41) the term $-\frac{1}{2}m\sin\varepsilon\cos\varepsilon\sin fv$ gives the direct effect of the earth's oblateness on the moon's latitude; while the term $(g^2-1)H\sin fv$ is its indirect action transmitted to the moon by means of the sun but decreased in the ratio of g^2-1 to unity. Now $\frac{1}{2}m\sin\varepsilon\cos\varepsilon = 0''.01654 = H$, and $(g^2-1)H = 0''.000133$. Whence it follows that the effect of the earth's oblateness on the moon's latitude is insensible.

The same result may be obtained by the method empl'd in my own work. For the same force which gives rise to the first term of the function (37), produces in the value of $\frac{d\delta_0\theta}{dt}$, eq'n (263) of my Theory of the Moon's Motion, the following terms

$$\frac{d\delta_0\theta}{dt} = mn\sin\varepsilon\cos\varepsilon\left[\tfrac{1}{2}\cos nt - nt\sin nt\right]; \qquad (42)$$

and this gives by integration

$$\delta_0\theta = m\sin\varepsilon\cos\varepsilon\left[-\tfrac{1}{2}\sin nt + nt\cos nt\right], \qquad (43)$$

which is the same as the corresponding terms of δs in equation (41), since $v = nt$, when we neglect the eccentricity of the moon's orbit.

It is easy to prove that equation (43) is correct, by means of the variation of the elements. For if we add a constant to the secular terms of these variations, which are given by equations (630) and (631) of my work, so that they may simultaneously vanish at the epoch, they will become

$$\left.\begin{aligned}\gamma\delta\Omega &= m\sin\varepsilon\cos\varepsilon\left[\tfrac{1}{2}\sin(2nt-\Omega)-(n\div a')(\sin\Omega-\sin\Omega_0)\right]\\ \delta\gamma &= m\sin\varepsilon\cos\varepsilon\left[\tfrac{1}{2}\cos(2nt-\Omega)+(n\div a')(\cos\Omega-\cos\Omega_0)\right]\end{aligned}\right\}. \qquad (44)$$

And if we substitute these values in the equation

$$\delta\theta = \sin(nt-\Omega_0)\delta\gamma - \cos(nt-\Omega_0)\gamma\delta\Omega, \qquad (45)$$

the secular terms will produce the following term,

$$\delta\theta = 2\frac{mn}{a'}\sin\varepsilon\cos\varepsilon\sin\tfrac{1}{2}(\Omega-\Omega_0)\cos nt. \qquad (46)$$

But since $\Omega-\Omega_0 = a't$, we have $\sin\frac{1}{2}(\Omega-\Omega_0) = \frac{1}{2}a't$, and eq. (46) becomes

$$\delta\theta = m\sin\varepsilon\cos\varepsilon\, nt\cos nt; \qquad (47)$$

which is the same as the last term of (43).

If we now substitute the periodic terms of equations (44) in equation (45) we get the first term of equation (43).

Finally La Place remarks, *Méc. Cél.*, [5398], that the inequality of the moon's motion in latitude arising from the oblateness of the earth is only

the reaction of the nutation of the earth's axis, discovered by Bradley;— a statement which seems strangely at variance with mechanical principles, since the one has a period of only twenty-seven days, while the other has a period of nearly nineteen years.

It is evident that this change in the equation of the moon's latitude affects the corresponding equation of the longitude; and I find that it changes its value from $\delta v = 6\frac{mn}{a'}\gamma \sin \varepsilon \cos \varepsilon \sin \Omega$, to $\delta v = \frac{13}{2}\frac{mn}{a'}\gamma \sin \varepsilon \cos \varepsilon \sin \Omega$. The equations of the moon's motion arising from the oblateness of the earth are, therefore

$$\delta v = 4''.814 \sin \Omega, \qquad \delta\theta = -0''.0165 \sin nt, \qquad (48)$$

instead of

$$\delta v = 4''.444 \sin \Omega, \qquad \delta\theta = -8''.226 \sin nt,$$

previously found.

But there is one point in Mr. Hill's "Review" which seems to be well made out, and for which I wish to thus publicly tender him my grateful acknowledgements. And that is in relation to the reduction of the moon's longitude from the plane of its orbit to the ecliptic, and which amounts to the correction which I have now applied to the equation of the longitude arising from the oblateness of the earth.

I have thus shown with considerable detail, not only that the lunar theories of my predecessors are erroneous by terms of the third order, but have also shown just how the errors were introduced into the several theories. The errors were first committed by La Place; and his methods of integration were blindly followed by his successors until his results came to be accepted by astronomers as the legitimate effects of the law of universal gravitation. Had La Place been a less skilful mathematician it is not probable that he would have made such mistakes; since he would have been content to consider one power of the disturbing force at a time, and he wo'd then have been obliged to regard the elements as constant in the integrat'ns; and then, in all the cases which I have pointed out, he would have found *zero* for a divisor and *infinity* for the perturbations. And such results could not have failed to indicate their origin in the improper application of a general formula of integration to a special case in which such formula fails to give correct results. I therefore confidently believe that existing theories when corrected for the sources of error which I have pointed out; together with a careful revision of the elem'ts of the lunar orbit, will give the moon's place in the heavens with all the precision required by observation, without carrying the approximat'n to terms of a higher order than have been already calculated.

Cleveland, Oct. 10, 1882.

NOTE.—The following equations (g) and (h), and concluding remark, was enclosed by Prof. Philbrick with the corrected "proof" of his paper, pp. 9–14, and should have appeared in connection with that paper; but by accident the proof was not rec'd till after the sheet was printed, hence their appearance here, and the corrections of the proof, in the *Errata* on p. 32.

"Also

$$\int\frac{\sin^m x\cos^r x dx}{(a+b\sec x)^n}=\int\frac{\sin^m x\cos^{r+n} x dx}{(b+a\cos x)^n}, \qquad (g)$$

which may be integrated by (a) or by (b); and

$$\int\frac{\sin^m x\cos^r x dx}{a+b\,\mathrm{co'c}\, x)^n}=\int\frac{\sin^{m+n} x\cos^r x dx}{(b+a\sin x)^n}. \qquad (h)$$

which may be integrated by (c) or (d).

Probably almost any combination of trigonometric functions may be integrated, directly or by transformation, by the general formulas above, or by others easily derived from them."

CALCULATION OF TRANSIT OF VENUS BY PROF. BARBOUR.—This calculation is made, by T. H. Safford, Jr.'s modification of Bessel's method, for the position of Louisville, Ky., N. Lat. 38° 14′ 57″.78, Long. 85° 45′ 52″.53 W.; transit to occur on Dec. 6th 1882.

The formulæ will be found in Chauvenet's Spher. and Pract. Ast., Vol. I, and in the Amer. Naut. Almanac for 1882.

$a=A-h\sin(\mu-\lambda)$, $\qquad b=B-EK+Gh\cos(\mu-\lambda)$,

$c=-C+FK-Hh\cos(\mu-\lambda)$, $m=\sqrt{(bc)}$. If $m=a$ the assumed time is correct.—*To find time of* 2^{nd} *contact.* Put φ' = geocentric latitude of the place; λ = longitude West from Greenwich; ρ = dist. from Earth's center; $h=\rho\cos\varphi'$, and $K=\rho\sin\varphi'$.

An easy method of calculating h and K is provided by Amer. Naut. Almanac for 1882, p. 499. $\rho\cos\varphi'=F'\cos\varphi$, and $\rho\sin\varphi'=G'\sin\varphi$, in which φ = geographical latitude, and logs F' and G' are given in a table.

$$\begin{array}{llll}
\log\cos\varphi & = 9.8950487 & \log\sin\varphi & = 9.7917506 \\
\log F' & = .000578 & \log G & = n\ .0024355 \\ \hline
\log h & = 9.8956267 & \log K & = n9.7941861
\end{array}$$

(n before a log. means that the number corresponding is negative.)

To find a. $\mu = 38°\ 13'\ 3''$ for epoch $2^h\ 24^m\ 57^s$ (Gr. mean time).

$\lambda = 85°\ 45'\ 52''.53$; $\therefore\ \mu-\lambda = -47°32'49''.53$ or $312°\ 27'\ 10''.47$. $\therefore\ \log\sin(\mu-\lambda)=n9.8679576$; $\log\cos(\mu-\lambda)=9.8292395$

$$\begin{array}{ll}
\log h & = 9.8956267 \\ \hline
\log h+\log\sin(\mu-\lambda) & = 9.7635843.
\end{array}\quad \therefore\ h\sin(\mu-\lambda)=.580209.$$

A, for given epoch, $= 18.8281658$; $+.580209 = 19.4083748 = a$.

To find b.

$\log E$, 2nd cont., $= 9.96859$ $\qquad$ $\log G$, 2nd cont, $= n9.58878$

$\log K \qquad\qquad = \underline{9.7941861}$ $\qquad$ $\log h \qquad\qquad = 9.8956267$

$\log (EK) \qquad = 9.7587761$ $\qquad$ $\log \cos(\mu-\lambda) \quad = \underline{9.8292935}$

$EK \qquad\qquad = .57382$ $\qquad$ $\text{l.}[Gh\cos(\mu-\lambda)] = 9.3137002$

$[Gh \cos (\mu-\lambda)-EK = -.205921-.57382 = -.779741.$

B, at 2nd contact $= 6.029408$; $b = 6.029408-.779741 = 5.249667$;

$\log b = 0.7201318.$

To find c.

$\log F$, at 2nd cont. $= 9.96689$ $\qquad$ $\log H$, at 2nd cont. $= n9.57533$

$\log K \qquad\qquad = \underline{9.7941861}$ $\qquad$ $\log h \qquad\qquad = 9.8956267$

$\log (FK) \qquad = 9.7610761$ $\qquad$ $\log \cos (\mu-\lambda) \qquad = \underline{9.8292589}$

$FK \qquad\qquad = .576867$ $\qquad$ $\log Hh \cos (\mu-\lambda) \quad = n9.3002156$

$Hh \cos (\mu-\lambda) \quad = -.199625$

C is intrinsically neg., $\therefore$ it becomes pos. here being produced by — sign.

$-C = 70.961592$; $c = 70.961592+.576867+.199625 = 71.738084$

$\log c = 1.8557498$; $\frac{1}{2} \log c = .9278749$

$\frac{1}{2} \log b = \underline{.3600659}$

$\log \surd(bc) = 1.2879408$; $\therefore \surd(bc)$ is 19.40621. Hence $a-\surd(bc) = 19.40837-19.40621 = +.00216$, error.

In the same manner we find for the epoch $2^h\ 24^m\ 58^s$, $a = 19.4056$ and $\surd(bc) = 19.4075$; consequently their difference is $-.0019$, error. So that $2^h\ 24^m\ 57^{\frac{1}{2}s}$ is a very close approximation.

In the present state of the ephemeris, the preceding calculation may seem needlessly precise. "The uncertainty of the tabular elements renders the computed times of contact doubtful by a large fraction of a minute." (Amer. Naut. Alm.) Yet by a very careful observ'n of the local or Wash. Mean Time, at Louisville, the ephemeris may be improved.

I have not taken the pains to compute the 3rd and 4th contacts, as they can be so easily observed—provided of course the sky be clear.

The dif. in time bet. Louisville and Greenwich is $5^h\ 43^m\ 3^{\frac{1}{2}s}$. Hence the Louisville Mean Time of 2nd contact should be $8^h\ 41^m\ 54^s$ A. M. of Dec. 6.

The 2nd contact ends for the earth generally $2^h\ 25^m\ 14^s.2$; and the 3rd ends at $8^h\ 1^m\ 52^s.7$ (G. M. T.). This gives a difference of $5^h\ 36^m\ 38^s.5$.

By the chart in Amer. N. Alm. the curve for $2^h\ 25^m$ runs nearly through Chicago, and sweeps around up to the extreme eastern part of Hudson's Bay. This curve is convex toward the Atlantic Ocean. The limiting curve, wh'h is the locus of the end of the 2nd contact, is concave toward the Atlantic, and passes through the Northern part of the Gulf of California and the middle of Hudson's Bay. But the dif. of time, $5^h\ 36^m\ 38^s.5$ will not vary more than a few seconds.—Richmond, Ky., Dec. 1, 1882.

DERIVATION OF THE FORMULA ON P. 96, VOL. III, VIZ.;

$$R = r\times\frac{(n+1)N+(n-1)r^n}{(n-1)N+(n+1)r^n},$$

where r is an approximate value of $\sqrt[n]{N}$ and R a much nearer approxima'n.

Let $N=r^n+a$, then, by the binomial formula,

$$N^{\frac{1}{n}} = r\left(1+\frac{a}{nr^n}-\frac{(n-1)a^2}{1.2.n^2r^{2n}}+\frac{(n-1)(2n-1)a^3}{1.2.3.n^3r^{3n}}-\text{\&c.}\right).$$

Beginning with the term $a\div nr^n$ and reducing to a continued fraction and stopping at the second term of the cont'd fract. gives approximately

$$\frac{\frac{a}{nr^n}-\frac{(n-1)a^2}{1.2.n^2r^{2n}}+\text{\&c.}}{1} = \cfrac{1}{\frac{nr^n}{a}+\cfrac{1}{\frac{2}{n-1}+\text{\&c.}}} = \frac{2a}{2nr^n+(n-1)a};$$

$$\therefore R = r\left(1+\frac{2a}{2nr^n+(n-1)a}\right) = \frac{2nr^n+(n+1)a}{2nr^n+(n-1)a}.$$

Substituting for a its value $= N-r^n$,

$$R = r\times\frac{(n+1)N+(n-1)r^n}{(n-1)N+(n+1)r^n} = N^{\frac{1}{n}} \text{ nearly.}$$

R. J. ADCOCK.

SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. IX.

SOLUTIONS of problems in No. 6, Vol. IX, have been received as follows:

From Florian Cajori, 422; Geo. E. Curtis, 419, 421; Prof. H. T. Eddy, 420; Geo. Eastwood, 422; Prof. A. Hall, 420; Henry Heaton, 419, 420, 422; Charles V. Kerr, 419; E. H. Moore, Jr., 419, 422; Levi W. Meech, 418; Thos. Spencer, 419; M. Updegraff, 419.

Prof. J. W. Nicholson sent elegant solutions of prob's 411 and 416, but his letter was accidentally misplaced, hence they were not included in notice of solutions in No. 6.

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418. *By Levi W. Meech, A. M., Norwich, Conn.*—"Required to express Lagrange's Theorem in terms of Finite Differences, as far as practicable, instead of the usual differentials."

SOLUTION BY THE PROPOSER.

Let θ denote an auxiliary, such that Lagrange's Theorem may take the form of the definite integral:

$$Fx = Ft + \int_0^1 d\theta \left\{ 1 + \theta \frac{d}{dt} eft + \frac{\theta^2 d^2}{1.2.dt^2} (eft)^2 + \dots \right\} eft. \frac{dFt}{dt}.$$

Compare ANALYST, Vol. III, pages 34, 38; and Boole's Finite Differences, page 18. Then, since $d \div dt = D = \log(1+\Delta)$;

$$Fx = Ft + \int_0^1 d\theta . \varepsilon^{\theta D eft} . eft. \frac{dFt}{dt} = Ft + \int_0^1 d\theta \, (1+\Delta)^{\theta eft} eft. \frac{dFt}{dt}.$$

Developing $1+\Delta$ by the binomial theorem, and then integrating with respect to θ, we obtain the required formula:

$$Fx = Ft + eft. \frac{dFt}{dt} + \frac{\Delta}{1.2} (eft)^2. \frac{dFt}{dt} + \frac{\Delta^2}{1.2.3} (eft)^2 (eft - 1\tfrac{1}{2}) \frac{dFt}{dt}$$
$$+ \frac{\Delta^3}{1.2.3.4} (eft)^2 (eft-2)^2 \frac{dFt}{dt} + \dots$$

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419. *By C. E. Everett, Spirit Lake, Iowa.*—"Find the locus of a point starting from the centre of a given circle and moving so that the arc included between any two positions of the point shall equal the arc of the circle intercepted by the radii drawn through the same positions."

SOLUTION BY GEO. E. CURTIS, BIRMINGHAM, CONN.

Let R be the radius of the given circle. The conditions of motion give at once the differential equation of the locus,

$$ds = \sqrt{(r^2 d\theta^2 + dr^2)} = R d\theta;$$

which by integration and reduction becomes

$$r = R \sin \theta.$$

This represents a circle half the radius of the given circle and internally tangent to it.

SOLUTION BY CHARLES V. KERR, ALLEGHENY, PENN'A.

Let O be the center of the given circle, EF a diameter, and let OPF be a circle described on the radius OF as diameter.

Through A and B, any two points on the circumf. of the inscribed circle, draw the radii OD and OC.

Now, a degree on the inscribed circle is equal in length to one-half of a degree on the given circle, since circumferences are to each other as their radii. But the arc AB is double the measure of the insc'd angle AOB, and hence contains twice as may degrees as the arc DC, which is the measure of the angle DOC; and since A and B are *any* two points the circle OHF is the req'd locus.

420. *By Prof. Asaph Hall.*—"Transform the definite integral

$$\int_b^a \varphi(x).dx,$$

so that the limits of integration shall be m and n."

SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

Put $x=py+q$. When $x=a$, $y=m$, when $x=b$, $y=n$. Hence we have $a=pm+q$ and $b\ pn+q$; whence

$$p=\frac{a-b}{m-n},\ \text{and}\ q=\frac{mb-an}{m-n};$$

$$\therefore \int_b^a \varphi(x)dx=p\int_n^m \varphi(py+q)dy=\frac{a-b}{m-n}\int_n^m \varphi\left(\frac{a-b}{m-n}y+\frac{mb-an}{m-n}\right)dy.$$

[Prof. Eddy and Prof. Hall obtain the same result as above, by a similar process, differing only slightly in the notation employed.]

421. *By George Eastwood, Saxonville, Mass.*—"In a Bicycle exercise on a level, circular course of given radius, what angle ought the plane of the machine to make with the vertical, so that the rider may move on the circumference of a perfect circle?"

SOLUTION BY FLORIAN CAJORI, MADISON, WISCONSIN.

Let $AO=r=$ radius of track; $m=$ the mass; $v=$ velocity; $g=$ gravity, and $x=$ the required angle. Also, let C denote the position of the centre of gravity of the bicycle and man, and put $a=AC$. Then $r-a\sin x=$ radius of circumference described by C. The centrifugal force equals the mass into the square of the velocity divided by the radius, or $mv^2\div(r-a\sin x)$.

Taking A as the origin of moments and equating the moment of the centrifugal force to the moment of grav., we obtain, after eliminating common factors, the equation

$$\frac{v^2}{r-a\sin x}\cos x=g\sin x,$$

wherein x is the required angle.

The last term of equation (32) is produced by the secular variation of the node and inclination of the moon's orbit, as may be shown in the following manner:—Since the principal term of the latitude is given by the equation

$$s = \gamma \sin(v - \Omega_0), \tag{33}$$

we shall have

$$\delta s = \sin(v - \Omega_0)\delta\gamma - \cos(v - \Omega_0)\gamma\delta\Omega; \tag{34}$$

and if we substitute the values of $\delta\gamma$ and $\gamma\delta\Omega$, which are given by eq. (19) we shall find

$$\left.\begin{aligned}\delta s &= \tfrac{5}{4}e^2\gamma \sin(\omega_0 - \Omega_0 - \omega + \Omega)\cos(v - \omega_0 - \omega + \Omega)\\ &= -\tfrac{15}{8}m^2e^2\gamma\, v \cos(v - 2\omega + \Omega)\end{aligned}\right\}, \tag{35}$$

the same as the last term of equation (32).

The second term of the latitude which seems to be entirely erroneous is the one arising from the oblateness of the earth. La Place first computed this inequality; and I am especially desirous of calling attention to it, since the error is common to every theory of the moon's motion which I have examined, including my own. In this examination I shall first follow the method given by La Place in the *Mécanique Céleste*, making use of the translation by Bowditch, since it is more convenient for reference than the original work. I shall then show how the same results may be obtained by the method given in my own work.

If, in the function [5347], *Méc. Cél.*, we substitute for $u \div h^2$ its value $1 \div a^2$, and also put for abridgement

$$m = (a\rho - \tfrac{1}{2}a\varphi)\frac{D^2}{a^2}, \tag{36}$$

it will become

$$2m \sin\epsilon \cos\epsilon \sin fv + (g^2 - 1)H \sin fv, \tag{37}$$

ϵ denoting the obliquity of the ecliptic to the equator.

But in [5350] La Place gives

$$H = -\frac{2m}{g^2 - 1}\sin\epsilon\cos\epsilon; \tag{38}$$

consequently the function (37) is *identically equal to nothing.*

Now La Place remarks that the oblateness of the earth *adds* the function (37) to the differential equation of the latitude, by which means it becomes

$$\frac{d^2\delta s}{dv^2} + \delta s + [2m \sin\epsilon\cos\epsilon + (g^2 - 1)H]\sin fv = 0; \tag{39}$$

the integral of which he gives as

$$\delta s = -\frac{m}{g - 1}\sin\epsilon\cos\epsilon\sin fv. \tag{40}$$

In other words, if *nothing* be *added* to the differential equation of the latitude, its integral will be *increased* by the value of δs in equation (40);—a result which is exceedingly interesting as well as mysterious to a person unacquainted with the wonderful powers of the *calculus.*

But the correct value of the integral of equation (39) is

$$\left.\begin{aligned}\delta s = &-\tfrac{1}{2}[2m\sin\varepsilon\cos\varepsilon+(g^2-1)H]\sin fv\\ &+\tfrac{1}{2}[2m\sin\varepsilon\cos\varepsilon+(g^2-1)H]\,v\cos fv\end{aligned}\right\},\tag{41}$$

since $f=1$.

The function H is therefore not correctly given by equation (38), and the function (37) is not equal to *zero*.

In equation (41) the term $-\frac{1}{2}m\sin\varepsilon\cos\varepsilon\sin fv$ gives the direct effect of the earth's oblateness on the moon's latitude; while the term $(g^2-1)H\sin fv$ is its indirect action transmitted to the moon by means of the sun but decreased in the ratio of g^2-1 to unity. Now $\frac{1}{2}m\sin\varepsilon\cos\varepsilon = 0''.01654 = H$, and $(g^2-1)H = 0''.000133$. Whence it follows that the effect of the earth's oblateness on the moon's latitude is insensible.

The same result may be obtained by the method empl'd in my own work. For the same force which gives rise to the first term of the function (37), produces in the value of $\frac{d\delta_0\theta}{dt}$, eq'n (263) of my Theory of the Moon's Motion, the following terms

$$\frac{d\delta_0\theta}{dt} = mn\sin\varepsilon\cos\varepsilon\,[\tfrac{1}{2}\cos nt - nt\sin nt]\,;\tag{42}$$

and this gives by integration

$$\delta_0\theta = m\sin\varepsilon\cos\varepsilon\,[-\tfrac{1}{2}\sin nt + nt\cos nt],\tag{43}$$

which is the same as the corresponding terms of δs in equation (41), since $v=nt$, when we neglect the eccentricity of the moon's orbit.

It is easy to prove that equation (43) is correct, by means of the variation of the elements. For if we add a constant to the secular terms of these variations, which are given by equations (630) and (631) of my work, so that they may simultaneously vanish at the epoch, they will become

$$\left.\begin{aligned}\gamma\delta\Omega &= m\sin\varepsilon\cos\varepsilon\,[\tfrac{1}{2}\sin(2nt-\Omega)-(n\div a')(\sin\Omega-\sin\Omega_0)\\ \delta\gamma &= m\sin\varepsilon\cos\varepsilon\,[\tfrac{1}{2}\cos(2nt-\Omega)+(n\div a')(\cos\Omega-\cos\Omega_0)\end{aligned}\right\}.\tag{44}$$

And if we substitute these values in the equation

$$\delta\theta = \sin(nt-\Omega_0)\delta\gamma - \cos(nt-\Omega_0)\gamma\delta\Omega,\tag{45}$$

the secular terms will produce the following term,

$$\delta\theta = 2\frac{mn}{a'}\sin\varepsilon\cos\varepsilon\sin\tfrac{1}{2}(\Omega-\Omega_0)\cos nt.\tag{46}$$

But since $\Omega\text{-}\Omega_0 = a't$, we have $\sin\frac{1}{2}(\Omega\text{-}\Omega_0)=\frac{1}{2}a't$, and eq. (46) becomes

$$\delta\theta = m\sin\varepsilon\cos\varepsilon\; nt\cos nt;\tag{47}$$

which is the same as the last term of (43).

If we now substitute the periodic terms of equations (44) in equation (45) we get the first term of equation (43).

Finally La Place remarks, *Méc. Cél.*, [5398], that the inequality of the moon's motion in latitude arising from the oblateness of the earth is only

On the Composition of Errors from Single Causes of Error. By **Chas. H. Kummell**, of U. S. Coast and Geodetic Surv., Wash. D. C. [Rep. from *Astronomische Nach.*, No. 2460-61.

The Intersection of Circles and the Intersection of Spheres. 24 pp. By **Benjamin Alvord**, Brig. Gen. U. S. A. [Reprinted from *Amer. Jour. of Mathematics*, Vol. V, No. 1.]

In this Memoir Gen. Alvord solves geometrically all the questions in Intersections, by the same principle, in effect, as was used by him in the memoir on "The Tangencies," publish'd in the Smithsonian Contribution, Vol. 8, 1855.—The question in Intersections is reduced to one in tangencies and orthogonals. There is an evolution throughout the whole investigation from the principle of the radical center, that is; the radical axes of three given circles intersect each other in a point, called the radical center, which is also the center of the circle orthogonal to said circles. The radical center of four spheres is found in like manner.—All questions in Spheres are reduced to those in Circles.

Naming the General question (without considering the various cases), the following are the problems solved, and the number of solutions to each.

1. To draw a circle to cut each of three given circles at the same given ang., 8 solutions.
2. To draw a sphere to cut each of five given spheres at a given angle, 16 "
3. To draw a circle to cut each of five given circles at the same angle (angle being unknown), 96 "
4. To draw a sphere to cut each of five given spheres at the same angle (angle being unknown), 640 "

It is believed that the last two questions have never been solved geometrically heretofore; nor was it known that there were so many solutions. Some of them are imaginary. Thus in circles the required circle may often not intersect either of the given circles, but will be situated in a similar manner toward each.

Prof. Arthur Cayley, F. R. S., who happen'd to be in Baltimore when the paper was offered, appends a valuable note at page 10.

Mr. Marcus Baker in the Analyst for July, 1877, page 128, proposed the last question for solution. R. J. Adcock, in the Analyst for September, 1877, page 158, gave the equations for solution of that question, and Thomas Craig, in Analyst for Jan. 1880, p. 13, gave an analytical solution by the method of determinants.

If Steiner, who proposed the last two questions in the 1st Vol. of Crelle, 1826, ever solved them, or if he published such solution, such fact is unknown to the best accessible authorities.

ERRATA.

On page 176, Vol. IV, *dele* $\sqrt{2}$ in last term of the value of I_5.
" " 164, line 8, of Vol. IX, for "persons" read insured persons.
" " 178 of Vol. IX, lines 5 and 8 from bottom, insert sign of integration after $\times$ at the beginning of each line.
" " 3 of Vol. X, line 4 from bottom, for lower limit of int. read ax.
" " 10 line 7, for $6a$ read $6a^2$.
" " 12 " 4, for $\sin x$ under the sign of integration read $\sec x$.
" " 13 " 16, insert minus sign after the sign of equality.
" " 13 " 17, for "This", read These.
" " 14 " 5, for the exponent "$n+2$", read $n-2$.
In Table " " 120 of Vol. IX, in $\sqrt{2}$, after the 124th dec., read 3605585073721286441 and in $\sqrt[3]{2}$ omit the last 13 decimals; in $\sqrt{8}$, the 23rd decimal is 7; in $\sqrt{10}$, the 102nd dec. is 5, and in $\sqrt{15}$, the 24th decimal is 5.

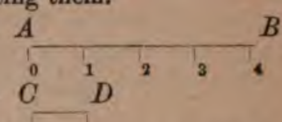
THE ANALYST.

VOL. X. MARCH, 1883. No. 2.

ON THE APPLICATION OF THE METHOD OF LEAST SQU'S TO THE REDUCT'N OF COMPARISONS OF LINE-MEASURES AND TO THE CALIBRATION OF THERMOMETERS.

BY T. W. WRIGHT, B. A., C. E., DETROIT MICH.

1. COMPARISONS of line-measures and observations for the calibration of thermometers may be discussed together, as there is no essential difference in the mode of making the observations or of reducing them.



Let AB be the line-measure divided into equal parts as nearly as may be at the points 1, 2, 3, ... The problem is to find the corrections to the graduation marks at the several points of division. In making the necessary observations, microscopes are mounted over the points 0 and 1 and readings made. The bar is then moved along until the points 1,2; 2,3,... are under the microscopes and readings are made each time. For the purpose of facilitating the work an extra scale CD may be employed and comparisons made between it and each of the spaces 0 1, 1 2, ... in succession. The correct's then are the amounts by which the marks should be changed in order to be in their true positions. When the corrections have been applied to the distances 0 1, 1 2,... these distances are all reduced to the same length.

With a thermometer AB, if 0, 1, 2, ... denote graduation marks, then by breaking off a column of mercury approximately equal in length to the distance 0 1, we may by reading the ends of this column when in the positions 0 1, 1 2,... determine the degree of uniformity of the bore of the tube, whence may be derived at the several points the *calibration corrections*, as they are called. The corrections to the graduation marks for calibration when applied to the observed values of the spaces 0 1, 1 2,... change them to what would have been found had the tube been of uniform bore throughout its whole length.

2. As the precision with which comparisons of line-measures can now be made is very great, it is fitting that the reduct'n of such comparisons should be carried out by the method of least-squares. With thermometers however such niceness of reading has not yet been attained and approximate methods of reduction may consequently be used to advantage. By such methods results will be found agreeing as closely with those found by the rigorous method as the quality of the observations will warrant. In investigating the reliance to be placed on approximate methods we must understand the rigorous method to see just what liberties we do take.

3. In a line-measure AB, let the spaces 0 1, 1 2, . . . be compared with the auxiliary space CD and let x_0, x_1, . . . be the graduation corrections at the points 0, 1, . . . Then, for the first comparison, if R_0, R_1 be the microscope readings at C, D and M_0, M_1 be the readings at 0, 1 we have

$$(M_1+x_1)-(M_0+x_0)-[(R_1-R_0)+y]=v,$$

where y is the amount by which CD differs from the space intended to be represented by CD and v is the residual error of observation.

Hence, for simplicity taking 4 spaces only, the observation equations may be written

$$\left.\begin{array}{r} x_1-x_0-y-(0\,1) = v_1 \\ x_2-x_1 \quad -y-(1\,2) = v_2 \\ x_3-x_2 \qquad -y-(2\,3) = v_3 \\ x_4-x_3 \qquad\quad -y-(3\,4) = v_4 \end{array}\right\}, \qquad (1)$$

where $(0\,1) = -(M_1-M_0)+(R_1-R_0)$

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As these equations contain 6 unknowns and are themselves only four in number, the number of unknowns must be arbitrarily reduced in order to carry out the solution by the method of least squares. The simplest supposition is to make $x_0=0$, $x_4=0$, $y=$ a known quantity. All of the readings being supposed to have been equally well made and the observation equations to be of the same weight, we have the normal equations

$$\left.\begin{array}{r} 2x_1-x_2 \qquad = (0\,1)'-(1\,2)' \\ -x_1+2x_2-x_3 = (1\,2)'-(2\,3)' \\ -x_2+2x_3 = (2\,3)'-(3\,4)' \end{array}\right\}, \qquad (2)$$

from which the unknowns are found.

Writing $[al]$, $[bl]$, $[cl]$ for the constant terms of the normal equations and solving

$$\left.\begin{array}{l} x_1 = \tfrac{1}{4}(3[al]+2[bl]+[cl]) \\ x_2 = \tfrac{1}{4}(2[al]+4[bl]+2[cl]) \\ x_3 = \tfrac{1}{4}([al]+2[bl]+3[cl]) \end{array}\right\} \qquad (3)$$

Hence weight of $x_1 = \frac{4}{3}$, of $x_2 = \frac{4}{4}$, of $x_3 = \frac{4}{3}$.

4. Generally if the number of spaces is n,

$$\begin{aligned}
n.x_1 &= (n-1)[al] + (n-2)[bl] + (n-3)[cl] + \cdots \\
n.x_2 &= (n-2)[al] + 2(n-2)[bl] + 2(n-3)[cl] + \cdots \\
n.x_3 &= (n-3)[al] + 2(n-2)[bl] + 3(n-3)[cl] + \cdots \\
&\cdots
\end{aligned}$$

Hence the weight of any correction as x_m is

$$\frac{n}{m(n-m)}.$$

Since this expression remains unchanged when $n-m$ is written for m, and is a minimum when $m = \frac{1}{2}n$, it follows that for graduation errors equally distant from the ends the weights are equal, and the weight decreases as we approach the center where it is a minimum.

5. It is in accordance with general experience and the principles of least squares that instead of spending all of the time of observation on the direct measurement of the spaces 0 1, 1 2, . . . better results would be obtained by spending part of it in reading combinations of these spaces as 0 2, 2 4, . . .; 0 3, 1 4, . . . Let then, for uniformity, each space and all possible combinations of spaces be equally well read. This will require instead of a single auxiliary space CD an auxiliary bar divided as nearly as possible the same as the bar AB. The observation equations therefore when written in full are

$$\left.\begin{array}{rrrrrrrrrl}
 & & & x_1 & -x_0 & -y_1 & & & & -(0\,1) = v_1 \\
 & & x_2 & -x_1 & & -y_1 & & & & -(1\,2) = v_2 \\
 & x_3 & -x_2 & & & -y_1 & & & & -(2\,3) = v_3 \\
x_4 & -x_3 & & & & -y_1 & & & & -(3\,4) = v_4 \\
 & & x_2 & & -x_0 & & -y_2 & & & -(0\,2) = v_5 \\
 & x_3 & & -x_1 & & & -y_2 & & & -(1\,3) = v_6 \\
x_4 & & -x_2 & & & & -y_2 & & & -(2\,4) = v_7 \\
 & x_3 & & & -x_0 & & & -y_3 & & -(0\,3) = v_8 \\
x_4 & & & -x_1 & & & & -y_3 & & -(1\,4) = v_9 \\
x_4 & & & & -x_0 & & & & -y_4 & -(0\,4) = v_{10}
\end{array}\right\} \quad (4)$$

10 equations with 9 unknowns.

6. Two cases may arise. In the first place the values of the spaces on the auxiliary bar CD may be known in terms of the accepted standard. We had an example of this on the Lake Survey when the $\frac{1}{2}$ mm. spaces on the Repsold Meter were compared with spaces on the Troughton & Simms Inch. The values of the spaces on the inch had been previously determined.

In this case then y_1, y_2, y_3, y_4 are known and the observation equations may be written more simply

$$\left.\begin{array}{r}
x_1-x_0-(0\,1) = v_1 \\
x_2-x_1 \quad -(1\,2) = v_2 \\
x_3-x_2 \qquad -(2\,3) = v_3 \\
x_4-x_3 \qquad\quad -(3\,4) = v_4 \\
x_2 \quad -x_0-(0\,2) = v_5 \\
x_3 \quad -x_1 \quad -(1\,3) = v_6 \\
x_4 \quad -x_2 \qquad -(2\,4) = v_7 \\
x_3 \quad -x_0-(0\,3) = v_8 \\
x_4 \quad -x_1 \quad -(1\,4) = v_9 \\
x_4 \quad -x_0-(0\,4) = v_{10}
\end{array}\right\}. \tag{5}$$

The normal equations are

$$\left.\begin{array}{rcl}
4x_0-x_1-x_2-x_3-x_4 & = & -(0\,1)-(0\,2)-(0\,3)-(0\,4) \\
4x_1-x_2-x_3-x_4 & = & (0\,1)-(1\,2)-(1\,3)-(1\,4) \\
4x_2-x_3-x_4 & = & (0\,2)+(1\,2)-(2\,3)-(2\,4) \\
4x_3-x_4 & = & (0\,3)+(1\,3)+(2\,3)-(3\,4) \\
4x_4 & = & (0\,4)+(1\,4)+(2\,4)+(3\,4)
\end{array}\right\} \tag{6}$$

Adding these equations there results

$$0 = 0.$$

The reason of the indeterminate form is that neither the interval nor the terminal point has been fixed. Consequently any arbitrary relation may be assumed between the corrections.

If as above we put $x_0=0$ and $x_4=0$ in the observation equations then the normal equations become

$$\left.\begin{array}{rcl}
4x_1-x_2-x_3 & = & (0\,1)-(1\,2)-(1\,3)-(1\,4) \\
4x_2-x_3 & = & (0\,2)+(1\,2)-(2\,3)-(2\,4) \\
4x_3 & = & (0\,3)+(1\,3)+(2\,3)-(3\,4)
\end{array}\right\}, \tag{7}$$

from which x_1, x_2, x_3 may be found.

However, by assuming the arbitrary relation

$$x_0+x_1+x_2+x_3+x_4 = 0 \tag{a}$$

between the corrections we obtain a much more simple solution. For by adding this relation to each of the normal equations (6) we have

$$\left.\begin{array}{rcll}
5x_0 & = & -(0\,1)-(0\,2)-(0\,3)-(0\,4) = [al] & \text{suppose} \\
5x_1 & = & (0\,1)-(1\,2)-(1\,3)-(1\,4) = [bl] & \text{"} \\
5x_2 & = & (0\,2)+(1\,2)-(2\,3)-(2\,4) = [cl] & \text{"} \\
5x_3 & = & (0\,3)+(1\,3)+(2\,3)-(3\,4) = [dl] & \text{"} \\
5x_4 & = & (0\,4)+(1\,4)+(2\,4)+(3\,4) = [el] & \text{"}
\end{array}\right\}. \tag{8}$$

The whole solution may therefore be conveniently arranged in tabular form. The sums of the horizontal rows are first formed and then plaoed in the proper vertical columns with the signs changed. The vertical columns are next added.

0	1	2	3	4	
	(0 1)	(0 2) (1 2)	(0 3) (1 3) (2 3)	(0 4) (1 4) (2 4) (3 4)	Sum_1 Sum_2 Sum_3 Sum_4
$-\mathrm{Sum}_1$	$-\mathrm{Sum}_2$	$-\mathrm{Sum}_3$	$-\mathrm{Sum}_4$		
$[al]$	$[bl]$	$[cl]$	$[dl]$	$[el]$	

7. The preceding method of transforming the normal equations (6) into (8) by means of the arbitrary relation (a) is allowable. For since the sum of the coefficients of the unknowns in each of equations (6) is zero, whatever values of $x_0, x_1, \ldots$ satisfy these equations, the values $x_0+\lambda, x_1+\lambda \ldots$ where λ is any constant will also satisfy them. Hence whatever set of values is taken to satisfy the equations the differences $x_1-x_0, x_2-x_0, \ldots$ will be the same. Therefore by arbitrarily fixing the initial point of graduation determinate values are found for the corrections at the other points.

8. In the second place if, as is usually the case, the values of the spaces on the auxiliary bar are unknown the last of equations (4) becomes without meaning and must be omitted. This gives 9 observation equations involving 8 unknowns from which equations, as the initial graduation point is not fixed, the normal equations resulting would be indeterminate. Putting $x_0 = 0$ and $x_4 = 0$ we have 9 equations and 6 unknowns from which follow the normal equations

$$\left.\begin{array}{rrrrrrl} +4x_1 & -x_2 & -x_3 & & +y_2 & +y_3 & =[al] \\ -x_1 & +4x_2 & -x_3 & & & & =[bl] \\ -x_1 & -x_2 & +4x_3 & & -y_2 & -y_3 & =[cl] \\ & & & 4y_1 & & & =[dl] \\ +x_1 & & -x_3 & & +3y_2 & & =[el] \\ +x_1 & & -x_3 & & & +2y_3 & =[fl] \end{array}\right\}, \tag{9}$$

where $$[al] = (0\,1)-(1\,2)-(1\,3)-(1\,4)$$

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This is a more general solution of the problem.

The elimination of the normal equations from (4) may be considerably shortened by an artifice employed by Hansen in his treatise *Von der Bestimmung der Theilungsfehler eines gradlinigen Maassstabes*, but the amount of work is still very considerable.

9. The Precision of the values of the Unknowns, x_1, x_2, x_3.

Eliminating y_2, y_3 from the normal equations (9) the resulting equations may be written

$$\left.\begin{aligned} 3\tfrac{1}{6}x_1 - x_2 - \tfrac{1}{6}x_3 &= [al]' \\ 4x_2 - x_3 &= [bl]' \\ +3\tfrac{1}{6}x_3 &= [cl]' \end{aligned}\right\}. \tag{10}$$

If the solution is finished by the Gaussian method of elimination the weights of the unknowns will be found to be

$$\text{weight of } x_1 = \tfrac{20}{7}, \text{ of } x_2 = \tfrac{10}{3}, \text{ of } x_3 = \tfrac{20}{7}.$$

Generally, if the number of graduation marks is $n+1$ the weights of the corrections to the two outside marks will be found to be $n(n+1) \div (2n-1)$ and of the center marks $\tfrac{2}{3}(n+1)$, where n is an odd number.

The p. e. of an observ'n of weight unity is found from the usual formula

$$r = .6745\sqrt{\frac{[vv]}{\text{no. of obs.—no. of indp't unknowns}}}$$

$$= .6745\sqrt{\frac{[vv]}{\frac{1}{2}(n-1)(n-2)}}, \tag{11}$$

where v is the residual error of observation.

Hence the p. e. of x_1, x_2, x_3 are known, since their weights are known.

10. The solution of equations (9) is quite complicated. The amount of labor increases very rapidly with the increase of the number of spaces read on. Accordingly whenever it is not possible to make use of the simple and symmetrical form (8), an approximate form of solution should be employed. This is more especially true of thermometers where the carrying out of the rigorous solution would entail an expenditure of labor altogether out of proportion to the increased gain in accuracy. "To devote punctilious attention to hundredths of a thermometric degree in results derived from data which are uncertain by ten or twenty times that amount is to 'strain at a gnat and swallow a camel.'" (Amer. Jour. of Science, Feb., 1882.)

Instead of finding the corrections to the graduation marks directly, the corrections to the several spaces between the graduation marks may first be found and thence the cerrections to the marks at the ends of the spaces. Thus if $z_1, z_2, \ldots$ denote the corrections to the spaces, then taking our example of four spaces

$$\left.\begin{aligned} z_1 &= x_1 - x_0 \\ z_2 &= x_2 - x_1 \\ z_3 &= x_3 - x_2 \\ z_4 &= x_4 - x_3 \end{aligned}\right\}. \tag{12}$$

Hence by addition

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 &= x_4 - x_0 \\ &= 0. \end{aligned}$$

From the observation equations (4) we have by subtracting in pairs

$$\left.\begin{array}{l} z_2-z_1-[(1\,2)-(0\,1)] = v^{\mathrm{i}} \\ z_3-z_2\quad-[(2\,3)-(1\,2)] = v^{\mathrm{ii}} \\ z_4-z_3\quad-[(3\,4)-(2\,3)] = v^{\mathrm{iii}} \\ z_3\quad-z_1-[(1\,3)-(0\,2)] = v^{\mathrm{iv}} \\ z_4\quad-z_2\quad-[(2\,4)-(1\,3)] = v^{\mathrm{v}} \\ z_4\quad-z_1-[(1\,4)-(0\,3)] = v^{\mathrm{vi}} \end{array}\right\}. \tag{13}$$

Hence considering (1 2)—(0 1), (2 3)—(1 2), ... as the independently observed quantities (which is not correct as they are entangled), and making the sum of the squares of v^{i}, v^{ii}, ... a minimum we have the normal eq'ns

$$\left.\begin{array}{r} 3z_1-z_2-z_3-z_4=-[(1\,2)-(0\,1)]-[(1\,3)-(0\,2)]-[(1\,4)-(0\,3)] \\ 3z_2-z_3-z_4=+[(1\,2)-(0\,1)]-[(2\,3)-(1\,2)]-[(2\,4)-(1\,3)] \\ 3z_3-z_4=\quad[(2\,3)-(1\,2)]-[(3\,4)-(2\,3)]-[(1\,3)-(0\,2)] \\ 3z_4=\quad[(3\,4)-(2\,3)]+[(2\,4)-(1\,3)]+[(1\,4)-(0\,3)] \end{array}\right\} \tag{14}$$

which when added give $0 = 0$ identically.

But $$z_1 + z_2 + z_3 + z_4 = 0.$$

Hence adding this relation to each of the normal equations we have the values of the unknowns. Thus

$$4z_1 = -[(1\,2)-(0\,1)]-[(1\,3)-(0\,2)]-[(1\,4)-(0\,3)]$$

- - - - - - - - - - - - - (15)

This form of approximation when applied to thermometers is known as Neumann's method. It is quite analogous to the system used by the Prussian Landestriangulation in the adjustment of horizontal angles and the principle involved is therefore really due to Gauss and Hansen.

For convenience the computation of the corrections may be arranged in tabular form as follows:

z_1	z_2	z_3	z_4	
	+(12)—(01)	+(13)—(02)	+(14)—(03)	Sum_1
		+(23)—(12)	+(24)—(13)	Sum_1
			+(34)—(23)	Sum_3
$-\mathrm{Sum}_1$	$-\mathrm{Sum}_2$	$-\mathrm{Sum}_3$		
$4z_1$	$4z_2$	$4z_3$	$4z_4$	

The corrections x_1, x_2, x_3 are then found from the relations

$$\left.\begin{array}{l} x_1 = z_1 \\ x_2 = x_1+z_2 \\ x_3 = x_2+z_3 \end{array}\right\} \tag{16}$$

From equations (15) and (16)

$$\left.\begin{array}{l} 4x_1 = -(12-01)-(13-02)-(14-03) \\ 4x_2 = -(13-02)-(14-03)-(23-12)-(24-13) \\ 4x_3 = -(14-03)-(24-13)-(34-23) \end{array}\right\} \tag{17}$$

11. **The Precision.**—The p. e. of the unit of weight is given by

$$r = .6745\sqrt{\frac{[v'v']}{\frac{1}{2}(n-1)(n-2)}}, \tag{18}$$

since the number of equations $= \frac{1}{2}n(n-1)$ and the number of independent unknowns $= n-1$, the number of spaces being n.

Also since z_1, z_2, z_3 result from the normal equations with the same w't they have the same p. e. From equations (17) it is evident that the p. e. of x_1 and x_3 is the same as that of z_1 but the p. e. of x_2 is $\sqrt{\frac{4}{3}}$ as great.

Generally when the number of spaces is an even number n, the p. e. of x_1 is to the p. e. of the center graduation mark as $\sqrt{(n-1)} : \frac{1}{2}n$.

Example.—The following were the observed values of the lengths of the 45°, 90° and 135° columns of Thermometer Green 4470, made to determine the calibration corrections at the 77°, 122° and 167° points. (Amer. Jour. Science, Vol. XXI, p. 374.)

45° col.	90° col.	135° col.
44.68	90.07	134.61
.71	.09	.68
.72	.11	
.75		

The corrections at the points 32° and 212° are assumed to be zero.

First Solution. By the method of Least Squares. The observation eq's are with the preceding notation, x_1, x_2, x_3 denoting the corrections at the 77°, 122° and 167° points respectively,

$$\begin{array}{rrrrrrl}
 & & x_1 & -y_1 & & -0.32 & = 0 \\
 & x_2 & -x_1 & -y_1 & & -0.29 & = 0 \\
x_3 & -x_2 & & -y_1 & & -0.28 & = 0 \\
-x_3 & & & -y_1 & & -0.25 & = 0 \\
 & +x_2 & & & -y_2 & +0.07 & = 0 \\
x_3 & & -x_1 & & -y_2 & +0.09 & = 0 \\
 & -x_2 & & & -y_2 & +0.11 & = 0 \\
x_3 & & & & -y_3 & -0.39 & = 0 \\
 & & -x_1 & & -y_3 & -0.32 & = 0
\end{array}$$

The Normal equations are

$$\begin{array}{rcl}
4x_1 - x_2 - x_3 + y_2 + y_3 & = & 0.33 \\
-x_1 + 4x_2 - x_3 & = & 0.05 \\
-x_1 - x_2 + 4x_3 - y_2 - y_3 & = & -0.20 \\
4y_1 & = & -1.14 \\
x_1 \qquad - x_3 + 3y_2 & = & 0.27 \\
x_3 \qquad - x_3 \qquad + 2y_3 & = & -0.71
\end{array}$$

Solving we find

$$x_1 = +0.031 \qquad y_1 = -0.285$$
$$x_2 = +0.028 \qquad y_2 = 0.090$$
$$x_3 = +0.031 \qquad y_3 = -0.355$$

Substitute these values in the observation equations and the residuals result. Squaring and adding,

$$[vv] = 0.0003.$$

Hence for the p. e. of a single observation

$$r = .6745\sqrt{\tfrac{.0003}{3}} = \pm 0.0007.$$

$$\therefore \text{ p. e. of } x_1 = .007\sqrt{.35} = 0^\circ.004$$
$$\text{" \quad " } x_2 = .007\sqrt{.3} = 0^\circ.004$$
$$\text{" \quad " } x_3 = .007\sqrt{.35} = 0^\circ.004$$

Second Solution. By Neumann's formula—Arranging according to table

z_1	z_2	z_3	z_4	Sums
	—.03	—.02	—.07	—.12
		—.01	—.02	—.03
			—.03	—.03
+.12	+.03	+.03		
+.12	00	00	—.12	

$$\therefore z_1 = +0^\circ.03 \qquad z_3 = +0^\circ.00$$
$$z_2 = +0^\circ.00 \qquad z_4 = -0^\circ.03$$

$$\therefore x_1 = z_1 = 0^\circ.03$$
$$x_2 = x_1 + z_2 = 0\ .03$$
$$x_3 = x_2 + z_3 = 0\ .03$$

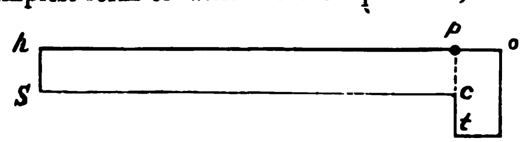
Also $$r = .6745\sqrt{\tfrac{.0004}{3}} = \pm 0.007$$

$$\text{and p. e. of } x_1 = .007\sqrt{\tfrac{1}{3}} = \pm 0^\circ.004$$
$$\text{" \quad " } x_2 = .007\sqrt{\tfrac{1}{4}} = \pm 0\ .004$$
$$\text{" \quad " } x_3 = .007\sqrt{\tfrac{1}{3}} = \pm 0\ .004.$$

THE MULTISECTION OF ANGLES.

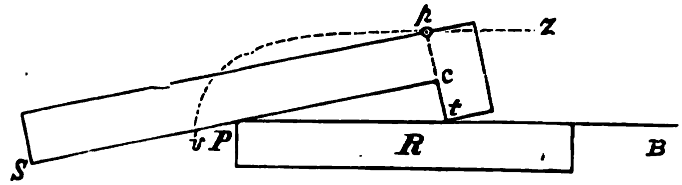
BY PROF. J. W. NICHOLSON, A. M., LA. STATE UNIV., BATON ROUGE, LA.

To divide an angle into any number of equal parts, we use an instrument called the *multisector*, the simplest form of which is the square *sot;* which consists of two perpendicular arms *os* and *ct*, the length (*ct*) of the latter being equal to the width (*cp*)



of the former. At p, the intersection of ro and the prolongation of tc, is an attachment for carrying a pencil.

THE POLYODE.—If we place the multisector on the rule R, or on the line AB, as in the diag'm, and slide it along on the rule or line so that sc will continua'y pass



through the point P, while t continues along the line BP, the pencil at p will describe the curve zpv, which is called a *polyode*, from *πολυσ*, many, and *οδοσ*, path.

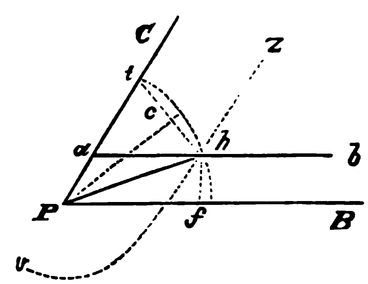
The line PB, along which the multisector moves, is the *directrix*, and may be either straight or curved.

The point P, through which the side sc continually passes is the *pole*, and may be either on or without the directrix.

The line, or distance, ct or cp, is the *modulus* of the multisector.

Let us now apply the multisector, or poliode, to the dividing of an angle into any number of equal parts.—1st, *To trisect an angle.*

Let CPB be the given angle. Draw ab parallel to PB and at a distance from it equal to the modulus of the multisector. With PC as a directrix, and P as a pole, describe the polyode vz cutting ab at p. Draw Pp, and we shall have



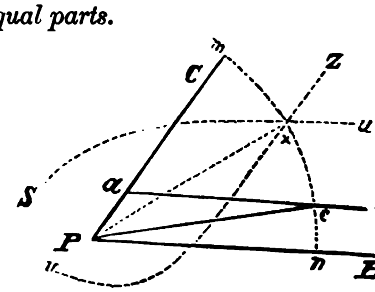
$$\angle BPp = \tfrac{1}{3}\angle BPC.$$

For, evidently

$$\angle tPp = 2\angle pPf.$$

2nd. *To divide an angle into five equal parts.*

Let CPB be the given angle. Draw ab and vz as in the preceding solution. With ab as a directrix and P as a pole, describe the polyode su cutting vz in x. From P as a center, with a radius $= Px$, describe the arc mn cuttirg ab at e. Draw Pe, and we shall have

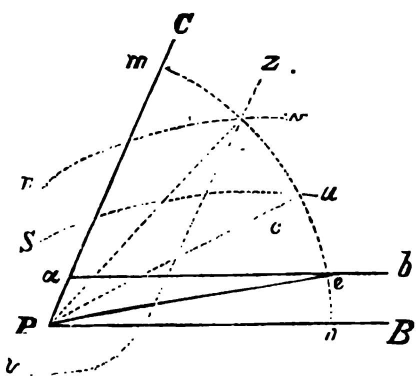


$$\angle BPe = \tfrac{1}{5}\angle BPC.$$

For, evidently ang. mPx = ang. xPe = 2ang. ePn.

3rd. *To divide an angle into seven equal parts.*

Let CPB be the given angle. Draw ab, rz and su as in the preceding solution. With su as a directrix, and P as a pole, describe the polyode rw cutting rz in x. From P as a center, with a radius $= Px$, describe the arc mn cutting ab at e, and su at c. Draw Pe and we sh'l evidently have



ang. $BPe = \frac{1}{7}$ ang. BPC.

In a similar manner we may const.

ang. $BPe = \frac{1}{2n+1}$ ang. BPC.

Equation of the Polyode.—The equation of the polyode evidently depends on the nature of thr directrix and the position of the pole; and hence may have an indefinite number of forms. Denoting the coordinates of the pole by a and b, the modulus by m, and the equation of the directrix by

$$y_1 = f(x_1), \qquad (1)$$

the eq. of the polyode may be found by combin'g (1) with the foll. relat'ns;

$$(y-b)^2+(y-a)^2 = (y_1-b)^2+(x_1-a)^2 \,..\, (2), \quad (y-y_1)^2+(x_1-x)^2 = 4m^2. \; (3)$$

Combining (2) and (3), we find, say,

$$y_1 = \varphi(x, y), \qquad x_1 = \theta(x, y).$$

Substituting, $$\varphi(x, y) = f[\theta(x, y)]. \qquad (4).$$

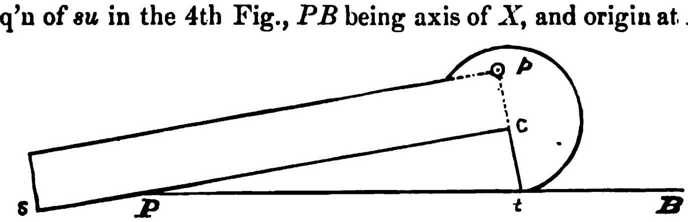
If the directrix coincides with the axis of X, or (1) is of the form $y_1 = 0x_1$, and $a = b = 0$, (4) becomes $x^2+y^2-x(x^2+y^2)^{\frac{1}{2}} = 2m^2$, or $x(4m^2-y^2)^{\frac{1}{2}} = 2m^2-y^2$; which is the equation of rz in the 2nd Fig. above, regarding PB as the axis of X, and P the origin. This is the *simplest* form of the polyode, and is identically the same curve as that so ingeniously empl'd by Dr. Hillhouse in trisectig an angle (Analyst, Vol. IX, No. 6).

Again if (1) is of the form $y_1 = 0x_1 + m$, and $a = b = 0$, (4) becomes

$$(y-m)^2+[\sqrt{(y^2+x^2-m^2)}-x]^2 = 4m^2;$$

which is the eq'n of su in the 4th Fig., PB being axis of X, and origin at P

A multis'r of this model may be made of pasteboard and used to advantage in the class-r'm.



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CORRESPONDENCE.

Editor Analyst:

OWING to absence from home, the answers published in the ANALYST in July 1881, to a query proposed by me escaped my notice until quite recently. I trust the subject will be found to possess sufficient interest for your readers to justify a recurrence to it. The query was as follows:—

"Let $u = \dfrac{\sin ax}{a}$. Now if $a = \infty$, $u = 0$ independently of x, therefore we should have $\dfrac{du}{dx} = 0$, when $a = \infty$. But $\dfrac{du}{dx} = \cos ax$, which is essentially indeterminate when $a = \infty$. What is the explanation of this paradox?"

Prof. Barbour, Mr. Adcock and Prof. Judson replied on page 129, ANALYST, Vol. VIII. Prof. Barbour does not seem to have considered the paradox as intended; for he does not make a infinite, but merely points out that $u = 0$ whenever ax is a multiple of 2π, while $\dfrac{du}{dx} = 1$, and remarks that u may $= 0$ while $\dfrac{du}{dx} = 1$. This is of course true, but the paradox consists in the fact that $u = 0$ for *all* values of x in the case considered; hence u being a constant so far as x is concerned, we should expect to find $\dfrac{du}{dx} = 0$ for all values of x.

Mr. Adcock remarks that "when $u = 0$ independently of x, it is not a function of x, and *therefore cannot be differentiated with respect to x*. Therefore the value of $du \div dx$ is $\cos ax$ independently of the value of a." I do not understand Mr. Adcock to mean by the words which I have italicized that $du \div dx$ ceases to have a meaning when u ceases to be a function of x. The natural conclusion would seem to be that the value of $du \div dx$ which is ordinarily a function of x, should become zero when u ceases to be a function of x. For example, $u = a^x$ ceases to be a function of x (at least for all finite values of x) when $a = 1$, viz., it takes the value 1 independently of x; accordingly we find that $\dfrac{du}{dx} = \log a \,.\, a^x$ becomes zero when $a = 1$. Assuming the a at the end of the last sentence to be misprinted for x, Mr. Adcock seems to draw only the conclusion that the value of $du \div dx$ should become independent of x; but he does not deny that the form assumed, namely $\cos \infty$, is essentially indeterminate, and not, as in the case instanced above, always equal to zero.

Prof. Judson, on the other hand, says "If $u=0$ independently of x, then u is not a function of x, and $du \div dx$ is without meaning." It is not necessary to discuss this point, for we already have an expression for the value of $du \div dx$, and the matter in hand is to account for the fact that this expression does not assume the value zero under the circumstances, as would naturally be expected. Prof. Judson, however, goes on to say "If a is a constant, then a cannot $=\infty$. If a is a variable independent of x, and $a=\infty$, i. e., a increases without limit, then $(\sin ax) \div a$ is an infinitesimal (not$=0$) *and* u *is therefore indeterminate*; $du \div dx$ is also indeterminate, and there is no paradox." It is of course sufficient to regard a as independent of x, and no objection will be made to Prof. Judson's phraseology. I have however italicized one clause for the sake of the remark that, if we admit that an infinitesimal is in a certain sense indeterminate, it is not necessarily or usually so in a sense that would imply that (as in the case of an indeterminate *finite* quantity) its derivative should admit of finite values. Wherein does this case differ from that of $u=a^x$, already mentioned? Not in the fact that u is infinitesimal for we might have taken $u=\dfrac{a+\sin ax}{a}$ whose value approaches indefinitely to 1 as a incereases without limit; nor in the fact that the critical case occurs when a (in the ordinary phraseology)$=\infty$; for we might have written $u=(a-1)\sin\dfrac{x}{a-1}$, and the critical case would have occurred when $a=1$.

If u be made the ordinate of a curve of which x is the abscissa, then for any finite value of a the curve is a sinusoid, and as a increases the waves become smaller both in length and amplitude. As a increases without limit, the curve approaches indefinitely to the axis of x, whose equation is $u=0$, whence we should naturally expect that $du \div dx$ would be zero, but since the waves do not change their shape when they become infinitesimal, we find that $du \div dx$ still admits of the same values as when a is finite, viz., all values between $+1$ and -1.

The analytical difficulty is connected with the occurrence of the form $\cos\infty$. I have not hesitated in the statement of the paradox to regard this as an essentially indeterminate form. In an interesting memoir "On the Sine and Cosine of an Infinite Angle", *Cambridge Philosophical Transactions*, Vol. VIII, p. 255, Mr. S. Earnshaw contends for the indeterminateness of these forms in opposition to writers upon Definite Integrals, who, he says, while admitting "that when x becomes infinite $\sin x$ and $\cos x$ cannot be said to be in one part of their periodicity rather than another" yet agree in "practically affirming that both the sine and cosine of an infinite angle are

equal to zero." In this memoir Mr. Earnshaw points out the falacies in the proofs that have been given of the equation $\cos\infty = 0$. In the same vol., De Morgan had, at page 191, given reasons why we should expect periodic functions, when indeterminate, to be represented by their mean values, and remarks that the indeterminate symbols, $\sin\infty$ and $\cos\infty$, are found in numberless cases to represent, each of them, 0, the mean value of both $\sin x$ and $\cos x$. Mr. J. W. L. Glaisher also discusses this question in the 5th volume of the first series of the *Messenger of Mathematics* with a view of determining the conditions under which these expressions may be taken equal to zero; or more generally, under which a periodic function may be assumed equal (when x is infinite) to its mean value, or φ being a rational function,

$$\varphi(\sin\infty, \cos\infty) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\sin x, \cos x)dx.$$

The paradox is not without interest inasmuch as the geometrical illustration renders it evident that the effective value of $du \div dx$ ought to coincide with its mean value.

In this connection I may also mention that the difficulty arising in the application of the theorem

$$\frac{f(x)}{\varphi(x)} = \frac{f'(x)}{f'(x)} \text{ to the example } \frac{x - \sin x}{x + \cos x},$$

when $f(x) = \infty$, $\varphi(x) = \infty$ when $x = \infty$, given in Bertrand's Calcul Differential, p. 476 (quoted in Rice and Johnson's Calculus, p. 114), disappears if we admit that $\sin\infty$ and $\cos\infty$ are each equal zero.

W. W. JOHNSON.

Annapolis, Md. Nov. 28, 1882.

GEOMETRICAL DETERMINATION OF THE SOLIDITY OF THE PARABOLA.

BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLAND.

LET AD be a parbola inscribed in the parallelogram $ALDR$. Suppose RCB to be one of an infinite number of inscribed triangles. Through C and B draw Mm and NK respectively parallel to RA, and draw CPH perpendicular to NK. Through E, the middle point of CB, draw the taugent ST to meet RS a perpendicular to it in S; also through E draw Oo parallel to Mm, and VUF perpendicular to RA.

From the similar triangles PCB and SRT we have

$$PC : CB :: RS : RT.$$

$$\therefore PC \times RT = RS \times CB.$$

The area of the triangle RCB $= \frac{1}{2}(CB \times RS) = \frac{1}{2}(PC \times RT)$.

Since $RT = RF + FT$,

$PC.RT = PC.RF + PC.FT$. (1)

Since $RF = OE$, and FT being subtangent $= 2AF = 2Eo$;

$\therefore PC \times RF = VU \times OE$, (2)

and $PC \times FT = VU.2Eo$. (3)

From (1), (2) and (3) we get, by substitution,

$$\text{area } RCB = \tfrac{1}{2}(VU \times OE) + VU \times Eo \qquad (4)$$

If the triangle RCB be revolved about RA as an axis it will generate a volume $= \frac{2}{3}\pi.2EF.RCB$, or by substituting from (4),

$$= \tfrac{2}{3}\pi \times EF \times VU \times OE + \tfrac{2}{3}\pi \times 2EF \times VU \times Eo. \qquad (5)$$

But $2\pi.EF.VU =$ the area generated by revolving the line VU with radius EF around RA as an axis, which if multiplied by OE will equal the solidity of a hollow cylinder formed by revolving $MNUV$ around RA with radius EF. And $2\pi.EF.VU$, if multiplied by Eo will equal the solidity of the hollow cylinder formed by revolving $VUKm$ around RA, with radius EF. Hence the volume generated by the revolution of the trian. RCB around $RA = \frac{2}{3}$ of half the volume generated by $MNUV + \frac{2}{3}$ of the volume generated by $VUKm$, when revolved about the same axis RA. (6)

Since the bases of the inscribed triangles are infinitely small, the sum of the volumes generated by $MNUV$ equals the paraboloid, which call P; and the sum of the volumes generated by $VUKm$ equals the solid formed by the revolution of $DEAL$ about RA, which call S.

Hence by (6) we have for the sum of the volumes generated by RCB, or

$$P = \tfrac{2}{3}(\tfrac{1}{2}P + S); \therefore P = S.$$

Since $S + P =$ the volume of a cylinder circumscribed about the paraboloid, therefore the volume of a paraboloid is one-half the volume of its circumscribing cylinder, the axis of abscissas being the axis of revolution.

2. Let the parabola revolve about RD as an axis. Then the volume generated by the revolution of RCB about $RD = \frac{2}{3}\pi.2EO.RCB$. (7)

Now $2EO = 2AR - 2AF$ (8), and by (4) $RCB = \frac{1}{2}VU.OE + VU.Eo$, $= \frac{1}{2}VU(RA - FA) + VU.FA$ (9). Substituting in (7) from (8) and (9) we get volume generated by the revolution of $RCB = \frac{2}{3}\pi(2RA - 2FA) \times [\frac{1}{2}VU(RA - FA) + VU.FA] = \frac{2}{3}\pi(2RA - 2FA)\frac{1}{2}(RA - FA)VU$

$$+ \tfrac{2}{3}\pi(2RA - 2FA)FA.VU$$

$$= \tfrac{2}{3}\pi(RA^2 - 2RA.FA + FA^2)VU + \tfrac{2}{3}\pi(2RA.FA - 2FA^2)VU$$

$$= \tfrac{2}{3}\pi(RA^2 - FA^2)VU = \tfrac{2}{3}\pi.RA^2.VU - \tfrac{2}{3}\pi.Eo^2.VU. \qquad (10)$$

Now $\pi.RA^2$ is the area of the base of a circumscribing cylinder the axis of which coincides with, and whose length is equal to, RD, which call C.

The factor $\pi.Eo^2$ in (10) represents the area of a circle, radius Eo and center on the line AL, and when multiplied by VU, and summed for all positions of the triangle RCB it represents the volume generated by the revolution of the area $AEDL$ about AL as an axis, which volume call S'.

Substituting in (10) we have $P' = \frac{2}{3}C - \frac{2}{3}S'$. (11)

Prolong IB to b. The revolution of $HIab$ about LA generates the vol. $\pi.2FA.HI.EF$ (12); while the rev. of $VUKm$ produces $\pi.FA^2.VU$. (13)

Since $EF.HI = 2FA.VU$ (see ANALYST, Vol. IX, p. 107); therefore by substitution (12) becomes $2\pi.FA.2FA.VU = 4\pi.FA^2.VU$. (14)

Comparing (13) and (14) (as $Eo = FA$) we see that the volume denoted by (12) = four times the volume denoted by (13); and as this relation is constant it follows that the volume generated by the revolution of the area $AEDR$ about $DL = 4S'$; therefore the circumscribing cylinder $C = 5S'$, and consequently $S' = \frac{1}{5}C$. Substituting this value of S' in (11) we have

$$P' = \tfrac{2}{3}C - \tfrac{2}{15}C = \tfrac{8}{15}C.$$

[If x and y represent the coordinates of the point R on the axis of the parabola, then is $\frac{3}{5}x$ the distance of its center of gravity from the vertex, A; and by the theorem of Guldinus we have $\frac{2}{3}xy \times \frac{4}{5}\pi x = \frac{8}{15}\pi x^2 y$ = the vol. generated by the revolution of the parabola about its limiting ordinate, y, $= \frac{8}{15}$ of the volume of a cylinder whose radius is x and altitude y. This agrees with the above result so ingeneously deduced by Mr. Mathiot.—Ed.]

NOTE BY HENRY HEATON.—To integrate $\dfrac{dx}{(a+b\tan x)^n}$, differentiate $\dfrac{1}{(a+b\tan x)^{n-1}}$ and we get

$$-\frac{(n-1)b\sec^2 x\,dx}{(a+b\tan x)^n} = -\frac{(n-1)(a^2+b^2)dx}{b(a+b)\tan x)^n} + \frac{2a(n-1)dx}{b(a+b\tan x)^{n-1}} - \frac{(n-1)dx}{b(a+b\tan x)^{n-2}}.$$

Hence

$$\int\frac{dx}{(a+b\tan x)^n} = \frac{-b}{(n-1)(a^2+b^2)(a+b\tan x)^{n-1}} + \frac{2a}{a^2+b^2}\int\frac{dx}{(a+b\tan x)^{n-1}} - \frac{1}{a^2+b^2}\int\frac{dx}{(a+b\tan x)^{n-2}}$$

In like manner, differentiating $\dfrac{\tan x}{(a+b\sec x)^{n-1}}$, we get

$$\int\frac{dx}{(a+b\sec x)^n} = \frac{-b^2\tan x}{a(n-1)(a^2-b^2)(a+b\sec x)^{n-1}} + \frac{(3n-2)a^2-(n-1)b^2}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-1}}$$
$$-\frac{3n-1}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-2}} + \frac{n}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-3}}.$$

ON THE VOLUME OF SOME SOLIDS.

From *Journal de Mathemtiques.*

TRANSLATED BY WILLIAM HOOVER, A. M., DAYTON, OHIO.

A BODY is comprised between two parallel planes, and a section made in this body by a plane parallel to the bases, at a distance x from the lower base, has an area equal to $\quad \alpha+\beta x+\gamma x^2+\omega x^3,$ (1)
$\alpha, \beta, \gamma, \omega$ being any four constant values, that is to say independent of x.

To determine the volume of this body.

Cut the body by an infinity of equidistant planes parallel to the given planes. In designating by $S_0, S_1, S_2 \ldots S_n$ the sections formed, S_0 and S_n designating the planes of the bases, the altitude h would be divided by these planes into n equal parts.

By virtue of the principle of Cavalieri (any body comprised between two parallel planes can be considered as the sum of an infinity of prisms or of cylinders of infinitely small altitude) the volume of the body considered would be expressed by

$$V=\frac{h}{n}\left(S_0+S_1+S_2\ldots+S_{n-1}\right), \text{ or } V=\frac{h}{n}\,\Sigma S. \tag{2}$$

This fixed, evaluate the sections $S_0, S_1, S_2 \ldots\ldots$ If $x=0$, $S_0=\alpha$,
$x=\frac{h}{n}$, $S_1=\alpha+\beta\frac{h}{n}+\gamma\frac{h^2}{n^2}+\omega\frac{h^3}{n^3}$, $x=\frac{2h}{n}$, $S_2=\alpha+\beta\frac{2h}{n}+\gamma\frac{4h^2}{n^2}+\omega\frac{8h^3}{n^3}$,

$$x=\frac{3h}{n}, \qquad S_3=\alpha+\beta\frac{3h}{n}+\gamma\frac{9h^2}{n^2}+\omega\frac{27h^3}{n^3},$$

- - - - - - - - - - - -

$$x=\frac{(n-1)h}{n}, S_{n-1}=\alpha+\beta\frac{(n-1)h}{n}+\gamma\frac{(n-1)^2h^2}{n^2}+\omega\frac{(n-1)^3h^3}{n^3};$$

thence $\Sigma S=n\alpha+\beta\frac{h}{n}[1+2+3..+(n-1)]+\gamma\frac{h^2}{n^2}[1^2+2^2+3^2..+(n-1)^2]$

$$+\omega\frac{h^3}{n^3}[1^3+2^3+3^3\ldots+(n-1)^3], = n\alpha+\beta\frac{h}{n}\left[\frac{n(n+1)}{2}\right]$$
$$+\gamma\frac{h^2}{n^2}\left[\frac{n(n+1)\,(2n+1)}{6}\right]+\omega\frac{h^3}{n^3}\left[\frac{n^2(n+1)^2}{4}\right].$$

Replacing in (2) ΣS by this value, we have

$$V=\alpha h+\beta\frac{h^2}{2}\left(1-\frac{1}{n}\right)+\gamma\frac{h^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\omega\frac{h^4}{4}\left(1-\frac{1}{n}\right)^2.$$

When n increases indefinitely, we have

$$V=\alpha h+\beta\,\tfrac{1}{2}h^2+\gamma\,\tfrac{1}{3}h^3+\omega\,\tfrac{1}{4}h^4. \tag{A}$$

REMARK.—If we designate by B, B_1 the upper and lower bases of a body, and by B_2 the section equidistant from the bases we have

$$x = 0,\ B = a,\ x = h,\ B_1 = a+\beta h+\gamma h^2+\omega h^3,$$
$$x = \tfrac{1}{2}h,\ B_2 = a + \beta\tfrac{1}{2}h + \gamma\tfrac{1}{4}h^2 + \omega\tfrac{1}{8}h^3;\ \text{whence}$$
$$4B_2 = 4a + 2\beta h + \gamma h^2 + \tfrac{1}{2}\omega h^3,\ \text{and consequently}$$
$$B+B_1+4B_2 = 6a + 3\beta h + 2\gamma h^2 + \tfrac{3}{2}\omega h^3,$$

which multiplied by $\frac{1}{6}h$ produces formula (A). We then have

$$V = \tfrac{1}{6}h(B + B_1 + 4B_2),\ \text{a known formula.}$$

For the Sphere.—If in (1) we make $a = 0$, $\beta = 2\pi R$, $\gamma = -\pi$, $\omega = 0$, that function reduces to

$$2\pi Rx-\pi x^2 = \pi x(2R-x),$$

which represents the area of a circle made in a sphere, at the altitude x.

In this case $B = 0$, $B_1 = 0$, $B_2 = \pi R^2$, $h = 2R$; thence

$$V = \tfrac{1}{3}R.4\pi R^2 = \tfrac{4}{3}\pi R^3.$$

For the spherical segment.—Making $B = 0$, $B_1 = \pi h(2R - h)$, $B_2 = \pi.\frac{1}{2}h(2R-\frac{1}{2}h)$, we have

$$V = \pi.\tfrac{1}{6}h^2(2R-h+4R-h) = \tfrac{1}{3}\pi h^2(3R-h).$$

For the cone.—If h is the altitude of a cone, R the radius of the base, the radius made at a height x is $R' = R-(R\div h)x$; whence

$$\pi R'^2 = \pi R^2-\frac{2\pi R^2}{h}x+\frac{\pi R^2}{h^2}x^2,$$

the expression to which (1) reduces itself when $a = \pi R^2$, $\beta = -2\pi R^2\div h$, $\gamma = \pi R^2\div h^2$, $\omega = 0$. Thence the cone $= \frac{1}{6}h(\pi R^2+\pi R^2) = \frac{1}{3}\pi R^2h$.

In the same way we obtain for the volume of the frustum of a cone

$$V = \tfrac{1}{6}h[\pi R^2+\pi R^2+\pi(r+R)^2] = \tfrac{1}{3}\pi h(R^2+r^2+Rr).$$

NOTE ON BILINEAR TANGENTIAL COORDINATES.

BY PROF. F. H. LOUD, COLORADO SPRINGS, COLORADO.

TANGENTIAL equations are usually written as homogeneous, and so analogous to trilinear equations; but interesting relations to the ordinary Cartesian coordinates are thus unnoticed. If the general equation of the first degree in two variables be divided through by its absolute term, and signs changed, it takes the form

$$px + qy - 1 = 0.$$

In the Cartesian system the two coefficients of this equation, p and q, are the reciprocals of the intercepts of its locus upon the two axes; and a recip-

rocal tangential interpretation will be put upon it if the same meanings be assigned to x and y, when p and q will take the significations of the Cartesian variables, viz., the distance of the point (the envelope denoted by the equation) from either axis in directions parallel in each case to the other axis. We are thus led to the system of bilinear tangential coordinates, wherein the two coordinates of a line are defined as the reciprocals of its intercepts on two given axes.

The origin of this system, being the envelope of all lines whose coordinates are infinite, takes the place which in the Cartesian system is occupied by the locus at infinity; and by the aid of this analogy it becomes possible to write the reciprocals of theorems involving measurement, as readily as those involving descriptive relations only.

For instance the equation of condition

$$AB' = A'B$$

which connects the coefficients of the Cartesian equations of parallels, will hold for the coefficients of the equations of points, if the latter are in a line with the origin. Points thus situated may therefore be spoken of as parallel to one another.

All formulæ relating to the Cartesian equations of right lines, which involve the angle ω of the axes, may be transcribed without the change of a letter as analogous formulæ for the equations of points, provided that ω in the tangential formulæ be defined to mean the angle between the positive direction of one axis and the negative direction of the other, and so to be the supplement of the angle denoted by the same letter in the Cartesian system. Thus two points are *perpendicular* to each other (their directions from the origin are at right angles), in case the coefficients of their equations fulfill the condition

$$AA' + BB' - (AB' + A'B)\cos\omega = 0.$$

In like manner, the formula for the tangent of the angle between the lines drawn from the origin to two points whose equations are given is the same as that which denotes the angle between two lines given by their Cartesian equations.

The quantity $[(x''-x')^2 + 2(x''-x')(y''-y')\cos\omega + (y''-y')^2]^{\frac{1}{2}}$, the familiar formula for the distance between two points, is found, when x', y', x'', y'' are the coordinates of lines, to denote a function of the position of the two lines in relation to the origin as well as to each other, which function, for analogy's sake, may be called the *punctual distance* of the two lines. It may be defined as the product of two ratios; first, that of the difference of the intercepts of the lines on an axis to the product of these intercepts; second, the ratio of the distance of the intersection of the lines from the

origin, to the distance of the same point (measured in a direction parallel to the other axis) from the axis formerly employed.

As an example of the reciprocation of metrical theorems, the following may be given; in which the left-hand theorem is the elementary proposition that the line parallel to the base of a triangle divides the sides proportionally.

If a line be drawn parallel to the base of a triangle, so as to cut the other sides, the ratio in which the linear distance on either side, between the vertex and the extremity of the base, is divided by the intersection of the parallel line, is equal to the ratio in which the other side is divided.

If a point be taken parallel to the vertex of a triangle, and joined to the extremities of the base, the ratio in which the punctual distance at either of these extremities, between the base and the adjacent side, is divided by the line drawn to the parallel point, is equal to the ratio in which the like distance is divided at the other ext'y.

In the subjoined figure, AX and AY are the axes, PMN the triangle, and Q the point parallel to P; the lines joining these four points two and two are produced to meet the axis of X, and the points M and N are joined both to the origin at A and (by lines parallel to AY) to M' and N' on the axis of X.

By writing out in full the values of the "punctual distances" referred to in the theorem, the proportion between them is put in the form:

$$\frac{A'B}{AA'.AB}\cdot\frac{AN}{N'N} : \frac{BC'}{AB.AC'}\cdot\frac{AN}{N'N} = \frac{A'C}{AA'.AC}\cdot\frac{AM}{M'M} : \frac{CB'}{AC.AB'}\cdot\frac{AM}{M'M}.$$

By cancellation of common factors this becomes:

$$A'B : \frac{BC'}{AC'} = A'C : \frac{CB'}{AB'};$$

which may be written

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B} = -1,$$

that is, the six points A, B, C, A', B', C' are in involution.

Since the theorem imposed no restriction upon the position of the triangle MNP relatively to the axes, it is apparent that, as a reciprocal to the proposition that a parallel to the base of a triangle divides the sides proportionally, we have proved that

"The three points in which any line cuts the sides of a triangle, and the projections, from any point in the plane, of the vertices of the triangle on the same line are six points in involution."

The fact that this proposition has thus been derived from another as its reciprocal, will not at all interfere with the deduction in the ordinary manner of the reciprocal theorem concerning six rays in involution joining any point to the vertices of a complete quadrilateral.

On the other hand, as no use whatever has been made of harmonic properties to obtain the theorem, these may all be deduced as special applications, by drawing the transversal AX through two of the points in which sides of the triangle MNP meet the lines joining Q to the opposite vertices, thus making these points foci of the involution.

Or, should it be preferred to proceed more nearly in the usual course, a definition of the punctual distance, equivalent to the one already stated, may be given in terms of angular functions. Thus, in the figure, the punctual distance between NC' and NB is

$$\frac{\sin NAX \sin BNC'}{N'N \sin ANB \sin ANC'};$$

and it is easily shown that the ratio in which the punctual distance of two out of three convergents is divided by the third is equal to the anharmonic ratio of the pencil formed by adding to the three convergents a line which joins the point of convergence to the origin.

A NEW AND USEFUL FORMULA FOR INTEGRATING CERTAIN DIFFERENTIALS.

BY PROF. J. W. NICHOLSON, A. M., BATON ROUGE, LA.

PROBLEM.—*To integrate* $v^n du$ *in terms of the descending powers of* v.

Let us assume

$$\int v^n du = yv^n + y_1 v^{n-1} + y_2 v^{n-2} + \ldots + y_t v^{n-t} + \int z v^{n-t-1} dv. \qquad (1)$$

Differentiating (1), and arranging with reference to v, we have

$$0 = \left.\begin{array}{r} -du \\ dy \end{array}\right| v^n + \left.\begin{array}{r} nydv \\ dy_1 \end{array}\right| v^{n-1} + \left.\begin{array}{r} (n-1)y_1 dv \\ dy_2 \end{array}\right| v^{n-2} + \ldots + \left.\begin{array}{r} (n-t)y_t dv \\ zdv \end{array}\right| v^{n-t-1} \qquad (2)$$

Now since (2) is true for every value of v, according to the principle of indeterminate coefficients, we have

$$\left.\begin{aligned}
dy &= du,\\
dy_1 &= -nydv,\\
dy_2 &= -(n-1)y_1dv,\\
dy_3 &= -(n-2)y_2dv,\\
&\cdots\cdots\cdots\cdots\\
dy_n &= -(n-t+1)y_{t-1}dv,\\
z &= -(n-t)y_t.
\end{aligned}\right\}\tag{3}$$

Integrating equations (3), supposing the constant to be added = 0, and denoting

$$\begin{array}{ll}
\int dv \int udv & \text{by } \int^2 udv^2,\\
\int dv \int dv \int udv & \text{by } \int^3 udv^3,\\
\&c. \quad \&c. & \&c.,
\end{array}$$

we have

$$\left.\begin{aligned}
y &= u\\
y_1 &= (-1)^1 n \int udv\\
y_2 &= (-1)^2 n(n-1)\int^2 udv^2\\
y_3 &= (-1)^3 n(n-1)(n-2)\int^3 udv^3\\
&- - - - - - - - - - - - -\\
y_t &= (-1)^t n(n-1)\ldots(n-t+1)\int^t udv^t\\
z &= (-1)^{t+1} n(n-1)\ldots(n-t)\int^t udv_t.
\end{aligned}\right\}\tag{4}$$

Substituting in (1), we have

$$\int v^n du = uv^n - nv^{n-1}\int udv + n(n-1)v^{n-2}\int^2 udv^2 - n(n-1)(n-2)v^{n-3}\int^3 udv^3$$
$$+\ldots(-1)^t n(n-1)\ldots(n-t+1)v^{n-t}\int^t udv^t + \ldots(-1)^{t+1}n(n-1)\ldots$$
$$(n-t)\int v^{n-t-1}dv\int^t udv^t. \tag{5}$$

If $t = n$ (5) becomes

$$\int v^n du = uv^n - nv^{n-1}\int udv + n(n-1)v^{n-2}\int^2 udv^2 \ldots (-1)^n n(n-1)\ldots$$
$$\ldots(3)(2)(1)\int^n udv^n. \tag{6}$$

Of course (6) is not so useful and general as (5), but the want of space will scarcely allow us to indicate the utility of the former in the applications of the formula which we purpose make to the integration of a few important differentials.

Ex. 1. $\int x^m \log^n x\, dx = ?$

Make $u = x^{m-1} \div (m+1)$, and $v = \log x$. Hence,

$$udv = \frac{x^m dx}{m+1},\quad \int udv = \frac{x^{m+1}}{(m+1)^2};$$

$$dv\int udv = \frac{x^m dx}{(m+1)^2},\quad \int^2 udv^2 = \frac{x^{m+1}}{(m+1)^3}.$$

Similarly,

$$\int^3 udv^3 = \frac{x^{m+1}}{(m+1)^4},$$
$$\cdots\cdots\cdots$$
$$\int^n udv^n = \frac{x^{m+1}}{(m+1)^n}.$$

Substituting in (6), we have

$$\int x^m \log^n x dx = \frac{x^{m+1}}{m+1}\left[\log^n x - \frac{n\log^{n-1}x}{m+1} + \frac{n(n-1)\log^{n-2}x}{(m+1)^2}\cdots\right.$$
$$\left.(-1)^n\frac{n(n-1)\ldots(3)(2)(1)}{(m+1)^n}\right]. \quad (7)$$

If in (7) $m=0$, and we take the integral between the limits $x=0$ and $x=1$, we have

$$\int_0^1 \log^n x dx = (-1)^n n(n-1)(n-2)\ldots 3.2.1.$$

Ex. 2. $\int a^x x^n dx = ?$

Make $u = a^x \div (\log a)$, and $v = x$. Hence,

$$\int udv = \frac{a^x}{\log^2 a},$$
$$\int^2 udv^2 = \frac{a^x}{\log^3 a},$$
$$- \; - \; - \; - \; - \; -$$
$$\int^n udv^n = \frac{a^x}{\log^{n+1}a}.$$

Substituting in (6),

$$\int a^x x^n dx = \frac{a^x}{\log a}\left[x^n - \frac{nx^{n-1}}{\log a} + \frac{n(n-1)x^{n-2}}{\log^2 a}\ldots(-1)^n\frac{n(n-1)\ldots 3.2.1}{\log^n a}\right].$$

In a similar manner we may integrate a number of other important differentials, such as

$$\sin^m x\cos^n x dx, \qquad (a+bx^n)^p x^m dx, \text{ \&c.}$$

Solution of Prob. 334, by W. E Heal.—(See pp. 60, 98, Vol. VIII)

Let the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The locus of intersection of tangents which intersect at a constant angle, φ, is

$$[(x^2+y^2)-(a^2+b^2]^2\tan^2\varphi = 4(b^2x^2+a^2y^2-a^2b^2). \qquad (A)$$

(Salmon's Conics, 5th edition, page 161.)

The polar of every point on this curve touches the required curve and conversely the polar of every point on the required curve touches the curve (A).

Let (x, λ) be the coordinates of any point on the required curve.

The polar of this point is

$$\frac{xx}{a^2} + \frac{\lambda y}{b^2} = 1. \qquad (B)$$

Making the equations homogeneous by writing in them $z = 1$,

$$[(x^2+y^2)-(a^2+b^2)z^2]^2\tan^2\varphi = 4z^2(b^2x^2+a^2y-a^2b^2z^2). \qquad (C)$$

$$\frac{xx}{a^2}+\frac{\lambda y}{b^2} = z. \qquad (D)$$

Eliminating z, the coordinates of the points in which (C) and (D) intersect are given by the equation

$$\left[(x^2+y^2)-(a^2+b^2)\left(\frac{xx}{a^2}+\frac{\lambda y}{b^2}\right)^2\right]^2\tan^2\varphi = 4\left(\frac{xx}{a^2}+\frac{\lambda y}{b^2}\right)^2$$

$$\times\left[b^2x^2+a^2y-a^2b^2\left(\frac{xx}{a^2}+\frac{\lambda y}{b^2}\right)^2\right].$$

Expanding and reducing, we get

$$Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4 = 0. \qquad (E)$$

Where

$$A = b^4[a^2-b^2(a^2+b^2)x^2]^2\tan^2\varphi-4a^2b^8x^2+4a^2b^{10}x^4,$$

$$B = 4[a^2b^6(a^2+b^2)^2x^3\lambda-a^4b^4(a^2+b^2)x\lambda]\tan^2\varphi-8a^4b^6x\lambda+16a^4b^8x^3\lambda,$$

$$C = [2a^4b^4-2a^2b^2(a^2+b^2)(a^4\lambda^2+b^4x^2)+6a^4b^4(a^2+b^2)^2x^2\lambda^2]\tan^2\varphi$$
$$-4(a^6b^4\lambda^2+a^4b^6x^2)+24a^6b^6x^2\lambda^2,$$

$$D = 4[a^6b^2(a^2+b^2)^2x\lambda^3-a^4b^4(a^2+b^2)x\lambda]\tan^2\varphi-8a^6b^4+16a^8b^4x\lambda^3,$$

$$E = a^4[b^2-a^2(a^2+b^2)\lambda^2]^2\tan^2\varphi-4a^8b^2\lambda^2+4a^{10}b^2\lambda^4.$$

But if (D) is tang't to (C), equation (E) must have equal roots and therefore its discriminant vanishes. Therefore

$$4(12AE-3BD+C^2)^3 = (72ACE+9BCD-27AD^2-27EB^2-2C^3)^2.$$

(Salmons Higher Algebra, 3rd edition, p. 306.)

This is an equation between the coordinates x, λ and is, therefore the required envelope.

NOTE ON AN INDETERMINATE EQN., BY WM. HOOVER, A. M.—The following quotation is from a communication by M. E. Catalan to *Journal de Mathematiques* for August, 1882.

"The identity

$$(a+b)^2(a-2b)^2(b-2a)^2+27a^2b^2 = 4(a^2-ab+b^2)^3,$$

easy of verification, gives an indefinite number of solutions, in entire numbers, of

$$x^2+3y^2 = z^3.$$

"We can take

$x = \frac{1}{2}(a+b)(a-2b)(b-2a$, $y = \frac{3}{2}ab(a-b)$, and $z = a^2-ab+b^2$."

DIFFERENTIATION OF $F(X) = LOG(X)$.

BY PROF. JAMES G. CLARK, LIBERTY, MO.

THE characteristic property of this function is expressed by the equation

$$F(x^m) = mF(x),$$

from which we have at once

$$dF(x^m) = mdF(x).$$

Let us assume $dF(x) = f(x)dx$, in which $f(x)$ is an unknown function. We have then

$$dF(x^m) = f(x^m)dx^m = mx^{m-1}f(x^m)dx.$$

Therefore by substitution,

$$mx^{m-1}f(x^m)dx = mf(x)dx, \text{ and}$$

$$f(x^m) = \frac{f(x)}{x^{m-1}}, \text{ or } x^m f(x^m) = xf(x).$$

This relation being independent of the value of m, it follows that

$$xf(x) = M, \text{ a constant.}$$

Therefore $f(x) = \dfrac{M}{x}$, and $dF(x) = d\log(x) = \dfrac{M}{x}dx$.

In this equation M is of unknown value. Its value cannot be directly determined from the general properties of logarithms, but it may be observ'd that in whatever system the logarithms be taken, $\log x \div M$ will be the logarithm of x in some other system, whose base may be denoted by e. W have, therefore, from the equation

$$d\log(x) = \frac{M}{x}dx,$$

$$d\left\{\frac{\log x}{M}\right\} = d\log_e x = \frac{dx}{x}.$$

There is therefore one system for which the multiplier of $dx \div x$ is unity, and it readily follows that the multipliers of $dx \div x$ in different systems are proportional to their moduli. It may be shown in the usual way by Maclaurin's theorem that $e =$ Napierian base.

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in No. 1 have been received as follows:

From Marcus Baker, 423, 426; Florian Cajori, 423, 424, 425, 426, 427; George E. Curtis, 423; Prof. W. P. Casey, 423, 426; G. W. Evans, 423;

Prof. A. B. Evans, 427; Prof. Wm. Hoover, 423, 425, 426; H. Heaton, 423, 424, 425, 426, 427; Prof. H. P. Manning, 426, 427; Prof. J. W. Nicholson, 426, 427; A. F. Parsons, 423, 424, 426; Prof. P. H. Philbrick, 423, 424, 425, 426, 427; P. Richardson, 423, 426; Prof. J. Scheffer, 423, 426, 427; Prof. E. B. Seitz, 423, 426, 427; R. S. Woodward, 423, 424, 425, 426, 427.

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423. *By E. Millwee, Add-Ran College, Granbury, Texas.*—"Given the hypothenuse of a right-angled triangle and the difference of the two lines drawn from the acute angles to the centre of the insribed circle, to find the triangle."

SOLUTION BY A. F. PARSONS, ADELAIDA, CAL.

Let ABC be the triangle, AI, BI the bisecting lines of which the difference is given. Draw IR perpendicular to AB.

Put $AB=c$, and the given difference $=d$. Also let $r=IR$, $x=AI$ and $y=AR$. Then $y+r=AC$, $c-y+r=BC$, $x+d=IB$, and $c-y=RB$.

By right triangles we have

$$x^2=y^2+r^2, \tag{1}$$

$$(x+d)^2-r^2=(c-y)^2, \text{ or } x^2+2xd+d^2-r^2=c^2-2cy+y^2, \tag{2}$$

$$(y+r)^2+(c-y+r)^2=c^2, \text{ or } y^2-cy=-(r^2+cr). \tag{3}$$

Substitute the values of x^2 and x from (1) in (2) and we get

$$d\sqrt{(y^2+r^2)}=\tfrac{1}{2}(c^2-d^2)-cy. \tag{4}$$

From (3) and (4) we get a quadratic whose root is

$$r=\tfrac{1}{2}(c^2-d^2)(\sqrt{2}-1). \tag{5}$$

Substituting values of r and r^2 from (5) in (3) we get the quadratic,

$$y^2-cy=\tfrac{1}{4}(c^2-d^2)[(d^2\div c^2)(3-2\sqrt{2})-1]; \tag{6}$$

$$\therefore y=\tfrac{1}{2}\{c\pm d\sqrt{[\tfrac{1}{2}(c^2-d^2)(3-2\sqrt{2})+1]}\}. \tag{7}$$

and the sides are

$$\frac{c^2+(c^2-d^2)(\sqrt{2}-1)}{2c}\pm\frac{d}{2}\sqrt{\left[\frac{c^2-d^2}{c^2}(3-2\sqrt{2})+1\right]}. \tag{8}$$

Or, take $ID=IA$ and connect AD. Then $DB=d$, $\angle AID=135^\circ$, $\angle IDA=22\tfrac{1}{2}^\circ$ and $\angle BDA=157\tfrac{1}{2}^\circ$; $\therefore DAB=\sin^{-1}[(d\div 2c)(2-\sqrt{2})]$.

Therefore DBA and consequently ABC become known. Then

$$AC=c\sin ABC; \qquad BC=c\cos ABC.$$

SOLUTION BY P. RICHARDSON, BROOKLYN, N. Y.

Let $2x$ and $2y$ be the sides, $r =$ the radius of the inscribed circle, $2h =$ the hypothenuse, and $2d =$ the given difference; then the following equations can be easily found;

$$x+y-r = h \ldots (1), \quad 2xy = r(x+y+h), \qquad (2)$$

$$x^2+y^2 = h^2 \ldots (3), \quad \sqrt{[(2x-r)^2+r^2]}-\sqrt{[(2y-r)^2+r^2]} = \pm d. \quad (4)$$

In (4) put value of r from (1), perform the operations indicated under the radical signs and substitute for (x^2+y^2) its value from (3), reduce and transpose second radical, and there results

$$\sqrt{(h^2-hy)} = \pm d + \sqrt{(h^2-hx)}.$$

Square this equation, transpose and substitute h^2 for x^2+y^2 and we have

$$h^4-4h^2d^2+d^4 = 2h^2xy-2hd^2(x+y). \qquad (5)$$

From (1) and (2) we get $x+y = r+h$ and $2xy = r(r+2h)$, put these values in (5), transpose and we have he quadratic equation

$$h^2r^2 + (2h^3-2hd^2)r = h^4-2h^2d^2+d^4, \text{ whence}$$

$$r = (\pm\sqrt{2}-1)(h^2-d^2)\div h; \qquad (6)$$

$$\therefore \; 2x = r+h+\sqrt{(h^2-r^2-2rh)}, \qquad 2y = r+h-\sqrt{(h^2-r^2-2rh)}.$$

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424 *By Prof. De Volson Wood.*—"Required the equation to the locus which is at a constant, internal, normal dist. from the four cusp epicycloid."

SOLUTION BY R. S. WOODWARD, C. E., DETROIT, MICHIGAN.

The equations of this epicycloid are

$$\left.\begin{aligned} x &= \tfrac{1}{4}r(5\cos\theta-\cos 5\theta) \\ y &= \tfrac{1}{4}r(5\sin\theta-\sin 5\theta) \end{aligned}\right\}, \qquad (1)$$

r being the radius of the fixed circle. The equation to the normal to this curve is

$$\eta-y = -\frac{dx}{dy}(\varepsilon-x),$$

and the distance of (ε, η) from (x, y), which in this problem is constant, is defined by

$$(\eta-y)^2+(\varepsilon-x)^2 = \left(1+\frac{dx^2}{dy^2}\right)(\varepsilon-x)^2 = \left(1+\frac{dy^2}{dx^2}\right)(\eta-y)^2 = c^2 \text{ say.} \quad (2)$$

Since by (1) $\dfrac{dx}{dy} = \dfrac{\sin\theta-\sin 5\theta}{\cos\theta-\cos 5\theta} = \cot 3\theta$, (2) gives

$$\varepsilon = x \mp c\sin 3\theta = \tfrac{1}{4}r(5\cos\theta-\cos 5\theta) \mp c\sin 3\theta,$$

$$\eta = y \pm c\cos 3\theta = \tfrac{1}{4}r(5\sin\theta-\sin 5\theta) \pm c\cos 3\theta,$$

which define the locus.

425. "A cylinder rolls down the upper plane of a wedge, the wedge being upon a perfectly smooth horizontal plane; required the velocity of the cylinder when it shall have rolled a distance l on the plane."

SOLUTION BY WILLIAM HOOVER, A. M., DAYTON, OHIO.

Let m and m' be the masses of the cylinder and wedge, a the radius of the cylinder, mk^2 its moment of inertia, α the inclination of the upper face of wedge to the horizon, θ the angle through which the cylinder has rolled in any time t from the beginning of motion, F and R the reactions of the cylinder and inclined plane, parallel and perpendicular to the upper surface of the wedge. Let x' be the distance the wedge is at any time from an assum'd point in the horizontal plane, x and y the coordinates of the center of gravity C, of the cylinder, the axis of which is all the time supposed horizontal.

For the cylinder, resolving horizontally and vertically and taking moments about the center of gravity,

$$m\frac{d^2x}{dt^2} = -R\sin\alpha + F\cos\alpha, \tag{1}$$

$$m\frac{d^2y}{dt^2} = R\cos\alpha + F\sin\alpha - mg, \tag{2}$$

$$mk^2\frac{d^2\theta}{dt^2} = Fa. \tag{3}$$

For the single motion of the wedge,

$$m'\frac{d^2x'}{dt^2} = R\sin\alpha - F\cos\alpha. \tag{4}$$

We also have

$$y\cos\alpha = a + (x - x')\sin\alpha. \tag{5}$$

From (1) and (4),

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0. \tag{6}$$

From (5)
$$\cos\alpha\frac{d^2y}{dt^2} = \sin\alpha\left(\frac{d^2x}{dt^2} - \frac{d^2x'}{dt^2}\right), \tag{7}$$

which with (6) gives

$$m'\cos\alpha\frac{d^2y}{dt^2} = (m+m')\sin\alpha\frac{d^2x}{dt^2}. \tag{8}$$

Since the cylinder rolls,

$$\frac{dx}{dt} = \frac{dx'}{dt} - a\cos\alpha\frac{d\theta}{dt}, \text{ w. ence}$$

$$a\cos\alpha\frac{d^2\theta}{dt^2} = \frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}. \tag{9}$$

m (6) and (9), $m' a \cos a \frac{d^2\theta}{dt^2} = -(m+m')\frac{d^2x}{dt^2}$. (10)

minating R and F from (1), (2) and (3),

$$a \cos a \frac{d^2x}{dt^2} + a \sin a \frac{d^2y}{dt^2} = k^2\frac{d^2\theta}{dt^2} - ay \sin a. \quad (11)$$

stituting (8) and (10) in (11) gives

$$[m'a^2\cos^2a+(m+m')(a^2\sin^2a+k^2)]\frac{d^2x}{dt^2} = -m'a^2g \sin a \cos a. \quad (12)$$

stituting in (8) gives

$$[m'a^2\cos^2a+(m+m')(a^2\sin^2a+k^2)]\frac{d^2y}{dt^2} = -(m+m')a^2g \sin a. \quad (13)$$

$x = b$, $y = c$, when $t = 0$. Then integrating (12) and (13) twice,

$$x - b = -m'a^2gt^2 \sin 2a \div 4[m'a^2\cos^2a+(m+m')(a^2\sin^2a+k^2)], \quad (14)$$

$$y - c = -(m+m')a^2gt^2\sin^2a \div 2[m'a^2\cos^2a+(m+m')(a^2\sin^2a+k^2)] \quad (15)$$

)÷(15) gives $(m+m')\sin a\,(x-b) = m'\cos a\,(y-c)$ (16), the equation raight line, the path described in space by the center of gravity of the er.

ltiplying (12) by $2dx \div dt$ and integrating,

$$\frac{dx}{dt} = \sqrt{\left\{\frac{m'a^2g \sin 2a\,(b-x)}{m'a^2\cos^2a + (m+m')(a^2\sin^2a+k^2)}\right\}},$$

rizontal velocity of the center of gravity of the cylinder for any ab- x. Let s = the distance moved over by the wedge while the cylinder over the distance l; then $(m+m')s = ml \cos a$; hence when the cyl- has moved the distance l, the value of x is

$$b-\left(l \cos a - \frac{ml \cos a}{m+m'}\right) = b - \frac{m'l \cos a}{m+m'}, \text{ which gives}$$

$$\frac{dx}{dt} = \sqrt{\left\{\frac{m'^2a^2gl \sin 2a \cos a}{(m+m')[m'a^2\cos^2a+(m+m')(a^2\sin^2a+k^2)]}\right\}},$$

rizontal velocity of C when the cylinder has rolled the distance l on per surface of the wedge.

m (16) we have

$$\frac{\sqrt{(dx^2+dy^2)}}{dy} = \frac{\sqrt{[(m+m')^2\sin^2a+m'^2\cos^2a]}}{(m+m')\sin a},$$

s. of the angle which the locus of C makes with the axis of abscissas. refore, resolving the above velocity in the direction of the locus, the ed velocity is

$$\frac{m'a}{(m+m')\sin a}\sqrt{\frac{gl \sin 2a \cos a\,[(m+m')^2\sin^2a + m'^2\cos^2a]}{(m+m')[m'a^2\cos^2a + (m+m')(a^2\sin^2a+k^2)]}}.$$

e velocity of the cylinder will be less than if it rolled down the same ce on a fixed inclined plane.

426. *By William Hoover, A. M.*—"Eliminate θ from the equations

$$(a+b)\tan(\theta-\varphi)=(a-b)\tan(\theta+\varphi),$$
$$a\cos 2\varphi+b\cos 2\theta=c."$$

SOLUTION BY MARCUS BAKER, U. S. COAST SURV., LOS ANGELES, CAL.

From (1) we have

$$\frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(2\theta+2\varphi)}{\tan\frac{1}{2}(2\theta-2\varphi)}=\frac{\sin 2\theta+\sin 2\varphi}{\sin 2\theta-\sin 2\varphi}, \text{ whence } \frac{a}{b}=\frac{\sin 2\theta}{\sin 2\varphi},$$

and our two equations stand thus:—

$$b\sin 2\theta = a\sin 2\varphi, \qquad (1')$$
$$b\cos 2\theta = c-a\cos 2\varphi. \qquad (2')$$

Squaring and adding,

$$b^2 = a^2+c^2-2ac\cos 2\varphi,$$

an equation from which θ has been eliminated.

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427. *By Prof. J. W. Nicholson.*—"C and D are two fixed points (1 dist. apart) on the line AB; show that the locus of a point P, moving in such a manner that $\angle PDB=n\angle PCB$, the origin being at C, is

$$\frac{y\sqrt{-1}}{x-1}=\frac{(x+y\sqrt{-1})^n-(x-y\sqrt{-1})^n}{(x+y\sqrt{-1})^n+(x-y\sqrt{-1})^n}."$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $CD=1$, $CH=x$, $PH=y$, $CP=r$, $\angle PDB=\theta$ $\angle PCB=\varphi$.

Then $\tan\theta=\frac{y}{x-1}$, $x=r\cos\varphi$, $y=r\sin\varphi$, $\theta=n\varphi$.

$$\therefore \tan\theta=\tan n\varphi$$
$$=\frac{\sin n\varphi}{\cos n\varphi}=\frac{1}{\sqrt{-1}}\cdot\frac{(\cos\varphi+\sqrt{-1}\sin\varphi)^n-(\cos\varphi-\sqrt{-1}\sin\varphi)^n}{(\cos\varphi+\sqrt{-1}\sin\varphi)^n+(\cos\varphi-\sqrt{-1}\sin\varphi)^n}.$$

(See Chauvenet's Trigonometry, p. 128.)

Substituting the value of $\tan\theta$, and multiplying both terms of the last fraction by r^n, and then substituting x for $r\cos\varphi$, and y for $r\sin\varphi$, we have

$$\frac{y\sqrt{-1}}{x-1}=\frac{(x+y\sqrt{-1})^n-(x-y\sqrt{-1})^n}{(x+y\sqrt{-1})^n+(x-y\sqrt{-1})^n}.$$

[After deducing the given equation, Mr. Heaton remarks, "The polar equation may be found from the triangle CPD as follows:

$r\sin CPD=1.\sin CDP$, or $r\sin(n+1)\varphi=\sin n\varphi$, the polar equation.

$r\,[e^{(n+1)\phi\sqrt{-1}}-e^{-(n+1)\phi\sqrt{-1}}]=e^{n\phi\sqrt{-1}}-e^{-n\phi\sqrt{-1}}$; $\therefore\ r\,[(\cos\varphi+\sin\varphi\sqrt{-1})^{(n+1)}-(\cos\varphi-\sin\varphi\sqrt{-1})^{(n+1)}]=(\cos\varphi+\sin\varphi\sqrt{-1})^n-(\cos\varphi-\sin\varphi\sqrt{-1})^n$; $\therefore\ (x+y\sqrt{-1})^{(n+1)}-(x-y\sqrt{-1})^{(n+1)}=(x+y\sqrt{-1})^n-(x-y\sqrt{-1})^n$. A simpler form of the equation."]

PROBLEMS.

428. *By George Lilley, A. M.*—Two circles, of given radii, R and R_1, touch a straight line on the same side; a third circle of radius R_2 touches each of them; find the position of the circles R and R_1 and the radius of a fourth circle such that it shall touch the same straight line and each of the three given circles.

429. *By Prof. M. L. Comstock.*—A cone of given weight W, is placed with its base on an inclined plane, and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pulley in the inclined plane at the same height as the vertex. Determine the conditions of equilibrium.

430. *By Prof. Milwee, Add-Ran Col. Texas.*—Given two fixed points A and B, one on each of the axes of coordinates, at the respective distances a and b from the origin; if A' and B' be taken on the axes so that $OA'+OB'=OA+OB$, find the locus of the intersection of AB' and $A'B$.

431. *By Prof J. W. Nicholson.*—Required the area of a triangle whose sides are equal to the three roots respectively of the following equation:

$$x^3+mx^2+nx+r=0.$$

432. *By R. J. Adcock.*—Show that the quadrant of the ellipse equals

$$a\int_0^1\left(\frac{1-e^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx=\tfrac{1}{2}\pi\cos\theta\left[1+(\tfrac{1}{2}\tan\theta)^2-\frac{1}{3}\left(\frac{1.3}{2.4}\tan^2\theta\right)^2+\frac{1}{5}\left(\frac{1.3.5}{2.4.6}\tan^3\theta\right)^2-\frac{1}{7}\left(\frac{1.3.5.7}{2.4.6.8}\tan^4\theta\right)^2+\ \&c.\right],$$

where $a=$ semi transverse axis, b semi conjugate, $e^2=1-(b^2\div a^2)$, $\tan^2\theta=e^2\div(1-e^2)$.

433. *By Prof. W. P. Casey.*—Given the base of a triangle, to find the locus of the vertex, when the centre of the inscribed square moves on a given conic section.

434. *By Prof. De Volson Wood.*—Find a number, the mantissa of the logarithm of which equals the number.

435. *By Prof. E. B. Seitz.*—Find the average area of a triangle draw on the su.face of a given circle, having its base parallel to a given line, an its vertex taken at random.

436. *By Prof. W. W. Johnson.*—Integrate the equation

$$x^m y^n(aydx + bxdy) = x^{m'} y^{n'}(a'ydx + b'xdy).$$

CORRECTION.—The last two lines on page 48 should read as follows:

$$\frac{b^2\tan x}{a(n-1)(a^2-b^2)(a+b\sec x)^{n-1}}+\frac{(3n-4)a^2-(n-1)b^2}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-1}}$$
$$-\frac{3n-5}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-2}}+\frac{n-2}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-3}}.$$

PUBLICATIONS RECEIVED.

Annual Report of the Chief Signal Officer to the Secretary of War for the year 1880. 1096 pp. 8vo., with 119 maps. Washington. 1881.

Science. An Illustrated Weekly Journal. MOSES KING publisher. Boston, Mass.

The first number of this Journal bears date February 9, 1883, and is devoted, as its name imports, to current scientific news. Price $5.00 per year; single numbers 15 cents.

Universal Necessity. A Philosophical Essay, by WERNER STILLE, PH. D. 8v. 35 pages. St. Louis, Mo. 1881.

Acta Mathematica, edited by G. MITTAG-LEFFLER. F. & G. Beijer, Stockholm. 1882.

This new journal which has been founded through the generosity of King OSCAR II appears under the cooperation of several able Scandinavian mathematicians. The first number contains the following articles:

(1). Theory of Foxian Groups; by H. POINCARE', pp. 1-62.
(2). On the Theory of Annuities; by J. C. MALMSTEN, pp. 63-76.
(3). A Method of Approximation in the Problem of three Bodies; by HUGO GYLDE'N, [pp. 77-92.
(4), The Problem of Configurations; by TH. REYE, pp. 93-96.

This number presents therefore a variety of subjects which are discussed by able writers. The article by Professor Gylden in which he gives an account of his method of treating the famous problem of three bodies will be interesting to astronomers. The theoretical solution of this problem remains nearly as it was left by Lagrange and Laplace, although we owe to Hansen and Delaunay important improvements in the practical parts of the work. It will be interesting to see the out-come of Gylden's labors on this question.

A. H.

ERRATA.

On page 12, line 9, for + between the two members of last term, read —.
" " " " 2, from bottom, for $\frac{1}{2}$ as index of (a^2-b^2), read $\frac{3}{2}$.
" " 14, " 3, from bottom, insert (f) at end of line.

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KEPLER'S PROBLEM.

BY PROF. ASAPH HALL, NAVAL OBSERVATORY WASHINGTON, D. C.

AT the end of the Fourth Part of his work "De Motibus Stellæ Martis," Kepler proposes the problem so long known by his name. His statement, translated from the Latin, is as follows:—

"But having given the mean anomaly, there is no geometrical method of arriving at its coequal, that is, the eccentric anomaly. For the mean anomaly is made up of two parts of area, a sector and a triangle, the former of which is counted by means of the eccentric arc, the latter by the sine of that arc multiplied into the value of the maximum triangle. But the proportions between the arcs and their sines are infinite in number. Therefore the sum of both being given it cannot be told how great the arc is, or how great is the sine of that arc corresponding to this sum, unless we first inquire how great an area is swept by the given arc, that is, unless we shall have constructed tables and afterwards shall have worked from them.

"This is my opinion of it; and by as much less as it seems to have of geometrical beauty, so much the more do I exhort geometers to solve for me this problem:

"*Given the area of a part of a semicircle, and given a point of the diameter, to find the arc and the angle at that point, the given area being included by the legs of this angle and by this arc; Or: To cut the area of a semi circle in a given ratio from any point whatsoever of the diameter.*

"I am sufficiently satisfied that it cannot be solved *a priori*, on account of the different nature of the arc and the sine. But if I am mistaken, and any one shall point out the way to me, he will be in my eyes the great Apollonius."

This is Kepler's opinion put forth nearly three centuries ago. The attempts to solve this famous problem, however, still continue, and nearly every year gives us two or three new solutions.—

The equation to be solved is the well known transcendental one,

$$E = M + e \sin E,$$

from which M and e being given we are to find E. The direct solution of this equation being impossible, the solutions that are given take a variety of forms, and for convenience we may arrange them in three classes:

(1). The pedagogic methods.

(2). The methods by series.

(3). The indirect method.

In the first class we put all the attempts to reduce the solution to a fixed set of rules, according to which nothing is left to the judgment of the computer, and where simple numerical accuracy, such as one might find in a child, or in a machine, is all that is required. This class will include many of the solutions that have been published, from that of Seth Ward, Professor at Oxford in 1649, to some of the most recent ones. The chief value of these solutions, and at the same time their weakness, is that they leave nothing uncertain in the process. They are therefore adapted to beginners in astronomy, and to those who always work by rule of thumb. The amount of intelligence required in the computer is a minimum.

The methods by series comprise the solutions given by Lagrange, Poisson, Bessel, Hansen, and other astronomers. These solutions are remarkable on account of the elegant and peculiar analysis employed, and they have an important use in theoretical astronomy. The series proceed according to powers of the eccentricity which are multiplied by the sines of multiples of the mean anomaly. The first systematic solution of this kind seems to have been given by Lagrange, who employed his well known theorem for determining the roots of all kinds of equations by means of series. This theorem, published in 1770, can be applied directly to Kepler's problem.

The indirect method, or solution by trial, is that, I think, to which every working astronomer finally comes. Of course, when the eccentricity is very small, or where tables of a planet have been computed, the equation of the center is tabulated, and this disposes of the matter once for all. But in case the eccentricity is as great as $\frac{2}{10}$ or $\frac{3}{10}$, and no tables are at hand, the indirect method does not require more than half the time of the other methods, and its accuracy is as great as one pleases. The advantage of this method consists in the indeterminate form in which the solution is left, and in the opportunity that is thus given for the intelligence and skill of the computer to come into use. Now this is the point that writers who are continually bringing forward solutions of the first kind fail to see; and at the same time they seem to be shocked at the notion of making a good guess in solving an equation. But certainly it is not necessary to be stupid in order to be mathematical; and there are so many problems to be solved that we need not fear of taking an undue advantage of things, or that our ingenuity will ever grow dull for lack of opportunity.

ON AN UNSYMMETRICAL PROBABILITY CURVE.

[Second paper.]

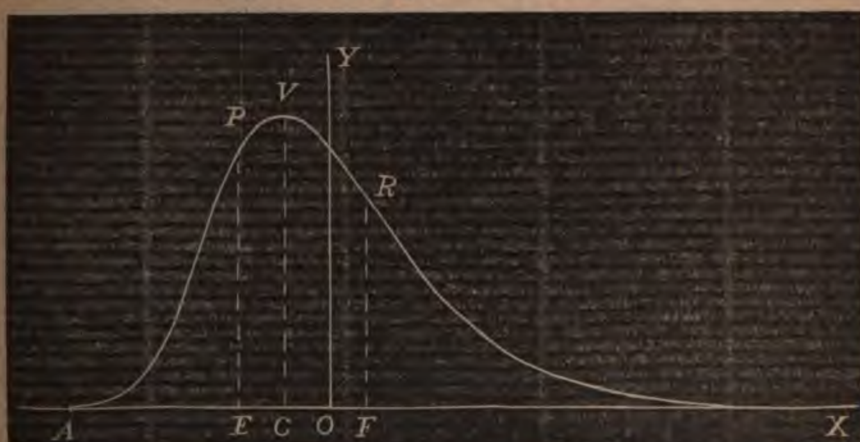
BY E. L. DE FOREST.

In my article on this subject ending at p. 7 of Vol. X of the ANALYST, some points of interest were omitted or but slightly touched upon. It is proposed to consider them here, and the formulas and tables obtained will be numbered consecutively with the previous ones.

The accompanying figure has been drawn to represent the curve as found for the example at page 1, its ordinates being proportional to those in column 9 of the table. For the equation of any curve of this nature, putting $Ydx = y$, we have by (52),

$$Y = \frac{1}{K\sqrt{(2\pi b)}}\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax}, \tag{81}$$

where a and b are constants determined by (39) from the q. m. error ε and the c. m. inequality ζ, and K is a given function of a^2b.



The curve is in general limited on one side, where it touches the X axis at A, and extends to infinity on the other side, the axis being an asymptote to it in the direction OX. We may say, therefore, that it has a long side and a short side. Since in (81) we have

$$\frac{x}{ab} = \frac{ax}{a^2b},$$

and a^2b is always positive, it appears that Y is a function of ax, and that its value is the same when a and x are both negative, as when they are both positive. Hence, changing the sign of a reverses the curve.

Our figure is drawn for positive a, but if it were negative the curve would be of the same form, only turned over so that the long side would come on the left hand and the short side on the right. From the nature of the curve as shown in (41) and elsewhere, it is evident that if the whole area included between it and the X axis could be cut out of cardboard or other material of uniform thickness, it would balance on a knife edge set to coincide with the Y axis. The greater leverage of the long side compensates for the greater area of the short side. The vertex of the curve does not lie in the Y axis. The maximum ordinate CV is separated from OY by the interval OC, and this is important, for it shows that when the law of error of a set of observations is such that ζ is not zero, the arithmetical mean is not the most probable value of the observed quantity. To get the most probable value of a single observation, we must apply to the arith. mean a correction $\varDelta$ equal to OC, and by (22) and (39) its amount will be

$$\varDelta = -\frac{1}{a} = \frac{-\zeta^3}{2\varepsilon^2}. \tag{82}$$

For the observations in Table II. we have

$$\varDelta = -\frac{1}{a} = -\frac{1}{1.095} = -0^\circ.91,$$

and this added to the arith. mean $5^\circ.20$ gives $4^\circ.29$ as the value of the amplitude most likely to occur. The probabilities that an error or deviation which occurs will fall on the + or — side of the most probable value, are not the same. The part VCX of the area of the curve lying on the long side is greater than the part ACV lying on the short side, as will appear hereafter. An ordinate dividing the total area into halves, would be between OY and CV. The probable error of the most probable value of an observation is greater toward the long side of the curve than toward the short side. The ordinate which divides the area ACV into halves is PE, and that which divides VCX into halves is RF. The probable error on the + side therefore is CF, and that on the — side is CE.

As found at page 3 and 4, putting $n = a^2b$ and $v = ax$, the areas on the short and long side of any ordinate Y whose abscissa x is reckoned from A as an origin will be respectively

$$\left.\begin{aligned} \int_0^x Ydx &= \frac{v^n e^{-v}}{\Gamma(n+1)}\left(1+\frac{v}{n+1}+\frac{v^2}{(n+1)(n+2)}+\&c.\right), \\ \int_x^\infty Ydx &= \frac{v^{n-1}e^{-v}}{\Gamma(n)}\left(1+\frac{n-1}{v}+\frac{(n-1)(n-2)}{v^2}+\&c.\right), \end{aligned}\right\} \tag{83}$$

and their sum is unity. The probability P_2 that an error which occurs will fall on the long side from VC is represented by the area VCX, and its

alue is found from the second integral by taking as the lower limit $x = ab$ -(1÷a), whence we have

$$v = ax = a^2b - 1 = n - 1,$$

$$P_2 = \frac{(n-1)^{n-1}e^{-n+1}}{\Gamma(n)}\left(2 + \frac{n-2}{n-1} + \frac{(n-2)(n-3)}{(n-1)^2} + \&c.\right), \qquad (84)$$

nd the series terminates if n is an integer. The values of P_2 have been ɔmputed thus, to 7 places of decimals, for $n = 4, 5, 6, 7, 8, 10, 14, 20, 30,$ 0, 100, 200. The corresponding values of $P_1 = 1 - P_2$, together with ıtermediate values found by interpolation, are given to 4 places in the sub- ›ined Table III. By means of the column of differences $\varDelta_1$, P_1 and conse- uently P_2 can be found for any given value of the argument, with suffic't ccuracy for all practical purposes. Now to get the position of the points

TABLE III.

a^2b	P_1	$\varDelta_1$	$-ar_1$	$\varDelta_1$	ar_2	a^2b	P_1	$\varDelta_1$	$-ar_1$	$\varDelta_1$	ar_2
4	.3528	102	.822	89	1.613	12	.4209	61	1.870	197	2.667
4.5	.3630	82	.911	84	1.703	14	.4270	72	2.067	263	2.865
5	.3712	69	.995	79	1.788	17	.4342	52	2.330	231	3.128
5.5	.3781	59	1.074	76	1.868	20	.4394	69	2.561	359	3.359
6	.3840	52	1.150	71	1.944	25	.4463	45	2.920	333	3.718
6.5	.3892	45	1.221	68	2.016	30	.4508	72	3.253	587	4.051
7	.3937	40	1.289	66	2.085	40	.4580	41	3.840	494	4.639
7.5	.3977	36	1.355	65	2.151	50	.4621	69	4.334	923	5.133
8	.4013	62	1.420	121	2.216	70	.4690	43	5.257	1220	6.056
9	.4075	51	1.541	113	2.337	100	.4733	43	6.477	1321	7.276
10	.4126	45	1.654	110	2.450	140	.4776	36	7.798	1323	8.597
11	.4171	38	1.764	106	2.561	200	.4812		9.121		9.920
						∞	.5000				

and F which mark the limits of probable error, we must find x from each f the conditions

$$\int_0^x Ydx = \tfrac{1}{2}P_1, \qquad \int_x^\infty Ydx = \tfrac{1}{2}P_2,$$

ıd by (83) this amounts to solving the numerical equations

$$\left.\begin{aligned} u_1 &= \left(1 + \frac{v}{n+1} + \frac{v^2}{(n+1)(n+2)} + \&c.\right) - \frac{n!}{2}P_1 e^v v^{-n} = 0, \\ u_2 &= \left(1 + \frac{n-1}{v} + \frac{(n-1)(n-2)}{v^2} + \&c.\right) - \frac{(n-1)!}{2}P_2 e^v v^{-n+1} = 0. \end{aligned}\right\} \quad (85)$$

Hence n is any assumed integer, P_1 and P_2 are numbers derived from it as above, and v is the unknown quantity. By Newton's method of solution, any assumed approximate values of v will be subject to the approximate corrections

$$h_1 = -u_1 \div \frac{du_1}{dv}, \qquad h_2 = -u_2 \div \frac{du_2}{dv}.$$

Differentiating (85), we can obtain

$$\left.\begin{aligned} h_1 &= \frac{\frac{(n+1)!}{2}P_1e^v v^{-n} - (n+1) - v\left(1+\frac{v}{n+2}+\frac{v^2}{(n+2)(n+3)}+\text{\&c.}\right)}{\frac{(n+1)!}{2}P_1e^v v^{-n}\left(\frac{n}{v}-1\right) + \left(1+\frac{2v}{n+2}+\frac{3v^2}{(n+2)(n+3)}+\text{\&c.}\right)} \\ h_2 &= \frac{\frac{v}{n-1}+\left(1+\frac{n-2}{v}+\frac{(n-2)(n-3)}{v^2}+\text{\&c.}\right)-\frac{(n-2)!}{2}P_2e^v v^{-n+2}}{\frac{1}{v}\left(1+\frac{2(n-2)}{v}+\frac{3(n-2)(n-3)}{v^2}+..\right)+\frac{(n-2)!}{2}P_2e^v v^{-n+2}\left(1-\frac{n-1}{v}\right)} \end{aligned}\right\} \quad (86)$$

Denote by v_1 and v_2 the desired roots of the equations (85). As rough approximations to start with, we may take

$$v_1 = n-1-.6745(2P_1)\sqrt{n}, \qquad v_2 = n-1+.6745(2P_2)\sqrt{n}.$$

The true values of v_1 and v_2 have generally been found without need to compute the corrections (86) more than once or twice for each, but to save labor, the calculation was made only for $n = 4, 5, 7, 10, 20, 50$ and 200. Now let r_1 and r_2 denote the probable errors on the short and long side of the curve respectively. They are CE and CF in our figure, the first being negative and the second positive. If a were negative, r_1 would be positive and r_2 negative. Thus r_1 and a have unlike signs, while r_2 and a have like signs. Hence ar_1 is always $-$ and ar_2 always $+$. We have therefore

$$-ar_1 = n-1-v_1, \qquad ar_2 = v_2-(n-1), \qquad (87)$$

From data thus obtained with the above values of n, intermediate values of $-ar_1$ and ar_2 have been interpolated, mostly by Lagrange's formula. All were carried to four places of decimals, only three of which are shown in Table III. As the data were not very near together, it is of course possible that some of the interpolated values may not be perfectly correct, but the errors, if any, are too small to have any practical importance. The first differences in the series $-ar_1$ are almost identical with those for ar_2, never differing more than a single unit of the last decimal place, so that one Δ_1 column serves well enough for both. If there should be occasion to take $n > 200$, we can come very near the truth with the empirical formulas

$$\left.\begin{aligned} P_1 &= \tfrac{1}{2} - (.266 \div \sqrt{n}), \\ r_1 &= \mp .6745\varepsilon + (2 \div 5a), \\ r_2 &= \pm .6745\varepsilon + (2 \div 5a), \end{aligned}\right\} \qquad (88)$$

the upper or lower sign being used according as a is $+$ or $-$.

'o illustrate the application of the table, take the amplitudes in Table for which we have obtained 4°.29 as the most probable value. At p. ɜ find $a^2b = 5.57$. With this argument, Table III. gives

$$-ar_1 = 1.085, \qquad ar_2 = 1.879;$$

$$\therefore\ r_1 = \frac{-ar_1}{-a} = \frac{1.085}{-1.095} = -.99,$$

$$r_2 = \frac{ar_2}{a} = \frac{1.879}{1.095} = 1.72.$$

ıese are *CE* and *CF* in the figure. It is an even chance whether amdes below 4°.29 will be over or under 3°.30, and an even chance her those above 4°.29 will be over or under 6°.01.

› show what appears to be the proper method of combining observa- when the c. m. inequality is of accidental origin, we will now find the probable mean value and probable error of an angle observed with a lolite by the method of repetition. The example is taken from Mey- *Wahrscheinlichkeitsrechnung*, p. 274, where it is treated, of course, in sual way, by the symmetrical law of probability. Fourteen observa- are taken, each giving the angle 17° 56′, plus a number of seconds a stand in our Table IV. in the column headed u. The number of rep- ns to each observation gives its weight, which stands in column p. next column shows the products pu. The general arith. mean is

$$u_0 = \frac{[pu]}{[p]} = \frac{1830}{46} = 39''.78.$$

btracting this from each u, we get the series of residuals v. These are

TABLE IV.

No.	u	p	pu	v	pv^2	p^2v^3
1	45″.00	5	225.0	5.22	136·25	3556.0
2	31 .25	4	125.0	— 8.53	291.04	— 9930.4
3	42 .50	5	212.5	2.72	37.00	503.0
4	45 .00	3	135.0	5.22	81.75	1280.2
5	37 .50	3	112.5	— 2.28	15.60	— 106.6
6	38 .33	3	115.0	— 1.45	6.30	— 27.5
7	27 .50	3	82.5	—12.28	452.40	—16666.2
8	43 .33	3	130.0	3.55	37.80	402.7
9	40 .63	4	162.5	.85	2.88	9.8
10	36 .25	2	72.5	— 3.53	24.92	— 176.0
11	42 .50	3	127.5	2.72	22.20	181.1
12	39 .17	3	117.5	— .61	1.11	— 2.1
13	45 .00	2	90.0	5.22	54.50	569.0
14	40 .83	3	122.5	1.05	3.30	10.4
		46	1830.0		1167.05	—20396.6

to be reduced to unit weight, before combining them for the purpose of finding the law of error of u_0. According to (60), the q. m. error of unit weight is $\sqrt{p}$ times that of weight p, and the c. m. inequality of unit wei't is $\sqrt[3]{p^2}$ times that of weight p. In other words, in a system of errors whose weig t is unity, the mean of the squares of the errors is p times, and the mean of the cubes of the errors taken with their signs is p^2 times what they would be in a system of errors having the weight p. Hence, as is well known, in order to find the approximate q. m. error ε of unit weight from the squared residuals, each v^2 of weight p must be multiplied by p. Likewise, to find the c. m. inequality ζ of unit weight from the cubed residuals, each v^3 of weight p must be multiplied by p^2. The last two columns in the table show the products pv^2 and p^2v^3. The number of observations being $m = 14$, we have for the square of the true q. m. error of unit weight

$$\varepsilon_1^2 = \frac{[pv^2]}{m-1} = \frac{1167.05}{13} = 89.77,$$

and as shown at p. 6, by virtue of the relation

$$\zeta_1^3 = \frac{[p^2v^3]}{m} + \frac{\zeta_1^3}{m^2},$$

the cube of the true c. m. inequality of unit weight is

$$\zeta_1^3 = \frac{m[p^2v^3]}{m^2-1} = \frac{14(-20396.6)}{195} = -1464.4. \tag{89}$$

Then for the arith. mean, whose weight is $[p] = 46$,

$$\left.\begin{aligned} \varepsilon_0^2 &= \frac{\varepsilon_1^2}{[p]} = \frac{89.77}{46} = 1.9515, \\ \zeta_0^3 &= \frac{\zeta_1^3}{[p]^2} = \frac{-1464.4}{46^2} = -.6921. \end{aligned}\right\} \tag{90}$$

These are the (q. m. e.)^2 of u_0 and the (c. m. i.)^3 of its possible errors, and from them we can construct the unsymmetrical curve which represents approximately the probabilities of all the possible errors of u_0.

By (39) we have

$$\left.\begin{aligned} a &= \frac{2\varepsilon_0^2}{\zeta_0^3} = 2\left(\frac{1.9515}{-.6921}\right) \doteq -5.640, \\ b &= \varepsilon_0^2 = 1.9515. \end{aligned}\right\} \tag{91}$$

Since a is negative, the curve has the reversed position. By (82) the most probable error of u_0 here is

$$\Delta_0 = -\frac{1}{a} = \frac{-\zeta_0^3}{2\varepsilon_0^2}, \;\therefore\; \Delta_0 = \frac{1}{5.640} = 0''.18. \tag{92}$$

This is more probable than the error zero, or any other. Hence, for the best value of the mean we must take not u_0 alone, but

$$u_0 + \Delta_0 = 39''.78 + 0''.18 = 39''.96.$$

To find the probable errors of this result, we enter Table III. with the argument

$$a^2 b = 1.9515\,(5.640)^2 = 62.1,$$

and there obtain

$$-ar_1 = 4.892, \qquad ar_2 = 5.691.$$

The probable errors are therefore

$$r_1 = \frac{-ar_1}{-a} = \frac{4.892}{5.640} = 0''.87,$$

$$r_2 = \frac{ar_2}{a} = \frac{5.691}{-5.640} = -1''.01,$$

and the most probable value of the mean is written

$$17^\circ\ 56'\ 39''.96 \begin{smallmatrix}+0.87\\-1.01\end{smallmatrix}. \tag{93}$$

We have this final result instead of $17^\circ\ 56'\ 39''.78 \pm 0.94$ as obtained by the ordinary method. Table III. also gives the probabilities that the error of u_0 will fall on the short or the long side of the curve from $u_0 + \varDelta_0$, meaning in this case error in excess or defect respectively. They are

$$P_1 = .4663, \qquad P_2 = 1 - P_1 = .5337.$$

While (93) shows an even chance for the error to fall within the limits $+0.87$ and -1.01, the chance of its being between 0 and 0.87 is not the same as that of its being between 0 and -1.01. The probabilities of the two events are

$$\tfrac{1}{2}P_1 = .2331, \qquad \tfrac{1}{2}P_2 = .2669.$$

We may observe that when the law of probability is unsymmetrical, the ordinary rule that probable error varies inversely as the square root of the weight, will not hold good. The probable errors depend not only on ε but on ζ, and these, as (60) shows, vary with the weight by different laws. An observation of weight p is supposed to represent the arith. mean of p observations chosen at random, out of a set of observations of unit weight sufficiently numerous to show the unsymmetrical law of error.

If the c. m. inequality of a set of observations of equal weight, is purely accidental, the probable value of ζ^2 is roughly expressed in termrs of ε^3 by the relation (80). When observations are of unequal weight, ζ^3 and ε^3 varying with the weight in different ratios, we may consider (80) as approximately valid for the average weight of the observations, so that ε and ζ denote the q. m. error and c. m. inequality for the weight $[p] \div m$. Then according to (60)

$$\varepsilon^2 = \left(\frac{m}{[p]}\right)\varepsilon_1^2, \qquad \zeta^3 = \left(\frac{m}{[p]}\right)^2 \zeta_1^3.$$

Substituting these in (80) we get

$$(\zeta_1^3) = \pm \frac{.6745\,\varepsilon_1^3}{m} \sqrt{(15[p])}. \qquad (94)$$

This is the limit within which there is about an even chance for ζ_1^3 to fall if the inequality is fortuitous. In our present example it gives

$$(\zeta_1^3) = \pm \frac{.6745}{14} \sqrt{[15\times 46(89.77)^3]} = 1076.4.$$

The actual value by (89) is $\zeta_1^3 = -1464.4$. The excess is not very great, and taken in connection with what we know of the nature of theodolite observations, it does not justify the belief that there was any real want of symmetry in the law of error.

Granting that the true law is symmetrical, the most probable value of the observed quantity is the ultimate arith. mean, that is, the mean of all the possible values, each taken a number of times proportional to the probability of its occurrence. But this ultimate mean cannot be found without taking a number of observations which is very large indeed, or infinite. We have to infer its probable value from a moderate number, in the foregoing case only $m = 14$. Owing to the small number of observations, the distribution of the residual errors is irregular and unsymmetrical, and the law of error of the mean, so far as it can be deduced from the observations, is an unsymmetrical law. Hence, the most probable value of the ultimate arith. mean is not the apparent mean u_0, but u_0 plus the correction Δ_0 due to the c. m. inequality ζ_0. The resulting value of the mean and its probable errors as exemplified in (93) constitute, as it seems to me, the best determination we can make of the most probable value of the observed quantity.

CORRESPONDENCE.

Editor Analyst:

I desire to make an additional remark in regard to the paradox in the Query by Professor Johnson. (See ANALYST, No. 2, p. 44.)

The principle which seems to be violated in the Ex. given is this:—If $u = f(a,x)$,.. (1) and $du \div dx = f'(a,x)$,.. (2) and we assign to a such value as will reduce u to a constant or to zero, then the same value ought to reduce $f'(a,x)$ to zero. If it does not, the paradox must arise from a misinterpretation of the result, or of some of the *singular forms* that enter into the work.

Now although $\sin\infty$ and $\cos\infty$ may each be regarded as zero, in the sense that any constant may be regarded as zero in comparison with another quantity which is infinitely greater, we cannot admit 0 as an absolute value in violation of the principle $\sin^2\infty + \cos^2\infty = 1$.

The paradox clearly arises from a misinterpretation of the form $a \div \infty$.

If $u = a^{-1}\sin ax$, then $du \div dx = \cos ax$ is true for all definite values of a, *however great;* and there is no difficulty in the in erpretation of the result. But when $a = \infty$ both u and $du \div dx$ become indefinite.

If however $a = \infty$ renders u *infinitesimal,* it still remains a function of x, and there seems to be no good reason why we should then expect $du \div dx$ to equal zero, *independently* of x. Let $u = a^{-1}\tan ax$, then $du \div dx = \sec^2 ax$ which is always greater than unity, though u may be infinitesimal.

If we write $u = a\sin(x \div a) \ldots (1)$, and $du \div dx = \cos(x \div a)$ and then make $a = 0$, the paradox appears in another form. If $a = 0$ (absolutely) then $u = 0$ independently of x; and if $x \div 0$ is a *quantity* $= \infty$ the paradox remains. But if, as I have attempted to show (see ANALYST, for July 1881), $x \div 0$ is only *a symbol* of *impossible conditions,* and not one of quantity, the paradox disappears.

Reject the forms $a.0^{-1} = \infty$, and $a.\infty^{-1} = 0$, and substitute $a.\circ^{-1} = \infty$, and $a.\infty^{-1} = \circ$, $a.0^{-1} =$ sign of impossible conditions, and a host of paradoxes will vanish.

Note. ∞ and $\circ$ are employed as symbols of variables, one increasing without limit and the other decreasing without limit.

C. H. JUDSON.

Greenville, S. C., March 10, 1883.

CORRECTION OF AN ERROR IN BARTLETT'S MECHANICS.—A remarkable error occurs in Bartlett's Analytical Mechanics, ninth edition, p. 178. The solution of the differential equation

$$\frac{d^2v_y}{dt^2} - m^2v_y = 0$$

is given as

$$v_y = a e^{mt+c},$$

(in which I have written m^2 for the constant $n^2(A-C)(C-B) \div (A.B)$, supposed positive) and a and c for a_y and c_y the constants of integration. The apparent presence of two constants appears to have induced the belief that the integral thus written was complete; whereas it can of course be written

$$v_y = a_y e^c e^{mt},$$

in which $a_y e^c$ is equivalent to a single constant. Thus the value given is only a particular integral, the complete integral being of the form

$$v_y = Ae^{mt} + Be^{-mt}.$$

The differential equation arises in the solution of simultaneous equations of the first order for the determination of v_x and v_y, and it may be remarked that both this and the preceding solution are faulty in presenting different sets of constants in the values of v_y and v_x, or in all four constants instead of two.

WM. WOOLSEY JOHNSON.

NOTE ON ANHARMONIC RATIOS.

BY PROF. WILLIAM WOOLSEY JOHNSON.

THE present note is a continuation of my paper "New Notation for Anharmonic Ratios", p. 185, ANALYST, Vol. IX; and the sections and equations are for convenience numbered consecutively with those of that paper.

12. As in section 2, the anharmonic ratio which we have denoted by

$$x = \begin{matrix} P & A \\ B & Q \end{matrix}$$

is regarded as the coordinate of P in an anharmonic system in which the coordinates of A, B and Q, which we may call *the fundamental points*, are 0, ∞, and 1, respectively. Now if p, a, b and q denote ordinary coordinates of the same points, that is, their distance from an assumed origin measured with an assumed unit of length, we may also write

$$x = \begin{matrix} p & a \\ b & q \end{matrix},$$

in which p, a, b and q are numbers; and, in accordance with the definition of the symbol in section 3, we have

$$x = \begin{matrix} p & a \\ b & q \end{matrix} = \frac{(p-a)(b-q)}{(p-b)(a-q)}. \tag{7}$$

13. The results of sections 4, 7, and 8 may then be expressed as follows:

1°. If, in this symbol, we interchange a and b, we convert the coordinate x into its reciprocal, interchanging the fundamental points 0 and ∞, and leaving the point 1 unmoved.

2°. If we interchange a and q, we convert x into its complement, interchanging the fundamental points 0 and 1, and leaving the point ∞ unmoved.

3°. If we interchange b and q, we take the conjugate, interchanging the points 1 and ∞, and leaving the point 0 unmoved. Thus, if

$$x = \begin{matrix} p & a \\ b & q \end{matrix}, \qquad Rx = \frac{1}{x} = \begin{matrix} p & b \\ a & q \end{matrix},$$

$$Cx = 1-x = \begin{matrix} p & q \\ b & a \end{matrix}, \qquad Jx = \frac{x}{x-1} = \begin{matrix} p & a \\ q & b \end{matrix}.$$

In like manner we have

$$CRx = \frac{x-1}{x} = \begin{matrix} p & q \\ a & b \end{matrix}, \qquad RCx = \frac{1}{1-x} = \begin{matrix} p & b \\ q & a \end{matrix},$$

in which the fundamental points are cyclically interchanged.

14. We have three special varieties of coordinates.

I. When the fundamental point B is at an infinite distance; the coordi

nates become ordinary coordinates or distances measured from A, AQ being the unit of length.

II. When the fundamental point A is at infinity; the coordinates become the reciprocals of the distances from B, BQ being the unit of length.

III. When the fundamental point Q is at infinity; the coordinates become the "position ratios" of section 1.

In transforming the system I to the system II, the operation is R; the points A and B interchanging: in transforming I to III, the operation is J; B and Q interchanging: in transforming II to III, the operation is C; A and Q interchanging.

15. The symbol $\begin{matrix} p & a \\ b & q \end{matrix}$ is obviously, from equation (7), a function of p, a, b and q, whose value is unaltered by adding the same quantity to each of the letters, or by multiplying them by the same quantity; that is to say, by changing the position of the origin in assigning the ordinary coordinates of P, A, B and Q, or by changing the scale with which those coordinates are measured, which is equivalent to changing the position of the unit point.

16. But furthermore the value of the symbol is unchanged when if for its elements we substitute any anharmonic coordinates of the same points; or

$$\begin{matrix} x & y \\ z & u \end{matrix} = \begin{matrix} x' & y' \\ z' & u' \end{matrix}, \tag{8}$$

if
$$x = \begin{matrix} x' & a \\ b & q \end{matrix} \qquad y = \begin{matrix} y' & a \\ b & q \end{matrix} \qquad \text{etc.}$$

For, since $x = \dfrac{(x'-a)(b-q)}{(x'-b)(a-q)}$ etc., we may write

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta}, \qquad y = \frac{\alpha y' + \beta}{\gamma y' + \delta}, \quad \text{etc.,} \tag{9}$$

whence
$$x - y = \frac{\alpha\delta x' + \beta\gamma y' - \alpha\delta y' - \beta\gamma x'}{(\gamma x' + \delta)(\gamma y' + \delta)}$$
$$= \frac{(\alpha\delta - \beta\gamma)(x' - y')}{(\gamma x' + \delta)(\gamma y' + \delta)}.$$

Employing like expressions for the other differences, we have

$$\begin{matrix} x & y \\ z & u \end{matrix} = \frac{(x-y)(z-u)}{(x-z)(y-u)} = \frac{(x'-y')(z'-u')}{(x'-z')(y'-u')} = \begin{matrix} x' & y' \\ z' & u' \end{matrix}.$$

In particular, we may in the equation

$$x = \begin{matrix} p & a \\ b & q \end{matrix}$$

substitute for p, a, b, q the anharmonic coordinates of the points P, A, B and Q in the very system which this equation defines; we thus have

$$x = \begin{matrix} x & 0 \\ \infty & 1 \end{matrix},$$

as may be easily verified by equation (7).

17. It is obvious that in equation (8) both x, y, z, u and x', y', z', u' may be anharmonic coordinates of the same four points referred to different fundamental points; for each member will be equal to the corresponding function of the ordinary coordinates of the same points. In other words, the anharmonic ratio of the anharmonic coordinates of four points, referred to any fundamental points, is constant, for it is simply the anharmonic ratio of the four points. It follows that the relation between the anharmonic coordinates of the same point in different systems is of the form expressed by equation (9).

18. If two lines L and L' be homographically divided, and corresponding points A, B, Q and A', B', Q' be taken as fundamental points; then, by the definition of homographic division, the anharmonic coordinates x and x' of corresponding points P and P' are equal. Hence, if non-corresponding points are taken as fundamental points, x and x' are the same as the anharmonic coordinates of the same point referred to different fundamental points; for x' is the same as the coordinate of P when referred to the points on L which correspond to the fundamental points taken on L'. x and x' have therefore, in this case also, a relation of the form expressed by equation (9).

19. When the lines L and L' coincide, and P and P' are referred to the same fundamental points, the case is equivalent to that last mentioned, since three pairs of corresponding points cannot coincide, unless all corresponding points coincide. It is possible however to make two pairs of corresponding points coincide, and one of these may be a pair selected at random: for if this pair be brought into coincidence (the lines L and L' intersecting at any angle), the corresponding points are in perspective, and a line drawn through the centre of perspective perpendicular to either of the bisectors of the angles between L and L' will determine a pair of corresponding points equidistant from the coincident pair at the intersection. When the lines are revolved into coincidence, one of these pairs of corresponding points comes into coincidence, and we have two pairs of coincident corresponding points, which are in fact the double points of the homographic system.

20. If two of the fundamental points are taken at the double points, the relation between x and x' is simplified. Thus, if A and B are at the double points, so that

$$x = \begin{matrix} P & A \\ B & Q \end{matrix}, \qquad x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & A \\ B & Q_1 \end{matrix},$$

where Q_1, see section 18, is the point of L corresponding to Q regarded as a point of L'; then the definition of x, equation (7), shows that the relation is of the form

$$x' = mx. \tag{10}$$

This is the simplest relation possible, being the form assumed by (9) when $\beta = 0$ and $\gamma = 0$; and is such that x and x' vanish simultaneously, and become infinite simultaneously, as should be the case, since P and P' arrive simultaneously at A, and also simultaneously at B.

Equation (10) is by equation (1) equivalent to

$$\frac{AP'}{BP'} = m\,\frac{AP}{BP},$$

whence

$$\frac{P\;\;A}{B\;\;P'} = \frac{1}{m},$$

that is, the anharmonic ratio of two corresponding points and the double points, is constant.

21. If however we take B and Q at the double points, so that

$$x' = \frac{P'\;\;A}{B\;\;Q} = \frac{P\;\;A_1}{B\;\;Q},$$

then the complements of the coordinates,

$$1 - x = \frac{P\;\;Q}{B\;\;A}, \quad \text{and} \quad 1 - x' = \frac{P\;\;Q}{B\;\;A_1}$$

have the relation given in equation (10); that is,

$$1 - x' = m(1 - x),$$

or

$$x' = 1 - m + mx, \tag{11}$$

a relation of the form (9), in which $\gamma = 0$ and $\alpha + \beta = \delta$; and in which x and x' take the value unity simultaneously, and become infinite simultaneously.

22. Again, if A and Q are taken at the double points, so that

$$x' = \frac{P'\;\;A}{B\;\;Q} = \frac{P\;\;A}{B_1\;\;Q},$$

then the conjugates of the coordinates,

$$\frac{x}{x-1} = \frac{P\;\;A}{Q\;\;B}, \quad \text{and} \quad \frac{x'}{x'-1} = \frac{P\;\;A}{Q\;\;B_1},$$

have the relation given in (10); thus

$$\frac{x'}{x'-1} = m\,\frac{x}{x-1},$$

whence

$$x' = \frac{mx}{(m-1)x+1}, \tag{12}$$

a relation of the form (9), in which $\beta=0$ and $\alpha=\gamma+\delta$; and in which x and x' vanish simultaneously, and take the value unity simultaneously.

23. When two points of L, whose distance is the same as that of their corresponding points on L' have been found, as in section 19, the lines L and L' may be made to coincide in such a manner that each of these points falls upon the point corresponding to the other. If now the fundamental points A and B are taken at these *doubly corresponding* points; in other words, if A as a point of L corresponds with B as a point of L', and A as a point of L' corresponds with B as a point of L, we have

$$x = \begin{matrix} P & A \\ B & Q \end{matrix}, \qquad x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & B \\ A & Q_1 \end{matrix};$$

since, as in section 18, x' is the same as the coordinate of P, referred to the points of L corresponding to the fundamental points A, B, Q regarded as points of L'. Hence x' is now the reciprocal of the x' in equation (10), and the relation is of the form

$$xx' = k, \tag{13}$$

in which $x=\infty$ gives $x'=0$.

24. In like manner, if B and Q are at the doubly corresponding points,

$$x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & A_1 \\ Q & B \end{matrix};$$

and x' is the conjugate of its value in equation (11). Replacing it therein by $x' \div (x'-1)$, the relation is of the form

$$xx' = x + x' + k, \tag{14}$$

in which $x=\infty$ gives $x'=1$.

25. Again, if A and Q are at the doubly corresponding points,

$$x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & Q \\ B_1 & A \end{matrix},$$

which is the complement of x' in equation (12); and replacing x' by $1-x'$ in that equation, we have a relation of the form

$$1 = x + x' + kxx', \tag{15}$$

in which $x=0$ gives $x'=1$.

26. The infinitely distant points on L and L' may be regarded as always coinciding when L' is superposed upon L; hence, if the points each of which corresponds on one line to the infinitely distant point upon the other line are made to coincide, this point of coincidence and the point at infinity are doubly corresponding points. Hence B, in equation (13), may be at infinity; but in this case the coordinates become distances from A. The two ranges of points now form a system in involution; equation (13) expressing that the product of the distances of corresponding points from A, the centre of involution, is constant.

INTEGRATION BY AUXILIARY INTEGRALS.

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

WHEN the integration of a function of the form $dy = f(x).dx$ is required in finite terms, we must reduce the function to some one or more of the well known fundamental forms. By far the most fruitful method of so doing is the substitution of a new variable; and the real difficulty to be overcome consists in the judicious choice of the form of the new variable in terms of x. Very often the reduction to any of the fundamental forms is found impossible, or at least all our attempts to the purpose prove futile and we have no resource left but the expansion of the function into a converging series of which the single terms are then to be integrated.

It would seem at first sight that, although the integration in finite terms may be found impracticable, yet the opposite mode of procedure must yield an abundance of new results and must lead to new fundamental forms, viz., the assumption of some new form of the variable instead of the variable of the fundamental integrals, which subst. would immediately yield new and useful primitive integrals which might in their turn be used as we now use the primitive forms of the text-books. The great obstacle, however, to the invention of new primitive integrals in this manner lies in the fact that upon differentiating the new variable, the differential is commonly found to consist of more than one term, so that, although the aggregate value of the integrals so arising is known, yet the values of the integrals each singly, are not known.

The result of this state of affairs has been that to the present day the number of differentials integrable in finite terms is quite limited. This is especially true of the so-called irrational functions; and an examination of any collection of tables of integrals reveals the fact that of irrational functions scarcely any are integrable in finite terms save such where the variable occurs at no higher power than the second. Indeed besides these radicals of the second degree scarcely any but the "binomial" differentials seem to have been studied. I purposely leave out of consideration the elliptic integrals and other transcendental functions, as not falling under the forms proposed, namely those integrable in finite form; my intention being, in the present paper, to treat of an extension of the well known methods of reduction of differentials to the primitive forms by means of other differentials of which the integrals are already known.

In order to give, at the outset, some idea of the method here chiefly to be

followed, I will illustrate the same by the following simple example. Let it be required to integrate

$$dy = \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}.$$

This form readily suggests two of the primitive forms, namely:

$$\frac{dx}{1+x^2} \text{ and } \frac{dx}{\sqrt{(1-x^2)}}.$$

We will put

$$dz = \frac{dx}{1+x^2}, \text{ so that } z = \operatorname{arc\,tg} x + C.$$

Since we are at liberty to assign to z any value compatible with the last equation, we can put $C = 0$, so that

$$z = \operatorname{arc\,tg} x, \quad \therefore x = \operatorname{tg} z.$$

We will now compare dy with dz, thus

$$\frac{dy}{dz} = \frac{1}{\sqrt{(1-x^2)}} = \frac{1}{\sqrt{(1-\operatorname{tg}^2 z)}};$$

$$\therefore dy = \frac{dz}{\sqrt{(1-\operatorname{tg}^2 z)}} = \frac{\cos z.dz}{\sqrt{(1-2.\sin^2 z)}}, \text{ hence}$$

$$y = \tfrac{1}{2}\sqrt{2}.\operatorname{arc\,sin}(\sqrt{2}.\sin z) + C.$$

On the other hand, we might also have integrated in this way: Putting $dt = dx \div \sqrt{(1-x^2)}$, $\therefore t = \operatorname{arc\,sin} x$, $\therefore x = \sin t$.

$$\frac{dy}{dt} = \frac{1}{1+x^2} = \frac{1}{1+\sin^2 t}, \; dy = \frac{dt}{1+\sin^2 t};$$

$$\therefore y = \tfrac{1}{2}\sqrt{2}.\operatorname{arc\,tg}(\sqrt{2}.\operatorname{tg} t) + C.$$

Now, in order to avail ourselves of the method just illustrated we will first extend somewhat the range of the primitive integrals of the books and then make use of the functions so found for the purpose just indicated. It will be seen in the sequel that we can easily by our method find a great number of integrals which to find in finite terms by the methods of the books would be impracticable.

For the sake of convenience I will arrange the new primitive functions into classes, or types, each corresponding to one of the fundamental forms of the text-books. I would beg leave, for this purpose, to retain the continental notation, namely arc sin x for $\sin^{-1}x$ and arc tg x for $\operatorname{tang}^{-1}x$ &c., which notation, it will be seen, is more suitable for our purpose.

A. The type arc sin x.

The following integrals are easily verified by differentiation. The constant of integration is omitted. The inversion of the integral function is subjoined in each case, since with it we shall chiefly have to deal.

$$\int \frac{dx}{\sqrt{x}\sqrt{(1-x)}} = \frac{2}{1}\text{arc sin}\, x^{1/2} = y;\ \therefore x = \left(\sin \frac{y}{2}\ \right)^2.$$

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \frac{2}{2}\text{arc sin}\, x\ = y;\ \therefore x = \left(\sin \frac{2}{2}y\right)^{\frac{2}{2}} = \sin y.$$

$$\int \frac{\sqrt{x}.dx}{\sqrt{(1-x^3)}} = \frac{2}{3}\text{arc sin}\, x^{\frac{3}{2}} = y;\ \therefore x = \left(\sin \frac{3}{2}y\right)^{\frac{2}{3}}.$$

$$\cdots\cdots\cdots\cdots$$

$$\int \frac{x^{1/2 n-1}dx}{\sqrt{(1-x^n)}} = \frac{2}{n}\text{arc sin}\, x^{1/2 n} = y;\ \therefore x = \left(\sin \frac{n}{2}y\right)^{\frac{2}{n}};$$

here n is any positive or negative integer or fraction.
Similarly:

$$\int \frac{dx}{x\sqrt{(x-1)}} = -2\ \text{arc sin}\, x^{-1/2} = y;\ \therefore x = \frac{1}{(\sin \frac{1}{2}y)^2}.$$

$$\int \frac{dx}{x\sqrt{(x^2-1)}} = -\frac{2}{2}\ \text{arc sin}\, x^{-1} = y;\ \therefore x = \frac{1}{(\sin y)}.$$

$$\int \frac{dx}{x\sqrt{(x^3-1)}} = -\frac{2}{3}\text{arc sin}\, x^{-\frac{3}{2}} = y;\ \therefore x = \frac{1}{(\sin \frac{3}{2}y)^{2/3}}.$$

$$\cdots\cdots\cdots\cdots$$

$$\int \frac{dx}{x\sqrt{(x^n-1)}} = -\frac{2}{n}\text{arc sin}\, x^{-1/2 n} = y;\ \therefore x = \frac{1}{(\sin \frac{n}{2}y)^{\frac{2}{n}}};$$

here n may be any positive or negative integer or fraction.
Similarly, when putting (as is customary) $i = \sqrt{-1}$,

$$\int \frac{dx}{\sqrt{x}\sqrt{(1+x)}} = -2i.\text{arc sin}\,(i.x^{1/2}) = y;\ \therefore x^{1/2} = -i \sin \tfrac{1}{2}iy.$$

$$\int \frac{dx}{\sqrt{(1+x^2)}} = -\frac{2}{2}i.\text{arc sin}\,(i.x\ ) = y;\ \therefore x\ = -i \sin iy.$$

$$\cdots\cdots\cdots\cdots$$

$$\int \frac{x^{1/2 n-1}dx}{\sqrt{(1+x^n)}} = -\frac{2}{n}i.\text{arc sin}\,(i.x^{1/2 n}) = y;\ \therefore x^{1/2 n} = -i \sin \tfrac{n}{2}iy.$$

And so also:

$$\int \frac{dx}{x\sqrt{(x+1)}} = 2i.\text{arc sin}\,(i.x^{-1/2}) = y;\ \therefore x^{-1/2} = i.\sin\,(i.\tfrac{1}{2}y).$$

$$\int \frac{dx}{x\sqrt{(x^2+1)}} = \frac{2}{2}i.\text{arc sin}\,(i.x^{-1}) = y;\ \therefore x^{-1} = i.\sin\,(i.\ y).$$

$$\int \frac{dx}{x\sqrt{(x^3+1)}} = \frac{2}{3}i.\text{arc sin}\,(i.x^{-\frac{3}{2}}) = y;\ \therefore x^{-3/2} = i.\sin\,(i.\tfrac{3}{2}y).$$

$$\cdots\cdots\cdots\cdots$$

$$\int \frac{dx}{x\sqrt{(x^n+1)}} = \frac{2}{n}i.\text{arc sin}\,(i.x^{-1/2 n}) = y;\ \therefore x^{-\frac{n}{2}} = i.\sin\,(i.\tfrac{n}{2}y).$$

The following forms belonging to the same type are also easily verified by differentiation:

$$\int\frac{dx}{(1+x)\sqrt{x}\sqrt{(1-x)}}=-\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x}{1+x}}=y;\ \therefore\ x=\frac{1-\sin^2(\frac{1}{2}y\sqrt{2})}{1+\sin^2(\frac{1}{2}y\sqrt{2})}$$

$$\int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}}=-\tfrac{1}{2}\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}}=y;\ \therefore\ x^2=\frac{1-\sin^2(\ y\sqrt{2})}{1+\sin^2(\ y\sqrt{2})}$$

$$\int\frac{\sqrt{x}.dx}{(1+x^3)\sqrt{(1-x^3)}}=-\tfrac{1}{3}\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x^3}{1+x^3}}=y;\ \therefore\ x^3=\frac{1-\sin^2(\frac{3}{2}y\sqrt{2})}{1+\sin^2(\frac{3}{2}y\sqrt{2})}$$

. . . .

$$\int\frac{x^{\frac{1}{2}n-1}dx}{(1+x^n)\sqrt{(1-x^n)}}=-\tfrac{1}{n}\sqrt{2}\text{ arc sin}\sqrt{\frac{1-x^n}{1+x^n}}=y;\ \therefore\ x^n=\frac{1-\sin^2(\frac{n}{2}y\sqrt{2})}{1+\sin^2(\frac{n}{2}y\sqrt{2})}$$

Again:

$$\int\frac{dx}{(1-x)\sqrt{x}\sqrt{(1+x)}}=-\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x}{1-x}}=y;\ \therefore$$
$$x=\frac{\sin^2(\frac{1}{2}y\sqrt{-2})-1}{\sin^2(\frac{1}{2}y\sqrt{-2})+1}.$$

$$\int\frac{dx}{(1-x^2)\sqrt{(1+x^2)}}=-\tfrac{1}{2}\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x^2}{1-x^2}}=y;\ \therefore$$
$$x^2=\frac{\sin^2(\ y\sqrt{-2})-1}{\sin^2(\ y\sqrt{-2})+1}$$

. . . .

$$\int\frac{x^{\frac{1}{2}n-1}dx}{(1-x^n)\sqrt{(1+x^n)}}=-\tfrac{1}{n}\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x^n}{1-x^n}}=y;\ \therefore$$
$$x^n=\frac{\sin^2(\frac{1}{n}y\sqrt{-2})-1}{\sin^2(\frac{1}{n}y\sqrt{-2})+1}.$$

Again:

$$\int\frac{dx}{\sqrt{(2ax-x^2)}}=\text{ arc sin}\frac{x-a}{a}=y;\ \therefore\ x=a(1+\sin\ y).$$

$$\int\frac{dx}{\sqrt{(2a-x^2)}}=\tfrac{1}{2}\text{arc sin}\frac{x^2-a}{a}=y;\ \therefore\ x^2=a(1+\sin 2y).$$

. . . .

$$\int\frac{x^{\frac{1}{2}n-1}dx}{\sqrt{(2a-x^n)}}=\tfrac{1}{n}.\text{arc sin}\frac{x^n-a}{a}=y;\ \therefore\ x^n=a(1+\sin ny).$$

And again:

$$\int\frac{dx}{x\sqrt{(2x-1)}}=\text{ arc sin}\left(1-\frac{1}{x}\right)=y;\ \therefore\ x=\frac{1}{1-\sin\ y}.$$

$$\int\frac{dx}{x\sqrt{(2x^2-1)}}=\tfrac{1}{2}\text{arcsin}\left(1-\frac{1}{x^2}\right)=y;\ \therefore\ x^2=\frac{1}{1-\sin 2y}.$$

. . . .

$$\int\frac{dx}{x\sqrt{(2x^n-1)}}=\tfrac{1}{n}\text{arcsin}\left(1-\frac{1}{x^n}\right)=y;\ \therefore\ x^n=\frac{1}{1-\sin ny}.$$

B. THE TYPE ARC TG X.

$$\int\frac{dx}{(2+x)\sqrt{(1+x)}} = 2\,\text{arc tg}\sqrt{(1+x)} = y;\ \therefore x = \text{tg}^2(\tfrac{1}{2}y)-1.$$

$$\int\frac{x^{n-1}dx}{(2+x^n)\sqrt{(1+x^n)}} = \frac{2}{n}\,\text{arc tg}\sqrt{(1+x^n)} = y;\ \therefore x^n = \text{tg}^2(\tfrac{1}{n}y)-1.$$

$$\int\frac{dx}{(2x+1)\sqrt{x}\sqrt{(x+1)}} = -\ \text{arc tg}\left(1+\frac{1}{x}\right)^{\frac{1}{2}} = y;\ \therefore x = \frac{1}{\text{tg}^2\,y-1}.$$

$$\int\frac{dx}{(2x^2+1)\sqrt{(x^2+1)}} = -\tfrac{1}{2}\text{arc tg}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} = y;\ \therefore x = \frac{1}{\text{tg}^2 2y-1}.$$

. . . .

$$\int\frac{x^{\frac{1}{2}n-1}\,dx}{(2x^n+1)\sqrt{(x^n+1)}} = -\tfrac{1}{n}\text{arc tg}\left(1+\frac{1}{x^n}\right)^{\frac{1}{2}} = y;\ \therefore x^n = \frac{1}{\text{tg}^2 ny-1}.$$

C. THE TYPE LOG X.

The following primitive integrals are also easily verified by differentiation.

$$\int\frac{dx}{x(x-1)} = \lg\left(1-\frac{1}{x}\right) = y;\ \therefore x = \frac{1}{1-e^y}.$$

$$\int\frac{dx}{x(x^2-1)} = \tfrac{1}{2}\lg\left(1-\frac{1}{x^2}\right) = y;\ \therefore x = \frac{1}{1-e^{2y}}.$$

$$\int\frac{dx}{x(x^3-1)} = \tfrac{1}{3}\lg\left(1-\frac{1}{x^3}\right) = y;\ \therefore x = \frac{1}{1-e^{3y}}.$$

. . . .

$$\int\frac{dx}{x(x^n-2)} = \tfrac{1}{n}\lg\left(1-\frac{1}{x^n}\right) = y;\ \therefore x = \frac{1}{1-e^{ny}}.$$

$$\int\frac{dx}{x(x+1)} = -\ \lg\left(1+\frac{1}{x}\right) = y;\ \therefore x = \frac{e^y}{1-e^y}.$$

$$\int\frac{dx}{x(x^2+1)} = -\tfrac{1}{2}\lg\left(1+\frac{1}{x^2}\right) = y;\ \therefore x = \frac{e^{2y}}{1-e^{2y}}.$$

. . . .

$$\int\frac{dx}{x(x^n+1)} = -\tfrac{1}{n}\lg\left(1+\frac{1}{x^n}\right) = y;\ \therefore x = \frac{e^{ny}}{1-e^{ny}}.$$

$$\int\frac{dx}{x\sqrt{(x^n-1)}[x^{\frac{1}{2}n}+\sqrt{(x^n-1)}]} = \tfrac{2}{n}\lg\left\{1+\left(1-\frac{1}{x^n}\right)^{\frac{1}{2}}\right\} = y;\ \therefore$$

$$x^n = \frac{1}{e^{\frac{1}{2}ny}(2-e^{\frac{1}{2}ny})}.$$

$$\int\frac{dx}{x\sqrt{(x^n+1)}[x^{\frac{1}{2}n}+\sqrt{(x^n-1)}]} = -\tfrac{2}{n}\lg\left\{1+\left(1+\frac{1}{x^n}\right)^{\frac{1}{2}}\right\} = y;\ \therefore$$

$$x^n = \frac{e^{ny}}{1-2e^{ny}}.$$

The application of the foregoing formulæ I begin with two functions commonly treated in the text-books. This will lead to a comparison of methods. Next I shall proceed to the integration of functions not found in the books.

The function

$$dy = \frac{x^m dx}{\sqrt{(1-x^2)}}$$

is usually treated in the books by means of a formula of reduction and the two cases where m is odd or even are separately considered. This function at once calls to mind the elementary form

$$dz = \frac{dx}{\sqrt{(1-x^2)}},$$

from which $z = \text{arc}\sin x$; $x = \sin z$. Hence

$$\frac{dy}{dz} = x^m = \sin^m z\,; \;\therefore\; dy = \sin^m z\, dz.$$

First let m be an odd number, for example $m = 5$, then

$$dy = \sin z\,(1-\cos^2 z)^2 dz = (1-2\cos^2 z+\cos^4 z)\sin z\, dz\,;$$

$$y = -\cos z + \tfrac{2}{3}\cos^3 z + \tfrac{1}{5}\cos^5 z.$$

This mode of treatment of $\int \sin^m z\, dz$ evidently applies to all cases where m is odd. Now let m be an even number, for example $m = 4$, then

$$dy = \sin^4 z\, dz = \tfrac{1}{4}(1-\cos 2z)^2 dz = \tfrac{1}{4}(1-2\cos 2z+\cos^2 2z)dz,$$

and expanding $\cos^2 2z$ as before,

$$dy = \tfrac{1}{4}(\tfrac{3}{2}-2\cos 2z+\tfrac{1}{2}\cos 4z)\, dz,$$

the integration of which is obvious. The method just employed for m=an even number is applicable to all cases of that kind. And let it be noticed by the way that all integrals of the forms

$$\int \cos^n x\, dx,\;\; \int \sin^n x\, dx\,;\;\; \int \sin^m x \cos^n x\, dx$$

may be found with equal facility as the above and without recourse to formulæ of reduction when m and n are positive. And I would mention also, in passing, that the same simple methods are still applicable in $\int \sin^m x\, dx$ and $\int \cos^m x\, dx$ when m is negative. For when, in this case, m is odd, the multiplication of numerator and denominator by $\cos x$ and by $\sin x$ respectively, readily suggests the further proceeding; and when m is an even number the passage to the cognate imaginary angles by the substitution $\sin x = -i \operatorname{tg} i\varphi$, $\therefore \cos x = 1 \div \cos i\varphi$, $\therefore dx = d\varphi \div \cos i\varphi$ reduces this case to that where m is odd, so that the formula of reduction given in the books becomes needless.

For the sake of comparison of methods let us (lastly) consider the funct.:

$$dy = \frac{x^n dx}{\sqrt{(2ax-x^2)}}$$

(n being any integer) which is commonly also treated by a formula of reduction, as is well known. We take in connection with this function one of our primitive forms, viz.:

$$dz = \frac{dx}{\sqrt{(2ax-x^2)}}; \therefore z = \text{arc sin}\,\frac{x-a}{a};\ x = a(1+\sin z);$$

$$\therefore \frac{dy}{dz} = x^n = a^n(1+\sin z)^n;$$

$$\therefore \quad y = a^n \int (1+\sin z)^n\,dz,$$

which evidently comes back to the forms just treated and is integrable without any difficulty.

We now proceed to the application of our elementary integrals to other forms not commonly found in the books.

Let it be required to integrate

$$dy = \frac{dx}{\sqrt{(1-x^4)}\,[\sqrt{(1+x^2)}+x\sqrt{2}]}.$$

This function suggests the elementary form

$$dz = \frac{\sqrt{2}.dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{\sqrt{2}.dx}{\sqrt{(1-x^4)}\sqrt{(1+x^2)}},$$

which gives

$$z = \text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}};\ \therefore \sin z = \sqrt{\frac{1-x^2}{1+x^2}};\ \cos z = x\sqrt{\frac{2}{1+x^2}}.$$

Now dy may be written thus

$$dy = \frac{dx\sqrt{(1+x^2)}}{(1+x^2)\sqrt{(1-x^2)}[\sqrt{(1+x^2)}+x\sqrt{2}]}$$

$$= \frac{dx}{(1+x^2)\sqrt{(1-x^2)}\left(1+\frac{x\sqrt{2}}{\sqrt{(1+x^2)}}\right)};$$

$$\therefore dy = \frac{1}{\sqrt{2}}\frac{dz}{1+\cos z};\ \therefore y = \tfrac{1}{2}\sqrt{2}.\text{tg}\,\tfrac{1}{2}z;$$

$$\therefore \int \frac{dx}{\sqrt{(1-x^4)}[\sqrt{(1+x^2)}+x\sqrt{2}]} = \frac{1}{\sqrt{2}}\,\text{tg}\left(\tfrac{1}{2}\,\text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}}\right).$$

Let it now be required to integrate the similar, but more general form:

$$dy = \frac{dx}{\sqrt{(1-x^4)}[a\sqrt{(1+x^2)}+c\,x]}.$$

Here, putting $c = b\sqrt{2}$, we can write

$$dy = \frac{dx}{\sqrt{(1-x^4)}\sqrt{(1+x^2)}\left(a+\frac{bx\sqrt{2}}{\sqrt{(1+x^2)}}\right)}.$$

Again comparing with

$$dz = \frac{\sqrt{2}.dx}{\sqrt{(1-x^4)}\sqrt{(1+x^2)}}; \therefore\ z = \text{arc}\sin\sqrt{\frac{1-x^2}{1+x^2}};\ \cos z = x\sqrt{\frac{2}{1+x^2}}$$

we have, when $a^2>b^2$,

$$dy = \frac{1}{\sqrt{2}}\frac{dz}{a+b\cos z}; \therefore\ y = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{(a^2-b^2)}}.\text{arc}\cos\left(\frac{b+a\cos z}{a+b\cos z}\right)$$

$$= \frac{1}{\sqrt{(2a^2-c^2)}}.\text{arc}\cos\left(\frac{c+a\sqrt{2}.\cos z}{a\sqrt{2}+c.\cos z}\right);$$

$$\therefore \int\frac{dx}{\sqrt{(1-x^4)}[a\sqrt{(1+x^2)}+cx]} = \frac{1}{\sqrt{(2a^2-c^2)}}$$

$$\times\ \text{arc}\cos\left(\frac{c+a\sqrt{2}.x\sqrt{[2\div(1+x^2)]}}{a\sqrt{2}+c.x\sqrt{[2\div(1+x^2)]}}\right)$$

$$= \frac{1}{\sqrt{(2a^2-c^2)}}.\text{arc}\cos\left(\frac{c\sqrt{(1+x^2)}+2ax}{a\sqrt{2}.\sqrt{(1+x^2)}+c\sqrt{2}.x}\right),$$

when $2a^2>c^2$. In case $2a^2<c^2$, we have:

$$\int\frac{dx}{\sqrt{(1-x^4)}[a\sqrt{(1+x^2)}+cx]} = \frac{1}{\sqrt{(c^2-2a^2)}}$$

$$\times\lg\left(\frac{2a+c\sqrt{(1+x^2)}+\sqrt{(c^2-2a^2)}\sqrt{(1-x^2)}}{a\sqrt{2}.\sqrt{(1+x^2)}+c.x}\right).$$

It is now readily seen that the method just employed amounts to the substitution of a new variable, for we have integrated in terms of z what was given in terms of x. Hence all functions integrable in terms of z become integrable also in terms of x. Thus, taking again

$$dz = \frac{\sqrt{2}.dx}{\sqrt{(1-x^2)(1+x^2)}}; \therefore\ z = \text{arc}\sin\sqrt{\frac{1-x^2}{1+x^2}}; \therefore\ \sin z = \sqrt{\frac{1-x^2}{1+x^2}};$$

$\therefore\ \cos z = x\sqrt{2}\div\sqrt{(1+x^2)}$, let it be required to integrate in terms of x.

$$dy = \frac{dz}{(a+b\cos z)^2}; \therefore\ y = \frac{b\sin z}{(b^2-a^2)(a+b\cos z)} - \frac{a}{b^2-a^2}\int\frac{dz}{a+b\cos z}.$$

This gives $dy = \dfrac{\sqrt{2}.dx}{\sqrt{(1-x^2)}\,(1+x^2)\,[a+bx\sqrt{2}\div\sqrt{(1+x^2)}]^2}$

$$= \frac{\sqrt{2}.dx}{\sqrt{(1-x^2)}\,[a^2+(2b^2+a^2)x^2+2\sqrt{2}.abx\sqrt{(1+x^2)}]};$$

$$\therefore \int\frac{dx}{\sqrt{(1-x^2)}[a^2+(2b^2+a^2)x^2+2\sqrt{2}.abx\sqrt{(1+x^2)}]} = \frac{1}{\sqrt{2}}\times$$

$$\frac{b\sqrt{(1-x^2)}}{(b^2-a^2)\,[a\sqrt{(1+x^2)}+b\sqrt{2}.x]}+\frac{1}{\sqrt{2}}\frac{a}{(a^2-b^2)^{\frac{3}{2}}}.\text{arc}\cos\left(\frac{b\sqrt{(1+x^2)}+a\sqrt{2}.x}{a\sqrt{(1+x^2)}+b\sqrt{2}.x}\right).$$

In this manner the function z which we have here chosen will lead to a considerable number of new integrals, especially when the trigonometric functions in terms of z are taken and expressed in terms of x, as above.

[To be continued.]

NEW RULE FOR CUBE ROOT.

BY J. B. MOTT, WORTHINGTON, MINNESOTA.

In the following example the ordinary method is pursued till the second figure of the root is found. We then find the triple product (t. p.) by placing the second figure of the root to the right of three times the preceding part of the root and multiplying this by said second figure, and so on for all the triple products, finding one from the other. Thus:

1st trial divisor + 1st t. p. = 1st complete divisor.

1st complete divisor + 1st t. p. + b^2 + 2nd t. p. = 2nd complete divisor.

2nd complete divisor + 2nd t. p. + c^2 + 3rd t.p. = 3rd com. divisor, &c., placing each t. p. two figures to the right under the preceding divisor; b, c &c., being 2nd, 3rd &c. figures of the root.

Each complete divisor is used as a trial divisor for the next figure of the root. The constant left hand figures of any divisor, used as in simple division, will determine as many more figures of the root.

EXAMPLE.—To find the cube root of 2 to fourteen places of decimals.

First trial divisor	= 3	2(1.259921
" t. p. = 32×2 =	64	1
" com. divisor =	364	)1000
1st t. p. + 4 + 2d t. p.,	8625	728
2nd complete divisor,	45025	)272000
2nd t. p + 25 + 3d t. p.,	218831	225125
3rd complete divisor,	4721331	)46875000
3rd t. p. + 81 + 4th t. p.,	3731211	42491979
4th complete divisor,	475864311	)4383021000
4th t. p. + 81 + 5th t. p.,	34765144	4282778799
5th complete divisor,	47621196244	)100242201000
5th t. p. + 4 + 6th t. p.,	79375561	95242392488
6th complete divisor,	4762198998961	)4999808512000
6th t. p. + 0 + 7th t. p.,	3779762	4762198998961
7th complete divisor,	4762202778723	)237609513039

Dividing, we have 2376095 ÷ 47622028 = .04989487+. This united to the root above gives $\sqrt[3]{2} = 1.25992104989487+$.

REMARKS ON PROFESSOR JOHNSON'S QUERY BY R. J ADCOCK.—Mr. Jonson's paradox involves two questions, the value of u, and the value of $du \div dx = \cos ax$, when $a = \infty$. He says "now if $a = \infty$, $u = 0$ independently of x". This I deny. 0 is used to represent both actual zero and

infinitesimal quantities. It cannot however be so used unless it can be shown that no error will result from such use in the particular case. In this case when $a = \infty$, u = actual zero or an infinitesimal, that is $u = 0 \div a$ or some infinitesimal between $1 \div a$ and $-1 \div a$, these infinitesimals depending for their values upon a and x. And the rate at which these infinitesimals change their values is $du \div dx = \cos ax$, for all values of a and x.

When a is infinite, that is greater than any assignable number, then a is indeterminately great, and by consequence indeterminate; $\cos ax$ is the cosine of an arc in a given circle, the arc being as indeterminate as a, and therefore $\cos ax$ is indetermiate both in "form" and value, and from the given conditions can no more be affirmed to be zero than any other value between $+1$ and -1. There being no preference or reason for the termination of the arc ax in one part of the circumference rather than another. Mathematics having to deal with truth, like the "Scripture is of no private interpretation".

DIFFERENTIATION OF THE LOGARITHM OF A VARIABLE.

BY PROF. LABAN E. WARREN, COLBY UNIV., WATERVILLE, ME.

To differentiate the logarithm of a variable, let $y = e^x$;

$$\therefore x = \log_e y; \ \therefore dx = d(\log_e y).$$

$$y+dy = e^{x+dx},\ dy = e^{x+dx} - e^x \text{ or } dy = e^x(e^{dx}-1),$$

$$e^{dx} = 1+dx+\frac{dx^2}{2!}+\frac{dx^3}{3!}+\&c.,\text{ or } e^{dx} = 1+dx;$$

$$dy = e^x(1+dx-1),\text{ or } dy = e^x dx = ydx,\text{ or } dx = dy \div y,$$

but $dx = d(\log_e y)$;

$$\therefore d(\log_e y) = dy \div y,\text{ differential of Napierian logarithm.}$$

$$\log_{10} y = m(\log_e y);\ \therefore d(\log_{10} y) = m\, d(\log_e y);$$

$$\therefore d(\log_{10} y) = m\frac{dy}{y},\text{ differential of common log.}$$

SOLUTIONS OF PROBLEMS NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From Prof. L. G. Barbour, 434; Prof. W. P. Casey, 430, 431, 433; G. E. Curtis, 429, 434; Geo. Eastwood, 431; Wm. Hoover, 429, 430; Prof. P. H. Philbrick, 429, 430, 431, 433, 434; Prof. E. B. Seitz, 430, 431, 433, 435; Prof. J. Scheffer, 428, 430, 431, 433.

428. *By Geo. Lilley, A. M.* — "Two circles, of given radii, R and R_1, touch a straight line on the same side; a third circle of radius R_2 touches each of them; find the position of the circles R and R_1 and the radius of a fourth circle such that it shall touch the same straight line and each of the three given circles."

SOLUTION BY PROF. J. SCHEFFER.

Denote the radius of the required circle by x, the distance of its point of contact with the straight line from the points of contact of the circles R and R_1 with the straight line respectively by y and z. From the centre of the circle R_2 let fall the perpendicular u on the straight line, and denote the distance of its foot from the point of contact of the circle x with the straight line by t, then we have at once for the determination of the five quantities x, y, z, t, u the five equations:

$$(R+R_2)^2 = (y+t)^2+(u-R)^2, \tag{1}$$

$$(R_1+R_2)^2 = (y-t)^2+(u-R_1)^2, \tag{2}$$

$$(R+x)^2 = y^2+(R-x)^2, \text{ whence } y^2 = 4Rx, \tag{3}$$

$$(R_1+x)^2 = z^2+(R_1-x)^2, \quad \text{"} \quad z^2 = 4R_1x, \tag{4}$$

$$(R_2+x)^2 = t^2+(u-x)^2, \quad \text{"} \quad t^2+u^2 = R_2^2+2R_2x+2ux. \tag{5}$$

Expanding (1) and (2) and substituting the values of y, z and t^2+u^2 from (3), (4), (5) we obtain the two equations:

$$2\sqrt{(Rx)}\times t + (x-R)u = RR_2-2Rx-R_2x, \tag{6}$$

$$2\sqrt{(R_1x)}\times t + (x-R_1)u = R_1R_2-2R_1x-R_2x, \tag{7}$$

whence, after some easy reductions:

$$t = \frac{(\sqrt{R}-\sqrt{R_1})x^2}{[\sqrt{(RR_1)}-x]\sqrt{x}}, \quad u = \frac{R_2x+2x\sqrt{(RR_1)}-R_2\sqrt{(RR_1)}}{\sqrt{(RR_1)}-x}. \tag{8}$$

Substituting these values of t and u in (5), we obtain the quadratic eq'n

$$(\sqrt{R}+\sqrt{R_1})^2.x^2+4R_2\sqrt{(RR_1)}.x = 4RR_1R_2,$$

whence

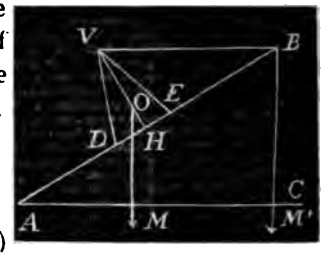
$$x = 2\sqrt{(RR_1R_2)}\times\frac{-\sqrt{R_2}+\sqrt{[R+R_1+R_2+2\sqrt{(RR_1)}]}}{(\sqrt{R}+\sqrt{R_1})^2}.$$

The values of y, z, t, u can now be derived respectively from (3), (4), (8).

If $R_2 = R_1 = R$, we get $x = \frac{1}{2}(\sqrt{5}-1).R$.

429. *By Prof. M. L. Comstock.*—"A cone of given weight W, is placed with its base on an inclined plane, and supported by a weight W' which hangs by a string fastened to the vertex of the cone and passing over a pulley in the inclined plane at the same height as the vertex. Determine the conditions of equilibrium."

SOLUTION BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY.



Let $i = \angle BAC$, $r = EH =$ radius of base of cone, $nr =$ altitude of cone, $\mu =$ coefficient of friction, $N =$ normal pressure of the cone on the plane, and $F =$ friction between the cone and the plane.

Resolving forces, normal to the plane,

$$N = W' \sin i + W \cos i; \therefore$$

$$F = \mu N = \mu(W' \sin i + W \cos i); \quad (1)$$

parallel to the plane, $\quad F + W' \cos i - W \sin i) = 0. \qquad (2)$

From (1) and (2) we get

$$\tan i = \frac{\mu W + W'}{W - \mu W'}. \qquad (3)$$

Taking moments about E, observing that $OH = \frac{1}{4} VH$, we get

$$W'(nr \cos i - r \sin i) = \text{or} < W(r \cos i + \tfrac{1}{4} nr \sin i), \text{ or}$$

$$W'(n - \tan i) = \text{or} < W(1 + \tfrac{1}{4} n \tan i), \qquad (4)$$

$$n \begin{array}{c} = \\ < \end{array} \frac{W + W' \tan i}{W' - \frac{1}{4} W \tan i}. \qquad (5)$$

If $W = 0$ (5) gives $n =$ or $< \tan i$, or $nr =$ or $< r \tan i = HD$, and the string passes through E or some point between E and H.

If $W' = 0$, $n =$ or $< -(4 \div \tan i)$ or $\frac{1}{4} nr \tan i =$ or $< -r = HD$ and the perpendicular through the center of gravity of the cone must pass thro' D. If $n = \infty$, that is, if the base of the cone is inappreciable in comparison with the height, either (4) or (5) gives $W' = \frac{1}{4} W \tan i$, or $W' \times nr \cos i = W \times \frac{1}{4} nr \sin i$, that is the moments of W' and W about H are equal.

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430. *By Prof. Milwee, Add-Ran Col. Texas.*—"Given two fixed points A and B, one on each of the axes of coordinates, at the respective distances a and b from the origin; if A' and B' be taken on the axes so that $OA' + OB' = OA + OB$, find the locus of the intersection of AB' and $A'B$."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $AO = a$, $BO = b$, $A'O = a + p$, then will $B'O = b - p$, from the conditions of the problem.

The equations of AB' and $A'B$ are respectively,

$$\frac{x}{a} + \frac{y}{b-p} = 1, \qquad \frac{x}{a+p} + \frac{y}{b} = 1,$$

or $\qquad bx + ay - ab + p(a - x) = 0, \; bx + ay - ab + p(y - b) = 0.$

Subtracting, $p(a - x) = p(y - b)$, or $x + y = a + b$, the eq. of the locus.

431. *By Prof J. W. Nicholson.*—"Required the area of a triangle whose sides are equal to the three roots respectively of the following equation:

$$x^3 + mx^2 + nx + r = 0." \qquad (1)$$

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let a, b, c be the sides of the triangle. The equation whose roots are a, b, c is $(x-a)(x-b)(x-c) = 0$, or

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0. \qquad (2)$$

To make (1) and (2) identical we must have

$$a + b + c = -m, \qquad (3)$$

$$ab+ac+bc = n, \qquad (4)$$

$$abc = -r. \qquad (5)$$

Subtracting (4)×2 from the square of (3), we have

$$a^2 + b^2 + c^2 = m^2 - 2n. \qquad (6)$$

Subtracting (3)×(5)×2 from the square of (4), we have

$$a^2b^2+a^2c^2+b^2c^2 = n^2 - 2mr. \qquad (7)$$

Subtracting the square of (6) from (7)×4, we have

$$2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 = 4m^2n-8mr-m^4. \qquad (8)$$

But the first member of (8) is equal to 16 times the square of the area of the triangle. Therefore we have

$$\text{area of triangle} = \tfrac{1}{4}\sqrt{(4m^2n - 8mr - m^4)}.$$

[This problem was solved in the same way, and the same result obtained, by Prof. Scheffer.]

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432. No solution received.

433. *By Prof. W. P. Casey.*—"Given the base of a triangle, to find the locus of the vertex, when the centre of the inscribed square moves on a given conic section."

SOLUTION BY PROF. J. SCHEFFER, HARRISBURG, PA.

Denote the given base AB by a. Let A be the origin of coordinates and AB the axis of x, and denote AI and CI respectively by x and y, and the side of the inscribed square by z. We easily obtain

$$z = \frac{ay}{a+y}, \text{ and } AD = \frac{ax}{a+y};$$

therefore the coordinates of the centre of the square are

$$AD+\tfrac{1}{2}z = \frac{2ax+ay}{2(a+y)} \ldots (1), \text{ and } \tfrac{1}{2}z = \frac{ay}{2(a+y)}. \qquad (2)$$

The values (1) and (2) substituted in the equation of the given curve will produce the equation of the locus required. Substituting them in the general equation of a conic $Ay^2+Bxy+Cx^2+Dy+Ex+F=0$, we obtain the equation of the locus,

$$(Aa^2+Ba^2+Ca^2+2Da+2Ea+4F)y^2+(2Ba^2+4Ca^2+4Ea)xy+4Ca^2x^2$$
$$+(2Da^2+2Ea^2+8Fa)y+4Ea^2x+4Fa^2=0,$$

which is the equation of a conic section, and is an ellipse, hyperbola or parabola according as $(Ba+2E)^2$ is $<$, $>$, or $= 4C(Aa^2+2Da+4F)$.

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434. *By Prof. De Volson Wood.*—"Find a number, the mantissa of the logarithm of which equals the number."

SOLUTION BY PROF. PHILBRICK.

If p is small and m the modulus of a system of logarithms, we have, approximately, $\log(1-p) = -mp = -1+(1-mp)$, $1-mp$ being the mantissa.

Now $(1-mp)-(1-p) = p(1-m) = e$ say, which approaches 0 as p approaches 0, or $1-p$ approaches 1; hence unity is the number sought.

In the common system $m = .4343$ and $e = .5657p$; in the Napierian system $m=1$ and $e=0$.

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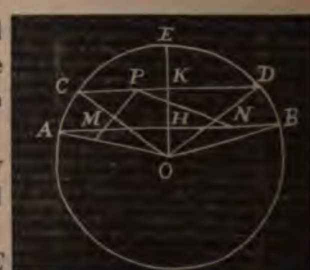
435. *By Prof. E. B. Seitz.*—"Find the average area of a triangle drawn on the surface of a given circle, having its base parallel to a given line, and its vertex taken at random."

SOLUTION BY THE PROPOSER.



Let O be the center of the given circle, and MNP a triangle drawn on the surface of the circle, having the side MN parallel to a given line.

Produce MN, forming the chord AB; draw the chord CD through P parallel to AB, and draw the radius OE perpendicular to AB.

Let $OA = r$, $BM = x$, $MN = y$, $\angle AOE = \theta$, and $\angle COE = \varphi$. Then $AB = 2r\sin\theta$, $CD = 2r\sin\varphi$, $OH = r\cos\theta$, $OK = r\cos\varphi$,

area $MNP = \frac{1}{2}MN\times HK = \frac{1}{2}ry(\cos\varphi - \cos\theta) = u$, when $\varphi < \theta$,

" " " $= \frac{1}{2}ry(\cos\theta - \cos\varphi) = u_1$, when $\varphi > \theta$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and θ and π; of x, 0 and $2r\sin\theta = x'$; and of y, 0 and x. Therefore the required average is

$$\frac{\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta 2ur^2\sin^2\varphi d\varphi + \int_\theta^\pi 2u_1r^2\sin^2\varphi d\varphi\right\}\int_0^{x'}\int_0^x r\sin\theta\, d\theta dx dy}{\int_0^{\frac{1}{2}\pi}\int_0^\pi\int_0^{x'}\int_0^x r\sin\theta . 2r^2\sin^2\varphi\, d\varphi dx dy}$$

$$= \frac{3}{2\pi r}\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta(\cos\varphi - \cos\theta)\sin^2\varphi d\varphi\right.$$

$$\left. + \int_\theta^\pi(\cos\theta - \cos\varphi)\sin^2\varphi d\varphi\right\}\int_0^{x'}\int_0^x \sin\theta d\theta dx dy$$

$$= \frac{2r^2}{\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^\theta(\cos\varphi - \cos\theta)\sin^2\varphi d\varphi + \int_\theta^\pi(\cos\theta - \cos\varphi)\sin^2\varphi d\varphi\right\}$$

$$\times\sin^4\theta d\theta$$

$$= \frac{r^2}{3\pi}\int_0^{\frac{1}{2}\pi}\Big[3(\pi - 2\theta)\cos\theta + 4\sin\theta + 2\sin\theta\cos^2\theta\Big]\sin^4\theta d\theta = \frac{512r^2}{525\pi}.$$

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436. No solution received.

NOTE BY W. E. HEAL.—The values of A, B, C, &c., given by me at page 56 (ANALYST, No. 2) should be

$$A = b^8[a^4 - (a^2+b^2)x^2]^2\tan^2\varphi - 4a^4b^{10}x^2 + 4a^2b^{10}x^4.$$

$$B = 4[a^2b^8(a^2+b^2)^2x^3\lambda - a^4b^6(a^2+b^2)x\lambda]\tan^2\varphi - 8a^6b^8x\lambda + 16a^4b^8x^3\lambda.$$

$$C = [2a^8b^8 - 2a^4b^4(a^2+b^2)(a^4\lambda^2+b^4x^2) + 6a^4b^4(a^2+b^2)^2x^2\lambda^2]\tan^2\varphi$$

$$-4(a^8b^6\lambda^2 + a^6b^8x^2) + 24a^6b^6x^2\lambda^2.$$

$$D = 4[a^8b^2(a^2+b^2)^2x\lambda^3 - a^6b^6(a^2+b^2)x\lambda]\tan^2\varphi - 8a^8b^6x\lambda + 16a^8b^4x\lambda^3.$$

$$E = a^8[b^4 - (a^2+b^2)\lambda^2]^2\tan^2\varphi - 4a^{10}b^4\lambda^2 + 4a^{10}b^2\lambda^4.$$

Also in lines 2 and 9 of same page "a^2y" should be a^2y^2.

PROBLEMS.

437. *By Prof. Casey.*—AE, AK are two indefinite given straight lines, C and H given points in them, and P a given point in their plane. Req'd to draw through P two straight lines, PB, PD, intersecting AE and AK in R and S, respectively, and containing a given angle RPS, so that $RC\times SH$ may be equal to a given magnitude.

438. *By Prof. F. H. Loud.*—Two equiangular parallelograms, $OACB$ and $OA'C'B'$ are so placed that the equal angles AOB and $A'OB'$ coincide. The sides of the former figure are constant, those of the latter are variable, subject to the condition $A'O+OB' : AO+OB ::$ *area* $OA'C'B'$: *area* $OACB$. $A'C'$ meets BC in D, and $B'C'$ meets AC in E. Show that DE passes through a fixed point, and determine the point.

439. *By Prof. Nicholson.*—Solve geometrically the following:

On a line whose length is a are two points x distance apart; what is the average value of x?

440 *Selected.*—A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two similar hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the coordinate axes. Prove that the product of inertia about the coordinate axes is

$$\frac{(a^2-a'^2)\,(\beta^2-\beta'^2)}{4(m^2+n^2)}$$

where aa', $\beta\beta'$ are the semi-major axes of the curves. (Routh's Rigid Dynamics.)

PUBLICATIONS RECEIVED.

Celestial Charts, made at the Litchfield Observatory of Hamilton College, Clinton, N. Y., by C. H. F. PETERS. Nos. 1–20. Each chart covering 20 min. in right ascension and 5° in declination.

The whole of the work involved in the production of this very valable series of charts, including the cost of publication, has been at the author's expense for gratuitous distribution.

Studies in Logic, by Members of the Johns Hopkins University. 203 pp. 12mo. Boston: Little, Brown, and Company. 1883.

Scientific Proceedings of the Ohio Mechanics' Institute. No. 1 of Vol. II., March, 1883. Cincinnati, Ohio.

An Extension of the Theorem of the Virial, and its Application to the Kinetic Theory of Gases. By H. T. EDDY C. E., PH. D. [From the Scientific Proceedings of the Ohio Mechanics' Institute for March, 1883.]

Report on the Michigan Forest Fires of 1881. [*Signal Service Notes, No. I.*] Office of Chief Signal Officer. Washington. 1882.

ERRATA.

On page 74, line 4, for "1076.4", read ± 1076.4.
" " 75, lines 12 and 13 from bottom, *dele* parethesis before "in" and after ["positive".
" " 86, line 12, for "2", read 1.

THE ANALYST.

VOL. X. JULY, 1883. No. 4

A METHOD OF DEMONSTRATING CERTAIN PROPERTIES OF POLYNOMIALS.

BY E. L. DE FOREST, WATERTOWN, CONN.

IN former articles of mine (see ANALYST, Vol. VII, p. 39 and Vol. IX, p. 141), it has been shown that when the coefficients of a polynomial are regarded as parallel forces acting positively or negatively in any coordinate direction at points whose rectangular coordinates are proportional to the exponents of the variables, the lever arm of the system of forces with respect to the coordinate planes, its "radius of gyration" or "quadratic radius" with respect to coordinate planes passed through the centre of forces of the system, and its "cubic radius" also, are such that if we multiply two or more polynomials together, the arm and the radii for the product can be found from those of the factors in a very simple manner. This has received an important application to the subject of probability. The coefficient or force at each point is regarded as the probability that an error which occurs will fall at that point. Since probabilities are always positive, and parallel forces acting in one direction may be represented by the force of gravity, and gravity is proportional to mass, the coefficients of the polynomial may be conc'd of as the masses of a system of material points. The centre of forces then becomes the centre of gravity of the system. It is the point whose coordinate in any direction is the *arithmetical mean* of the corresponding coordinates of all the points of error, each taken with a weight proportional to the probability of its occurrence. The sum of the probabilities of all possible errors is unity, that is, certainty. We shall assume that whether the coefficients of a polynomial represent probabilities or not, their algebraic sum is always unity. This does not impair the generality in any case, every polynomial being reducible to this form by dividing it through by the algebraic

sum of its coefficients. The methods of demonstrating the properties we have referred to have hitherto been special and restricted, but we shall now show that they can all be replaced by one which is both simple and comprehensive, giving many new properties which could not well be otherwise proved.

Let the polynomials have three variables r, s, t, with exponents a, b, c proportional to the coordinates x, y, z, locating the points anywhere in space of three dimensions. The properties for one and two dimensions are special cases under this, and will not need separate proof. Using a notation similar to that of ANALYST, IX. 34, we can write the first polynomial factor

$$u = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L_{a,b,c}\, r^a s^b t^c), \tag{1}$$

the coefficient L of each term being distinguished by sub-indices equal to the exponents of the variables in that term. The units of measure in the x, y, z directions are Δx, Δy, Δz, which may be taken of any convenient magnitude or even infinitesimals dx, dy, dz, and are not necessarily equal to each other; and a, b, c are numbers, positive or negative, integral or fractional, such that for any point

$$x = a\Delta x, \qquad y = b\Delta y, \qquad z = c\Delta z. \tag{2}$$

The number m is taken so large that a, b or c will not exceed it. The polynomial (1) can be understood to include all the points in a rectangular block. When any points are not to be included, we only have to suppose that for each of them the coefficient L is zero. We may take m large enough to include either of two polynomial factors u and v. Denoting the coefficients in v by L', we write

$$v = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L'_{a,b,c}\, r^a s^b t^c).$$

Let l denote the coefficients in the product of u and v, then

$$uv = \Sigma_{c=-2m}^{c=2m} \Sigma_{b=-2m}^{b=2m} \Sigma_{a=-2m}^{a=2m} (l_{a,b,c}\, r^a s^b t^c).$$

Since the limits of summation in either factor will remain the same throughout our present investigation, and those in the product likewise, we will omit writing them and merely suppose that the summation extends to all the terms in the polynomial. Also, to save repetition, we omit to write the sub-indices a, b, c after L, L' and l, and we denote $\Sigma\Sigma\Sigma$ by Σ_3 and $\Sigma\Sigma$ by Σ_2. The two polynomials now are

$$u = \Sigma_3(L r^a s^b t^c), \qquad v = \Sigma_3(L' r^a s^b t^c), \tag{3}$$

and their product is

$$uv = \Sigma_3(l r^a s^b t^c). \tag{4}$$

We shall not regard the coefficients exclusively as forces or masses, but as quantities which are located at points in space, and which are to be multiplied into certain powers or products of their coordinates. Adding together

all the products thus obtained in any one polynomial, we call their sum a *moment* of the system of coefficients. For instance, denoting the coordinates as in (2),

$$\Sigma_3(Lx) = \Sigma_3(aL)\Delta x$$

is the x moment of the coefficients L; that is, the sum of the products formed by multiplying each L into its abscissa x. Likewise

$$\Sigma_3(Lx^2y) = \Sigma_3(a^2bL)(\Delta x)^2\Delta y$$

is the x^2y moment; and so on.

The variables are independent of the coefficients, and of each other. Differentiating (3) and (4) with respect to r and multiplying the results by r, we get

$$\left.\begin{aligned} r\left(\frac{du}{dr}\right) = \Sigma_3(aLr^as^bt^c), \quad r\left(\frac{dv}{dr}\right) = \Sigma_3(aL'r^as^bt^c), \\ r\left\{ v\left(\frac{du}{dr}\right) + u\left(\frac{dv}{dr}\right) \right\} = \Sigma_3(alr^as^bt^c). \end{aligned}\right\} \tag{5}$$

As these relations are true for any values assigned to the variables, we may assume

$$r = 1, \qquad s = 1, \qquad t = 1, \qquad \therefore u = 1, \qquad v = 1, \tag{6}$$

since by hypothesis $\Sigma_3L = 1$, $\Sigma_3L' = 1$. Then, denoting by $(du \div dr)_1$ &c. what $du \div dr$ &c. become when $r = s = t = 1$, we get from (5)

$$\left.\begin{aligned} \left(\frac{du}{dr}\right)_1 = \Sigma_3(aL), \quad \left(\frac{dv}{dr}\right)_1 = \Sigma_3(aL'), \\ \left(\frac{du}{dr}\right)_1 + \left(\frac{dv}{dr}\right)_1 = \Sigma_3(al). \end{aligned}\right\} \tag{7}$$

The first member of the last equation being the sum of those of the two first, we have

$$\Sigma_3(al)\Delta x = \Sigma_3(aL)\Delta x + \Sigma_3(aL')\Delta x, \tag{8}$$

that is, giving $a\Delta x$ its value from (2), the x moment in the product is equal to the sum of the x moments in the two factors. This is the theorem respecting the lever arms. (ANALYST, Vol. VII, p. 80.) When the sum of the masses in a system is unity, the statical moment of the system is numerically the same as its lever arm. By differentiating (3) and (4) with respect to s and t, we could get relations like (8) for the y and z moments. But owing to the symmetry of (3) with respect to a, b and c, what is proved for one coordinate direction is proved, *mutatis mutandis*, for all. Hence we have

$$\left.\begin{aligned} \Sigma_3(lx) = \Sigma_3(Lx) + \Sigma_3(L'x), \quad \Sigma_3(ly) = \Sigma_3(Ly) + \Sigma_3(L'y), \\ \Sigma_3(lz) = \Sigma_3(Lz) + \Sigma_3(L'z). \end{aligned}\right\} \tag{9}$$

With polynomials of only two variables the points for each are all in one XY plane. Omitting t and c from (3) and (4), we get as above

$$\Sigma_2(lx) = \Sigma_2(Lx) + \Sigma_2(L'x), \quad \Sigma_2(ly) = \Sigma_2(Ly) + \Sigma_2(L'y). \tag{10}$$

(Compare Vol. VII, p. 45.) And with only one variable, the points being ranged along a straight line or x axis (Vol. VII, p. 22),

$$\Sigma(lx) = \Sigma(Lx) + \Sigma(L'x). \tag{11}$$

We have hitherto supposed that the origins of coordinates, or places of $L_{0,0,0}$ and $L'_{0,0,0}$ in (1) and (3), are taken anywhere at pleasure, but a great advantage will be gained by taking them so that the sum of the products Lx on one side of the origin is equal to the sum of those on the other side; and likewise for the $L'x$, the Ly, &c. This reduces the lever arms to zero, and locates the origins at the centres of forces of the two systems, which are the centres of gravity when the coefficients L and L' are positive and regarded as masses. It gives

$$\left.\begin{array}{lll} \Sigma_3(Lx) = 0, & \Sigma_3(Ly) = 0, & \Sigma_3(Lz) = 0, \\ \Sigma_3(L'x) = 0, & \Sigma_3(L'y) = 0, & \Sigma_3(L'z) = 0. \end{array}\right\} \tag{12}$$

The points thus determined will henceforth be taken as origins. Then by (2), (7) and (9) we have in the two factors

$$\left.\begin{array}{lll} \left(\frac{du}{dr}\right)_1 = 0, & \left(\frac{du}{ds}\right)_1 = 0, & \left(\frac{du}{dt}\right)_1 = 0, \\ \left(\frac{dv}{dr}\right)_1 = 0, & \left(\frac{dv}{ds}\right)_1 = 0, & \left(\frac{dv}{dt}\right)_1 = 0, \end{array}\right\} \tag{13}$$

and consequently in the product

$$\Sigma_3(lx) = 0, \quad \Sigma_3(ly) = 0, \quad \Sigma_3(lz) = 0, \tag{14}$$

the origin of coordinates or place of $l_{0,0,0}$ in the product being thus the centre of forces of the whole system of coefficients l, in the same sense as the origins in the factors are centres of forces for L and L'.

Now differentiating (5) with respect to r and multiplying the results by r, we get

$$\left.\begin{array}{l} r\left(\frac{du}{dr}\right) + r^2\left(\frac{d^2u}{dr^2}\right) = \Sigma_3(a^2Lr^as^bt^c), \\ r\left(\frac{dv}{dr}\right) + r^2\left(\frac{d^2v}{dr^2}\right) = \Sigma_3(a^2L'r^as^bt^c), \\ r\left\{ v\left(\frac{du}{dr}\right) + u\left(\frac{dv}{dr}\right)\right\} + r^2\left\{ v\left(\frac{d^2u}{dr^2}\right) + 2\left(\frac{du}{dr}\cdot\frac{dv}{dr}\right) + u\left(\frac{d^2v}{dr^2}\right)\right\} \\ \qquad = \Sigma_3(a^2lr^as^bt^c), \end{array}\right\} \tag{15}$$

and giving the quantities the values from (6) and (13),

$$\left.\begin{array}{l} \left(\frac{d^2u}{dr^2}\right)_1 = \Sigma_3(a^2L), \quad \left(\frac{d^2v}{dr^2}\right)_1 = \Sigma_3(a^2L'), \\ \left(\frac{d^2u}{dr^2}\right)_1 + \left(\frac{d^2v}{dr^2}\right)_1 = \Sigma_3(a^2l). \end{array}\right\} \tag{16}$$

Consequently

$$\Sigma_3(a^2l)(\Delta x)^2 = \Sigma_3(a^2L)(\Delta x)^2 + \Sigma_3(a^2L')(\Delta x)^2,$$

and applying (2), we have in the three coordinate directions,

$$\left.\begin{array}{c}\Sigma_3(lx^2) = \Sigma_3(Lx^2)+\Sigma_3(L'x^2),\ \Sigma_3(ly^2) = \Sigma_3(Ly^2)+\Sigma_3(L'y^2),\\ \Sigma_3(lz^2) = \Sigma_3(Lz^2)+\Sigma_3(L'z^2).\end{array}\right\} \quad (17)$$

The x^2 moment in the product of two polynomials is equal to the sum of the x^2 moments in the two factors, when the origins are at the centres of forces; and the like is true for the y^2 and z^2 moments. This is the theorem respecting the radii of gyration, or quadratic radii (AN., Vol. VII, p. 81). When the mass of a system is unity, its moment of inertia is numerically equal to the square of its radius of gyration. The theorem holds true also when the moments are taken not with respect to the coordinate planes, but with respect to the coordinate axes or to the origin. (Vols. VII, p. 82 and IX, p. 67.) If the points for each polynomial are all in one plane, (17) reduces to

$$\Sigma_2(lx^2) = \Sigma_2(Lx^2)+\Sigma_2(L'x^2), \quad \Sigma_2(ly^2) = \Sigma_2(Ly^2)+\Sigma_2(L'y^2) \quad (18)$$

(Vol. VII, p. 78); and if all are in one straight line (VII, p. 22),

$$\Sigma(lx^2) = \Sigma(Lx^2)+\Sigma(L'x^2). \quad (19)$$

This property of polynomials affords the simplest explanation of that "law of great numbers," so important in the theory of probability. It proves rigorously that with any given law of facility of error, the "quadratic mean error" of the arithmetical mean of n observations varies inversely as $\sqrt{n}$. (Vols. VIII, p. 4 and IX, p. 168.) Hence, by increasing the number of observations, we can diminish indefinitely the probable error of the mean.

Next, differentiating (15) with respect to r and then multiplying by r, we get

$$\left.\begin{array}{l} r\left(\frac{du}{dr}\right)+3r^2\left(\frac{d^2u}{dr^2}\right)+r^3\left(\frac{d^3u}{dr^3}\right) = \Sigma_3(a^3Lr^as^bt^c),\\ r\left(\frac{dv}{dr}\right)+3r^2\left(\frac{d^2v}{dr^2}\right)+r^3\left(\frac{d^3v}{dr^3}\right) = \Sigma_3(a^3L'r^as^bt^c),\\ r\left\{v\left(\frac{du}{dr}\right)+u\left(\frac{dv}{dr}\right)\right\}+3r^2\left\{v\left(\frac{d^2u}{dr^2}\right)+2\left(\frac{du}{dr}\cdot\frac{dv}{dr}\right)+u\left(\frac{d^2v}{dr^2}\right)\right\}\\ +r^3\left\{v\left(\frac{d^3u}{dr^3}\right)+3\left(\frac{dv}{dr}\cdot\frac{d^2u}{dr^2}+\frac{du}{dr}\cdot\frac{d^2v}{dr^2}\right)+u\left(\frac{d^3v}{dr^3}\right)\right\}\\ \qquad = \Sigma_3(a^3lr^as^bt^c). \end{array}\right\} \quad (20)$$

By means of (6) and (13) this is reduced to

$$3\left(\frac{d^2u}{dr^2}\right)_1+\left(\frac{d^3u}{dr^3}\right)_1 = \Sigma_3(a^3L),\ 3\left(\frac{d^2v}{dr^2}\right)_1+\left(\frac{d^3v}{dr^3}\right)_1 = \Sigma_3(a^3L'),$$

$$3\left\{\left(\frac{d^2u}{dr^2}\right)_1+\left(\frac{d^2v}{dr^2}\right)_1\right\}+\left(\frac{d^3u}{dr^3}\right)_1+\left(\frac{d^3v}{dr^3}\right)_1 = \Sigma_3(a^3l),$$

and consequently

$$\Sigma_3(a^3l)(\Delta x)^3 = \Sigma_3(a^3L)(\Delta x)^3+\Sigma_3(a^3L')(\Delta x)^3,$$

so that applying (2), we have

$$\left.\begin{aligned}\Sigma_3(lx^3) = \Sigma_3(Lx^3)+\Sigma_3(L'x^3),\ \Sigma_3(ly^3) = \Sigma_3(Ly^3)+\Sigma_3(L'y_3),\\ \Sigma_3(lz^3) = \Sigma_3(Lz^3)+\Sigma_3(L'z^3).\end{aligned}\right\}\ (21)$$

The x^3 moment in the product is equal to the sum of the x^3 moments in the two factors; and so too for the y^3 and the z^3 moments. In space of two dimensions (21) becomes

$$\Sigma_2(lx^3) = \Sigma_2(Lx^3)+\Sigma_2(L'x^3),\quad \Sigma_2(ly^3) = \Sigma_2(Ly^3)+\Sigma_2(L'y^3),\ (22)$$

and in space of one dimension

$$\Sigma(lx^3) = \Sigma(Lx^3)+\Sigma(L'x^3).\qquad (23)$$

This is the theorem respecting the "cubic radii". (ANALYST, IX, 161.)

Relations such as these can be proved for moments of the 4th and higher orders, only they will not be quite so simple. Differentiating (20) with respect to r, multiplying by r and applying (6) and (13), we get

$$7\left[\frac{d^2u}{dr^2}\right]_1+6\left[\frac{d^3u}{dr^3}\right]_1+\left[\frac{d^4u}{dr^4}\right]_1 = \Sigma_3(a^4L),$$

$$7\left[\frac{d^2v}{dr^2}\right]_1+6\left[\frac{d^3v}{dr^3}\right]_1+\left[\frac{d^4v}{dr^4}\right]_1 = \Sigma_3(a^4L'),$$

$$7\left\{\left[\frac{d^2u}{dr^2}\right]_1+\left[\frac{d^2v}{dr^2}\right]_1\right\}+6\left\{\left[\frac{d^3u}{dr^3}\right]_1+\left[\frac{d^3v}{dr^3}\right]_1\right\}+\left[\frac{d^4u}{dr^4}\right]_1$$

$$+\left[\frac{d^4v}{dr^4}\right]_1+6\left[\frac{d^2u}{dr^2}\right]_1\left[\frac{d^2v}{dr^2}\right]_1 = \Sigma_3(a^4l),$$

and consequently by help of (16),

$$\Sigma_3(lx^4) = \Sigma_3(Lx^4)+\Sigma_3(L'x^4)+6\Sigma_3(Lx^2)\Sigma_3(L'x^2).\qquad (24)$$

The x^4 moment in the product is equal to the sum of the x^4 moments in the two factors, plus 6 times the product of their x^2 moments; and similarly for the y^4 and z^4 moments. In space of only one or two dimensions the relation is of corresponding form. In the same way it can be shown that

$$\Sigma_3(lx^5) = \Sigma_3(Lx^5)+\Sigma_3(L'x^5)+10[\Sigma_3(Lx^2)\Sigma_3(L'x^3)+\Sigma_3(Lx^3)\Sigma_3(L'x^2)].\ \ldots\ (25)$$

In general, the moment of the nth order in the product will be expressed in terms of the moments of the nth and lower orders in the two factors. Hence, if some of the coefficients L and L' are negative, in such manner as to reduce to zero the moments of all orders up to and including the nth in both factors, the same moments in the product will also be zero. If they are zero in any one polynomial, they will be zero in all the powers of that polynomial. This is the property which I otherwise demonstrated in ANALYST, VI, 145 and VII, 105. The quantities there denoted by b_n, &c. are the same as $\Sigma(a^nL)$, &c., in our present notation.

Next let (5) be differentiated with respect to s. Multiplying the results by s we have

$$\left.\begin{array}{l} rs\left[\dfrac{d^2u}{dr\,ds}\right]=\Sigma_3(abLr^as^bt^c),\quad rs\left[\dfrac{d^2v}{dr\,ds}\right]=\Sigma_3(abL'r^as^bt^c),\\ rs\left\{v\left[\dfrac{d^2u}{dr\,ds}\right]+\dfrac{du}{dr}\cdot\dfrac{dv}{ds}+\dfrac{du}{ds}\cdot\dfrac{dv}{dr}+u\left[\dfrac{d^2v}{dr\,ds}\right]\right\}=\Sigma_3(ablr^as^bt^c),\end{array}\right\}\quad(26)$$

and applying (6) and (13),

$$\left.\begin{array}{l} \left[\dfrac{d^2u}{dr\,ds}\right]_1=\Sigma_3(abL),\quad \left[\dfrac{d^2v}{dr\,ds}\right]_1=\Sigma_3(abL'),\\ \left[\dfrac{d^2u}{dr\,ds}\right]_1+\left[\dfrac{d^2v}{dr\,ds}\right]_1=\Sigma_3(abl),\end{array}\right\}\quad(27)$$

$$\therefore\ \Sigma_3(abl)\Delta x\Delta y=\Sigma_3(abL)\Delta x\Delta y+\Sigma_3(abL')\Delta x\Delta y,$$

and for the three coordinates taken two and two,

$$\left.\begin{array}{l}\Sigma_3(lxy)=\Sigma_3(Lxy)+\Sigma_3(L'xy),\ \Sigma_3(lxz)=\Sigma_3(Lxz)+\Sigma_3(L'xz),\\ \Sigma_3(lyz)=\Sigma_3(Lyz)+\Sigma_3(L'yz).\end{array}\right\}\quad(28)$$

The xy moment in the product is equal to the sum of the xy moments in the two factors; and so also for the xz and yz moments. For points in a plane, (28) reduces to the single relation

$$\Sigma_2(lxy)=\Sigma_2(Lxy)+\Sigma_2(L'xy).\quad(29)$$

Again, differentiating either (15) with respect to s or (26) with respect to r, multiplying the results by s in the one case or r in the other, and applying (6) and (13), we find

$$\left.\begin{array}{l}\left[\dfrac{d^2u}{dr\,ds}\right]_1+\left[\dfrac{d^3u}{dr^2ds}\right]_1=\Sigma_3(a^2bL),\quad \left[\dfrac{d^2v}{dr\,ds}\right]_1+\left[\dfrac{d^3v}{dr^2ds}\right]_1\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=\Sigma_3(a^2bL'),\\ \left[\dfrac{d^2u}{dr\,ds}\right]_1+\left[\dfrac{d^2v}{dr\,ds}\right]_1+\left[\dfrac{d^3u}{dr^2ds}\right]_1+\left[\dfrac{d^3v}{dr^2ds}\right]_1=\Sigma_3(a^2bl),\end{array}\right\}\quad(30)$$

$$\therefore\ \Sigma_3(lx^2y)=\Sigma_3(Lx^2y)+\Sigma_3(L'x^2y).\quad(31)$$

The x^2y moment in the product is equal to the sum of the x^2y moments in the two factors. The same is true of the x^2z, xy^2, y^2z, xz^2 and yz^2 moments. For points in a plane these six are reduced to two,

$$\Sigma_2(lx^2y)=\Sigma_2(Lx^2y)+\Sigma_2(L'x^2y),\ \Sigma_2(lxy^2)=\Sigma_2(Lxy^2)+\Sigma_2(L'xy^2).\ (32)$$

Let us now differentiate (26) with respect to t, multiplying the results by t, and apply (6) and (13); this gives

$$\left.\begin{array}{l}\left\{\dfrac{d^3u}{dr\,ds\,dt}\right\}_1=\Sigma_3(abcL),\ \left\{\dfrac{d^3v}{dr\,ds\,dt}\right\}_1=\Sigma_3(abcL'),\\ \left\{\dfrac{d^3u}{dr\,ds\,dt}\right\}_1+\left\{\dfrac{d^3v}{dr\,ds\,dt}\right\}_1=\Sigma_3(abcl),\end{array}\right\}\quad(33)$$

$$\therefore\ \Sigma_3(lxyz)=\Sigma_3(Lxyz)+\Sigma_3(L'xyz).\quad(34)$$

The xyz moment in the product is equal to the sum of the xyz moments in the two factors.

It is evident that such properties as (17), (21), (28), (31) and (34) can be readily extended to the product of any number of polynomial factors, the moment in the final product being equal to the sum of those in all the factors. If a polynomial is raised to the nth power, the moment for that power is n times as great as for the first power.

We have hitherto supposed that the rectangular coordinate axes are taken in any convenient directions, provided only that the origin must be at the centre of forces, which is the centre of gravity when the coefficients L and L' are all positive and regarded as the masses of material points. But in finding the law of probability of errors in space of two or three dimensions, the equations are simplified when we assume the axes to coincide with the "free axes," or principal axes through the centre of gravity. (ANALYST, VIII, 43 and IX, 38.) All the moments in (28) or (29) are thereby reduced to zero. Then by (27)

$$\left.\begin{array}{lll} \left\{\dfrac{d^2u}{dr\,ds}\right\}_1 = 0, & \left\{\dfrac{d^2u}{dr\,dt}\right\}_1 = 0, & \left\{\dfrac{d^2u}{ds\,dt}\right\}_1 = 0, \\ \left\{\dfrac{d^2v}{dr\,ds}\right\}_1 = 0, & \left\{\dfrac{d^2v}{dr\,dt}\right\}_1 = 0, & \left\{\dfrac{d^2v}{ds\,dt}\right\}_1 = 0. \end{array}\right\} \tag{35}$$

The positions of the free axes can be found for any given polynomial by known methods applicable to bodies of three dimensions, for which see chapter II of Vol. II of Poisson's *Traité de Mécanique*. For points in a plane the formula will be, as in the ANALYST above cited,

$$\tan 2\varphi = \frac{2\Sigma_2(Lxy)}{\Sigma_2(Lx^2)-\Sigma_2(Ly^2)}, \tag{36}$$

where φ is the angle which a free axis makes with the assumed X axis, and has two values differing by 90°. The moment in the numerator of (36) is of the same form as those in (29), while the moments in the denominator are like those in (18). When (36) is applied to the nth power of a polynomial, both numerator and denominator will be, as we have shown, n times as great as they were for the first power, so that the value of φ will be unchanged; that is to say, the free axes of the material points l in the expans'n will make with the assumed axes, the same angles which those of the points L did in the given polynomial. This is an improved demonstration of the property which I inferred from somewhat different considerations in ANALYST, Vol. VIII, p. 48. A similar property holds true for powers of a polynomial of three variables, occupying points in space of three dimensions. (Vol. IX, p. 67.) According to the formulas given by Poisson, it is manifest that the position of the free axes is not changed when the moments

used as data, of the same form as those in (17) and (28), are all multiplied by a constant number.

The positions of the free axes in the final product of any number of different polynomial factors will be given by the same formulas as above, when for the moments used we substitute the sums obtained by adding together the corresponding moments for all the factors.

When the free axes are taken as coordinate axes, simple properties like those we have already found can be demonstrated for moments of still higher orders. By the same process of differentiation and multiplication, and then applying (6), (13) and (35), it is found that

$$\Sigma_3(lx^3y) = \Sigma_3(Lx^3y) + \Sigma_3(L'x^3y), \qquad (37)$$

the like being true of course for the x^3z, xy^3, y^3z, xz^3 and yz^3 moments. It is also found that

$$\Sigma_3(lx^2yz) = \Sigma_3(Lx^2yz) + \Sigma_3(L'x^2yz), \qquad (38)$$

the same being true of the xy^2z and xyz^2 moments.

For the form x^2y^2 the relation is more complex, namely

$$\Sigma_3(lx^2y^2) = \Sigma_3(Lx^2y^2) + \Sigma_3(L'x^2y^2) + \Sigma_3(Lx^2)\Sigma_3(L'y^2) + \Sigma_3(Ly^2)\Sigma_3(L'x^2). \qquad (39)$$

The like holds true for the x^2z^2 and y^2z^2 moments.

CORRESPONDENCE.

Editor Analyst:

If not trespassing too much upon your space permit me to reply in a few words to the letters of Mr. Adcock and Prof. Judson, in the last number of the ANALYST.

Mr. Adcock, after quoting from my original statement of the paradox, "now if $a = \infty$, $u = 0$ independently of x", adds "This I deny." His first answer begins, "When $u = 0$, independently of x it is not a function of x", from which it seems that he was at that time content to accept the equation $u = 0$. It is difficult to see what he wishes now to substitute for this eq'n, for he goes on to say, "In this case, u = actual zero, or an infinitesimal." Perhaps he means only to deny t e clause "independently of x"; for he remarks "the rate at which these infinitesimals change their value is $du \div dx = \cos ax$." But this is a mere restatement of the paradox; viz., that u has a finite rate of change, and yet no finite change in value.

Mr. Adcock's "private interpretation" of the form $\cos\infty$, as "indetermi-

nate both in form and value," agrees with my own, as expressed in my last letter as well as in the original statement of the paradox; viz., that $\cos \infty$ is an "*essentially* indeterminate form." Neverthelss he will not deny that there is some interest attaching to De Morgan's "private interpretation" of the same form.

In this connection, it should be stated, in reply to Prof. Judson's remark that we cannot admit zero as the value of $\sin \infty$ and $\cos \infty$ "in violation of the principle $\sin^2 \infty + \cos^2 \infty = 1$", that the writers who take this view expressly state that, by their principles of interpretation, $\sin^2 \infty$ is not to be regarded as the square of $\sin \infty$, but as the mean value of $\sin^2 x$, which is $\frac{1}{2}$.

Again, De Morgan finds $\sec \infty = 0$ (not equal to the reciprocal of $\cos \infty$) and $\tan \infty = \sqrt{-1}$; but he appears to violate these principles by citing as a verification of these last values the fact that they satisfy the equation $\sec^2 \infty = \tan^2 \infty + 1$.

Prof. Judson states very clearly, in his last letter, the "principle which seems to be violated in the example given," i. e. that, when we assign to a such a value as will reduce it to a constant or to zero then the same value ought to reduce the derivative with respect to x to zero. This is certainly what we naturally expect, and what we usually find to be the case. Nevertheless I maintain that the principle as stated not only "seems to be", but *is* violated, in opposition to Prof. Judson's opinion that the paradox must arise from a misinterpretation of the form $a \div \infty$. In support of this Prof. Judson says that when $a = \infty$ u "becomes indefinite." This certainly is not true in the sense in which $du \div dx$ becomes indefinite, and therefore I suppose means infinitesimal, in which case we are told "it still remains a function of x and there seems to be no good reason why we should then expect $du \div dx$ to equal zero, *independently* of x." The phrase "it still remains a function of x" is here misleading; for u is no longer a function of x in the sense that its value changes with x by any finite amount, and it is simply the fact that it does not change its value by any finite amount that leads us to expect that its derivative should reduce to zero.

If we examine the principle as stated above, we shall I think find that though generally, it is not universally true, and shall see why it fails in the case in question. When a function has an actual (not zero) rate of change, the ratio of its increment to that of x has an actual limiting value which measures this rate, and is the derivative found by the ordinary rules of differentiation; whence we naturally assume (what is in fact generally true) the converse theorem; that when the rate is zero the derivative is also zero. But to prove this in any case, it is necessary to show that the difference between the actual increment of the function and the product of the derivative

by the increment of x (the second term in the expansion by Taylor's Theorem) vanishes in comparison with this product. This is not necessarily true when the second derivative is infinite, which is the case with the function in question. This is made clear by writing out the development, thus

$$\frac{\sin a(x+h)}{a} = \frac{\sin ax}{a} + \cos ax.h - a\sin ax.\frac{h^2}{2} - \dots$$

The actual increment of the function, consisting of all the terms of the series after the first, is indeed zero, because the function does not change its value when x is replaced by $x+h$; but it is not true that the coefficent of h is the limit of

$$\left[\frac{\sin a(x+h)}{a} - \frac{\sin ax}{a}\right] \div h$$

because the result of dividing the terms which follow the second by h does not vanish with h.

Of course if with Mr. Todhunter and other writers we define the derivative or "first differential coefficient" as the limit of the ratio written above, we are compelled to reject the equation $du \div dx = \cos ax$ when $a = \infty$. If on the other hand we admit this equation, because it results from the rules of differentiation, we cannot deny that the first derivative is a measure of the rate, unless the higher derivatives are finite.

WM. WOOLSEY JOHNSON.

Editor Analyst:

I submit the following correction of an error in Bartlett's Mechanics, ninth edition, p. 398, equations (580).

From the equation

$$\frac{d^2\xi}{dt^2} = -n_x^2\xi,$$

the author deduces by integration

$$\frac{d\xi^2}{dt^2} = -n_x^2\xi^2,$$

whereas it should be

$$\frac{d\xi^2}{dt^2} = n_x^2(a_x^2 - \xi^2),$$

where a_x is the constant of integration. The corrected equation may be deduced directly from equations (530) of that work. The other two equat'ns of (580) should be corrected in the same manner.

DE VOLSON WOOD.

INTEGRATION BY AUXILIARY INTEGRALS.

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

[Continued from page 88.]

Let it now be required to integrate

$$dy = \frac{dx}{\sqrt{(\operatorname{tg} x)}}.$$

We know from our elementary forms that when

$$du = \frac{2x.dx}{1+x^4}, \text{ then}$$

$$u = \operatorname{arc tg} x^2; \therefore x = \sqrt{(\operatorname{tg} u)}.$$

On the other hand, when

$$dv = \frac{dx}{1+x^4}$$

v is a well known integral. Now

$$\frac{dv}{du} = \frac{1}{2x} = \frac{1}{2\sqrt{(\operatorname{tg} u)}}; \therefore dv = \frac{du}{2\sqrt{(\operatorname{tg} u)}}.$$

But $v = \frac{1}{4\sqrt{2}} \lg\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + \frac{1}{2\sqrt{2}} \operatorname{arc tg}\left(\frac{x\sqrt{2}}{1-x^2}\right)$, hence

$$\int \frac{du}{\sqrt{(\operatorname{tg} u)}} = \frac{1}{2\sqrt{2}} \lg\left(\frac{1+\sqrt{(2.\operatorname{tg} u)}+\operatorname{tg} u}{1-\sqrt{(2.\operatorname{tg} u)}+\operatorname{tg} u}\right) + \frac{1}{\sqrt{2}} \operatorname{arc tg}\left(\frac{\sqrt{(2.\operatorname{tg} u)}}{1-\operatorname{tg} u}\right).$$

This last equation, in connection with our elementary integrals, leads to a number of new integrals. Thus, for example take

$$du = \frac{dx}{\sqrt{(2a-x^2)}}, \therefore u = \tfrac{1}{2}\operatorname{arc sin} \frac{x^2-a}{a}, \therefore \operatorname{tg} 2u = \frac{x^2-a}{x\sqrt{(2a-x^2)}},$$

$$\therefore \frac{du}{\sqrt{(\operatorname{tg} 2u)}} = \frac{dx}{\sqrt{(2a-x^2)}} \cdot \frac{x^{\frac{1}{2}}(2a-x^2)^{\frac{1}{4}}}{\sqrt{(x^2-a)}}; \text{ hence}$$

$$\int \frac{\sqrt{x}.dx}{(2a-x^2)^{\frac{1}{4}}(x^2-a)^{\frac{1}{2}}} = \int \frac{du}{\sqrt{(\operatorname{tg} 2u)}}.$$

Taking the same integral in connection with the elementary form:

$$dy = \frac{2xdx}{\sqrt{(1-x^4)}}, \therefore y = \operatorname{arc sin} x^2, \therefore x^2 = \sin y, \therefore \operatorname{tg} y = \frac{x^2}{\sqrt{(1-x^4)}}$$

$$\frac{dy}{\sqrt{(\operatorname{tg} y)}} = \frac{2xdx}{\sqrt{(1-x^4)}} \cdot \frac{(1-x^4)^{\frac{1}{4}}}{x} = \frac{2dx}{(1-x^4)^{\frac{1}{4}}}; \text{ hence}$$

$$\int \frac{dx}{(1-x^4)^{\frac{1}{4}}} = \tfrac{1}{2}\int \frac{dy}{\sqrt{(\operatorname{tg} y)}}.$$

Taking the same integral in connection with the elementary form:

$$dy = \frac{dx}{(2x^2+1)\sqrt{(x^2+1)}}, \therefore y = -\tfrac{1}{2} \operatorname{arc} \operatorname{tg} \sqrt{1+\frac{1}{x^2}},$$

$$\therefore \sqrt{\frac{x^2+1}{x^2}} = -\operatorname{tg} 2y, \therefore i\sqrt{(\operatorname{tg} 2y)} = \frac{(x^2+1)^{¼}}{\sqrt{x}};$$

$$\frac{dy}{i\sqrt{(\operatorname{tg} 2y)}} = \frac{dx}{(2x^2+1)\sqrt{(x^2+1)}} \cdot \frac{\sqrt{x}}{(x^2+1)^{¼}}, \text{ hence}$$

$$\int \frac{\sqrt{x}.dx}{(2x^2+1)(x^2+1)^{¾}} = -i\int \frac{dy}{\sqrt{(\operatorname{tg} 2y)}}.$$

Before proceeding to the further employment of our elementary integrals for the integration of other functions, I propose to demonstrate the fruitfulness of our method by showing that all (or nearly all) the integrals known in finite form and contained in the larger collections of integrals are found by our method without resorting to the decomposition into partial fractions and without the process of rationalization. For these integrations the twelve or fifteen fundamental integrals of the text-books, when used as auxiliary integrals, suffice. It will be seen that our results are different in form from those found by the ordinary method, and at the same time better adapted to numerical computations, inasmuch as they allow of more extensive use of the tables of logarithms and of circular functions. I shall follow the arrangement of *Minding's* "Integraltafeln".

I°. Rational Functions.

The first six tables of *Minding* contain the functions of the form

$$dv = \frac{x^m dx}{(a+bx)^n},$$

where m and n are positive integers, m from 1 to 6 and n from 1 to 10. In all these cases we take

$$du = \frac{dx}{a+bx}; \therefore u = \frac{1}{b}\lg(a+bx).$$

$$\therefore a+bx = e^{bu}; \quad x = (1 \div b)(e^{bu} - a).$$

This gives

$$\frac{dv}{du} = \frac{x^m}{(a+bx)^{n-1}} = \frac{1}{b^m}\frac{(e^{bu}-a)^m}{e^{(n-1)bu}},$$

the integration of which presents no difficulty. In order to apply this to a special example, let it be required to integrate

$$dv = \frac{x^5 dx}{(a+bx)^8}, \text{ then } \frac{dv}{du} = \frac{x^5}{(a+bx)^7} = \frac{1}{b^5}\frac{(e^{bu}-a)^5}{e^{7bu}},$$

of which the integration is so simple as to require no further remarks; and

the lists of integrals of the above form already calculated become needless. This result, it is readily seen, is likewise found by the substitution $a+bx=u$.

Tables VII to XIII (of *Minding*) contain the functions of the form

$$dv = \frac{dx}{x^m(a+bx)^n},$$

where m and n are positive whole numbers, m from 1 to 7 and n from 1 to 8. In all these cases we take

$$du = \frac{dx}{x(a+bx)};\ \therefore\ u = -\frac{1}{a}\lg\left(\frac{a+bx}{x}\right);$$

$$\therefore\ a+bx = \frac{a}{1-b.e^{au}};\ x = \frac{a.e^{au}}{1-b.e^{au}}.$$

This gives

$$\frac{dv}{du} = \frac{1}{x^{m-1}(a+bx)^{n-1}} = \frac{(1-b.e^{au})^{m-1}}{a^{m-1}.e^{(m-1)au}}.\frac{(1-b.e^{au})^{n-1}}{a^{n-1}},$$

which is integrable without any difficulty. By way of an example of this kind, let it be required to integrate

$$dv = \frac{dx}{x^6(a+bx)^8},\ \text{then}\ \frac{dv}{du} = \frac{1}{x^5(a+bx)^7};$$

$$\therefore\ dv = \frac{1}{a^{12}}\frac{(1-b.e^{au})^{12}}{e^{5au}}du.$$

Here the integration is so simple as to require no further remark. Again it appears that the tables of integrals already calculated become superfluous when our method is employed; whereas the integration by the ordinary method of decomposition in such an example as we have just treated would involve considerable labor.

Tables XIV to XIX contain integrals of the form

$$v = \int\frac{x^{2m}dx}{(a+bx^2)^n},$$

where m and n are positive whole numbers, m from 1 to 5 and n from 1 to 8. In all these cases we take

$$du = \frac{dx}{a+bx^2};\ \therefore\ u = \frac{1}{\sqrt{(ab)}}\ \text{arc tg}\left(x\sqrt{\frac{b}{a}}\right);$$

$$\therefore\ a+bx^2 = a.\sec^2[\sqrt{(ab)}.u];\ x = \sqrt{(a\div b)}.\text{tg}[\sqrt{(ab)}.u].$$

This gives

$$\frac{dv}{du} = \frac{x^{2m}}{(a+bx^2)^{n-1}} = \left(\frac{a}{b}\right)^m.\text{tg}^{2m}[\sqrt{(ab)}.u].\frac{1}{a^{n-1}}.\cos^{2(n-1)}[u\sqrt{(ab)}].$$

For example, let it be required to integrate

$$dv = \frac{x^8dx}{(a+bx^2)^6};\ \therefore\ \frac{dv}{du} = \frac{x^8}{(a+bx^2)^5};$$

$$\therefore\; dv = \frac{1}{b^4 a}\sin^8[u\sqrt{(ab)}]\cos^2[u\sqrt{(ab)}]\,du,$$

the integration of which presents no difficulty.

Now let b be negative, so that (putting a^2 and b^2 as constants) it is req'd to integrate:

$$dv = \frac{x^{2m}dx}{(a^2-b^2x^2)^n}.$$

In this case we take

$$du = \frac{dx}{a^2-b^2x^2},\;\therefore\; u = \frac{1}{2ab}\lg\left(\frac{a+bx}{a-bx}\right);$$

$$\therefore\; x = \frac{a}{b}\left(\frac{e^{2abu}-1}{e^{2abu}+1}\right).$$

For our present purpose it is more convenient to employ the equivalent geometric functions; this gives

$$x^2 = -\frac{a^2}{b^2}\operatorname{tg}^2(i.abu);\; a^2 - b^2x^2 = a^2.\sec^2(i.abu).$$

Let it be required, for example, to integrate

$$dv = \frac{x^4dx}{(a^2-b^2x^2)^6};\;\therefore\;\frac{dv}{du} = \frac{x^4}{(a^2-b^2x^2)^5};$$

$$dv = \frac{1}{b^4a^6}\sin^4(i.abu)\cos^6(i.abu)\,du,$$

the integration of which presents no difficulty.

Tables XX to XXVI contain the integrals of the form

$$v = \int\frac{dx}{x^{2m}(a+bx^2)^n},$$

where m and n are positive integers, m from 1 to 5 and n from 1 to 7. In all these cases we may again take for auxiliary integral

$$u = \int\frac{dx}{a+bx^2} = \frac{1}{\sqrt{(ab)}}\operatorname{arc\,tg}\left(x\sqrt{\frac{b}{a}}\right);$$

$$\therefore\; x = \sqrt{(a\div b)}\operatorname{tg}[u\sqrt{(ab)}];\; a+bx^2 = a\sec^2[u\sqrt{(ab)}].$$

Let it be required to integrate, for example,

$$dv = \frac{dx}{x^6(a+bx^2)^3};\;\therefore\;\frac{dv}{du} = \frac{1}{x^6(a+bx^2)^2};$$

$$dv = \frac{b^3}{a^5}.\frac{\cos^{10}[u\sqrt{(ab)}]}{\sin^6[u\sqrt{(ab)}]},$$

and the integration presents no difficulty. I omit for the present the forms

$$\int\frac{x^m dx}{(a+bx^n)^p}\text{ and }\int\frac{dx}{x^m(a+bx^n)^p},$$

as we shall have to treat them in connection with auxiliary integrals not now to be employed.

Tables LII to LVI of Minding treat of the forms

$$v = \int \frac{x^m dx}{(a + bx + cx^2)^n},$$

where m and n are positive whole numbers, m from 0 to 6 and n from 1 to 6. In all these cases we take

$$du = \frac{dx}{a+bx+cx^2}; \therefore u = \frac{2}{\sqrt{(4ac-b^2)}} \operatorname{arc\,tg}\left(\frac{\sqrt{(2cx+b)}}{\sqrt{(4ac-b^2)}}\right)$$

so long as $4ac-b^2$ is positive. This gives

$$2cx+b = \sqrt{(4ac-b^2)}\,\mathrm{tg}\,[u\sqrt{(4ac-b^2)}] = a.\mathrm{tg}\,(ua);$$

$$x = \frac{1}{2c}[a.\mathrm{tg}\,(ua) - b]; \; a + bx + cx^2 = \frac{a^2}{4c \,.\cos^2(au)}.$$

Let it be required, for example, to integrate

$$v = \int \frac{x^6 dx}{(a+bx+cx^2)^3}, \text{ then}$$

$$\frac{dv}{du} = \frac{x^6}{(a+bx+cx^2)^2} = \frac{1}{(2c)^6}[a.\mathrm{tg}\,(au) - b]^6 \frac{1}{a^4}[4c.\cos^2(au)]^2;$$

$$v = \frac{(4c)^2}{(2c)^6 a^4} \int [a.\mathrm{tg}(au) - b]^6 \cos^4(au).du,$$ the integration of which needs no discussion.

When m is negative, so that

$$v = \int \frac{dx}{x^m(a+bx+cx^2)^n}$$

is required, the substitution $x = (1 \div t)$ gives

$$du = -\frac{dt}{at^2 + bt + c}; \therefore u = \frac{2}{\sqrt{(4ac-b^2)}} \operatorname{arc\,tg}\left(\frac{2c+bt}{t\sqrt{(4ac-b^2)}}\right);$$

$$t = \frac{2c}{a.\mathrm{tg}\,(\frac{1}{2}a.u) - b}; \therefore \frac{c + bt + at^2}{t^2} = \frac{a^2}{2c}\sec^2(au),$$

which gives very simple results where $m > 2n$ or $m = 2n$.

II. Irrational Functions.

The number of irrational functions integrable in finite form and found in the collections of integrals is quite small. The chief difficulty in the way is the rationalization of the functions, which is often attempted in vain. We will integrate without resorting to rationalization. *Minding* begins with integrals of the form

$$v = \int \frac{x^m(a+bx)^n\, dx}{\sqrt{(a+bx)}},$$

where m and n are positive integers.

In all these cases we take

$$du = \frac{dx}{\sqrt{(a+bx)}}; \therefore u = \frac{2}{b}\sqrt{a+bx};$$

$$\therefore a + bx = \tfrac{1}{4}b^2u^2;\ x = \tfrac{1}{4}bu^2-(a\div b).$$

This gives $dv\div du = x^m(a+bx)^n$. Therefore

$$v = \int\left(\frac{bu^2}{4}-\frac{a}{b}\right)^m\left(\frac{b^2u^2}{4}\right)^n du,$$

the integration of which is exceedingly simple.

When n is negative the same process applies without modification. Thus

when $$v = \int\frac{x^m dx}{(a+bx)^n\sqrt{(a+bx)}},$$ then

$$\frac{dv}{du} = \frac{x^m}{(a+bx)^n} = \left(\frac{bu^2}{4}-\frac{a}{b}\right)^m\div\left(\frac{b^2u^2}{4}\right)^n,$$

which presents no more difficulty than the foregoing form.

When x^m occurs in the denominator, we take

$$du = \frac{dx}{x\sqrt{(a+bx)}}; \therefore u = \frac{1}{\sqrt{a}}\lg\left(\frac{\sqrt{(a+bx)}-\sqrt{a}}{\sqrt{(a+bx)}+\sqrt{a}}\right);$$

$$\therefore \sqrt{(a+bx)} = \sqrt{a}\left(\frac{1+e^{u\sqrt{a}}}{1-e^{u\sqrt{a}}}\right).$$

It is more convenient, however, to operate with the equivalent goniometric functions, and we have

$$\sqrt{(a+bx)} = i.\sqrt{a}.\mathrm{tg}(i.\tfrac{1}{2}u\sqrt{a});\ x = -(a\div b)\sec^2(i.\tfrac{1}{2}u\sqrt{a}),$$

and the integration of

$$dv = \frac{dx}{x^m(a+bx)^n\sqrt{(a+bx)}}$$

is easily performed in terms of u.

We now come to the form

$$v = \int\frac{x^m dx}{(a+bx+cx^2)^n\sqrt{(a+bx+cx^2)}},$$

where m and n again are positive integers. Here we take the aux. integral

$$u = \int\frac{dx}{\sqrt{(a+bx+cx^2)}} = -\frac{i}{\sqrt{c}}\text{ arc sin}\left(i.\frac{2cx+b}{\sqrt{(4ac-b^2)}}\right).$$

This gives

$$2cx+b = -i\sqrt{(4ac-b^2)}\sin(i.u\sqrt{c}) = -i.\beta\sin(i.u\sqrt{c});$$

$$x = -\frac{1}{2c}[i.\beta\sin(i.u\sqrt{c})+b];\ \sqrt{(a+bx+cx^2)} = \sqrt{\left(\frac{4ac-b^2}{4c}\right)}$$

$$\times\cos(i.u\sqrt{c}) = a.\cos(i.u\sqrt{c}).$$

Now we have

$$\frac{dv}{du}=\frac{x^m}{(a+bx+cx^2)^n}=\left(-\frac{1}{2c}\right)^m\frac{[i.\beta\sin(i.u\surd c)+b]^m}{[a.\cos(i.u\surd c)]^n},$$

the integration of which can be performed by well known methods.

If, for example,

$$v=\int\frac{x^4dx}{(a+bx+cx^2)^3\surd(a+bx+cx^2)}$$

is required, we have

$$\frac{dv}{du}=\frac{x^4}{(a+bx+cx^2)^3}=\frac{1}{(2c)^4}\frac{[i.\beta\sin(i.u\surd c)+b]^4}{[a.\cos(i.u\surd c)]^3},$$

which is easily integrated by well known methods.

When

$$v=\int\frac{dx}{x^m(a+bx+cx^2)^n\surd(a+bx+cx^2)}$$

is required, we take the auxiliary

$$u=\int\frac{dx}{x\surd(a+bx+cx^2)}=\frac{i}{\surd a}\text{arc}\sin\left(i\frac{2a+bx}{x\surd(4ac-b^2)}\right),$$

which gives

$$\frac{1}{x}=\frac{-i\surd(4ac-b^2)}{2a}\sin(i.u\surd a)-\frac{b}{2a};$$

$$\frac{\surd(a+bx+cx^2)}{x}=\sqrt{\left(\frac{4ac-b^2}{4a}\right)}\cos(i.u\surd a),$$

and leads to very simple results when $m>2n$.

I will now present one example of the "binomial" class of integrals to show the treatment without rationalization, which example I take from *Sohnke's* Collection of Problems. It is required to integrate

$$v=\int\frac{32.dx}{x^5(1-x)^{\frac{5}{2}}}.$$

When rationalizing in the usual way this function gives rise to a rather complicated expression for the integral. We take as auxiliary

$$du=\frac{dx}{x\surd(1-x)};\ \therefore u=2\sin x^{\frac{1}{2}},\ \therefore x=\sin^2\tfrac{1}{2}u;\ \surd(1-x)=\cos\tfrac{1}{2}u.$$

$$\therefore\frac{dv}{du}=\frac{32}{x^4(1-x)^3}=\frac{32}{\sin^8\frac{1}{2}u.\cos^6\frac{1}{2}u}.$$

To integrate this last function in the simplest way we take

$$dy=\frac{du}{\sin^2\frac{1}{2}u};\ y=-\cot\tfrac{1}{2}u;\ \therefore\sin^2\tfrac{1}{2}u=\frac{1}{1+y^2};\ \cos^2\tfrac{1}{2}u=\frac{y^2}{1+y^2}.$$

$$\frac{dv}{dy}=64(1+y^2)^6\frac{(1+y^2)^6}{y^{12}};$$

$$v=64\int\frac{(1+y^2)^{12}}{y^{12}}dy.$$

[To be continued.]

CUBIC EQUATIONS.

BY PROF. L. G. BARBOUR, RICHMOND, KENTUCKY.

SEVERAL years ago the writer published in the ANALYST (Vol. V, pp 73–79) a method of solving numerical equations of the 3rd deg. with three real roots,—of which two might be equal. The present paper will extend the same general method, so as to solve equations having two imaginary roots. The equations considered are of the form

$$x^3 \pm px \pm q = 0.$$

I. $x^3 - px + q = 0$. The roots are of the form $A+B\sqrt{-1}$, $A-B\sqrt{-1}$, $-2A$. Construct an equation from these roots. We get

$$x^3 - (3A^2 - B^2)x + (A^2 + B^2)2A = 0.$$

Let $B^2 = mA^2$, then

$$x^3 - (3-m)A^2x + (2 + 2m)A^3 = 0;$$

$$\therefore p = (3-m)A^2; \quad q = (2+2m)A^3;$$

$$\therefore \frac{p^3}{q^2} = \frac{(3-m)^3}{(2+2m)^2};$$

$$\therefore 3 \log p - 2 \log q = 3 \log (3-m) - 2 \log (2+2m).$$

We now make a table with m as the argument, taking it $= .001$, $.002$, &c., successively, up to 3. In the second column write the differences between $3 \log (3-m)$ and $2 \log (2+2m)$. In this table we find the value of $3 \log p - 2 \log q$ in the 2nd column, and thus get the value of m in the 1st column, as we find a number from its logarithm in the ordinary logarithmic tables. Then since $(3-m)A^2 = p$, $A = \sqrt{[p \div (3-m)]}$; $B^2 = mA^2$; $B = A\sqrt{m}$. Test: $(2+2m)A^3 = q$, $\therefore \log A = \frac{1}{3} \log q - \frac{1}{3} \log (2-2m)$.

II. $x^3 - px - q = 0$. The roots are $-A-B\sqrt{-1}$, $-A+B\sqrt{-1}$, $+2A$. The equation constructed from these roots is

$$x^3 - (3-m)A^2x - (2+2m)A^3 = 0;$$

$\therefore p = (3-m)A^2$ and $q = (2+2m)A^3$, as before. The difference is that when, as in this case, q is negative, the signs of the roots will be reversed, as above indicated.

III. $x^3 + px + q = 0$. Construct the equation from the roots $A + B\sqrt{-1}$, $A-B\sqrt{-1}$, and $-2A$. We again get

$$x^3 - (3A^2-B^2)x + (A^2+B^2)2A$$
$$= x^3 - (3-m)A^2x + (2+2m)A^3.$$

But since p in this case is positive, it is necessary and sufficient to write

$$x^3 + (m-3)A^2x + (2+2m)A^3 = 0.$$

In cases I and II, $m < 3$, or at most, $m = 3$. In case III, $m > 3$ or at least, $m = 3$. In fact there will be one place common to the two tables, viz., when $m = 3$. Then $3 \log (3-m)$, or, in the other table, $3 \log (m-3)$ $= 3 \log 0 = \infty$; and the tabular method cannot be used. The solution in this instance, however, is easy enough, for $p = (3-m)A^2 = 0$, and the equation becomes $x^3 + q = 0$; from which the roots are readily found.

This second table is constructed from the formula $3 \log p - 2 \log q = 3 \log (m-3) - 2 \log (2+2m)$. Ascending values are assigned to m from 3 up as high as may be needed.

IV. $x^3 + 70x - q = 0$. The roots are $-A + B\sqrt{-1}, -A - B\sqrt{-1}$, and $+2A$; $\therefore x^3 + (m-3)A^2 - (2+2m)A^3 = 0$. The same table is used as in case III.

General Remarks.

If p is $+$, in the equation under consideration, there must be 2 imaginary roots. But if p is $-$, there may be 3 real, unequal roots, or 3 real roots, of which two are equal; or one real root and two imaginary and unequal; or one real root and, perhaps we may say, 2 equal imaginary roots. The discrimination is effected thus: If $(p^3 \div q^2) > \frac{27}{4}$, i. e., $3 \log p - 2 \log q > .8293038$, there are 3 real, unequal roots. If it be $< .8293038$ there are 2 imaginary unequal roots. But if it $= .8293038$, two of the roots are equal and are ordinarily considered real. However, if we choose to regard B as $= 0$, we may treat this as coming under the head of imaginary roots; $A + B\sqrt{-1}$, and $A - B\sqrt{-1}$ becoming equal when $B = 0$.

Proof:—It is evident that $3 \log (3-m) - 2 \log (2+2m)$ reaches a maximum when $m = 0$. Then $3 \log 3 - 2 \log 2 = \log \frac{27}{4}$.

In my former article, the table was constructed by the formula $3 \log (1 + a + a^2) - 2 \log (a + a^2)$; $\therefore$ by differentiating for maximum and minimum, $a = 1, -\frac{1}{2}$ or -2. It was shown that when $a = 1$, there are two equal and real roots; and the same is true when $a = 2$ or $-\frac{1}{2}$. In our present inquiry, when $m = 0$, $mA^2 = B^2 = 0$, $\therefore$ the roots are $A, A, -2A$; or $-A, -A, +2A$.

The method of procedure then is this: Observe the sign of p in the given equation; if it is $-$, then find $3 \log p - 2 \log q$, either in the old table for 3 real roots, which extends from $+\infty$ down to $\log \frac{27}{4} = .82930772831$; or else in the first of the new tables subjoined, which extends from .82930772-831 down to $-\infty$. If it falls in the old table, the argument is a; the roots were called r, s, and t; $r^2 = p \div (1 + a + a^2)$, $\therefore \log r = \frac{1}{2} \log p - \frac{1}{2} \log (1 + a + a^2)$; $s = ar$; $t = r + s$, but with the opposite sign.

The signs of r, s, and t are determined by that of q. If q is $+$, r and s

+, and t —. If q is —, r and s are —, and t +. In other words r l s always have the same sign as q; t has always the opposite sign. The gative values of a are not employed, since they give no new results.

In the same way if q is +, the imaginary roots are $+A+B\sqrt{-1}$, and $A-B\sqrt{-1}$. The real root, corresponding to t, is $-2A$. But if q is , the roots are $-A-B\sqrt{-1}$, $-A+B\sqrt{-1}$, and $+2A$.

Again if p is +, look only in the second subjoined table, No. III, exıding from $-\infty$ up as far as we need it.

It is interesting to note that if p is —, we may by changing q arbitrarily, value and sign, get an equation with three real roots, all unequal, or two ual; or one real and two imaginary; but if p is +, there is no help for ; two roots must be imaginary.

In table II, m is the argument; $(3-m)A^2 = p$; $mA^2 = B^2$;

$$\therefore \log A = \tfrac{1}{2}[\log p - \log(3-m)],$$
$$\log B = \log A + \tfrac{1}{2}\log m.$$

In table III, m is the argument; $(m-3)A^2 = p$;

$$\therefore \log A = \tfrac{1}{2}[\log p - \log(m-3)],$$
$$\log B = \log A + \tfrac{1}{2}\log m.$$

The use of the tables is best explained by actual examples. In the forer article one or two examples were worked out in full. It was found at the results were true for 5, 6, or 7 places in equations having three real ots. I have very recently tried the method on a few problems from Prof. 'entworth's "Complete Algebra." On p. 508, under the head of Trigonoetric Solution of Equations, he says "Take the difficult equation

$$x^3 - \tfrac{403}{441}x + \tfrac{46}{147} = 0."$$

Solving by trigonometry he gets $x' = 0.42855$, $x'' = 0.66670$, $x''' = 1.09525$.

By the method proposed in this paper, $3\log\frac{403}{441} - 2\log\frac{46}{147} = .8917182$. ·om the table we get, by proportional parts, $a = .6428582$, $\therefore r = .666626$; $s = .42857182$; $t = -1.09523808$. The exact roots are

$r = \frac{2}{3} = .666$; $s = \frac{3}{7} = .428571428571$; $t = \frac{23}{21} = 1.09523809$.

The value of t is usually the most exa.t. By a simple interpolation in : table, the value of a can be found more closely, giving the roots a little ırer. Since $s = ar$, $a = s \div r$, we find the exact value of $a = \frac{3}{7} \div \frac{2}{3} =$ ¦28571428571+.

On p. 510 Prof. W. gives as "the true values," 0.42857, 0.66667, —1.024.

One defect of the method of this article is that it gives results true to only ır 6 or 7 places. By an extension of the method closer results could be

obtained, but they are seldom needed. It is seen above that the actual results are nearer true than those which Prof. W. deems sufficiently exact.

On p. 496 he says "In rare cases two of the roots are so nearly equal that Horner's Method carried out as above will not find them both. Take for example $x^3 + 11x^2 - 102x + 181 = 0$,
in which we have found that there are two roots between 3 and 4. Horner's method gives, for the first transformed equation,

$$y^3 + 20y^2 - 9y + 1 = 0.$$

Since the coefficient of y^2 is large, it is best to neglect y^3 only." That is, the equation $20y^2 - 9y + 1 = 0$ must be solved,

This plan, if my memory serves me, is employed by Newton in similar cases. From curiosity I tried this problem by the aid of the tables. At the outset we encounter the unavoidable difficulty of removing the second term of the equation. This defect is inseparable from the method. The removal can be accomplished, however, without much trouble, by Horner's Synthetic Divisor, or by substitution. Then by the regular use of the table the roots are found to about 5 or 6 places. One root is 3.22953. Prof. W. gives 3.22952. A curious application of the tabular method has been made to a problem in Newcomb's Algebra, p. 192.

$$\frac{\sqrt{(x+a)}}{\sqrt{(x-a)}} = \frac{x}{a}$$

This gives $x^3 - ax^2 - a^2x - a^3 = 0$,
or removing the second term,

$$y^3 - \tfrac{4}{3}a^2y - \tfrac{38}{27}a^3 = 0,$$

in which case $x = y + \frac{1}{3}a$. Also $p = -\frac{4}{3}a^2$; $q = -\frac{38}{27}a$. Therefore

$$p^3 \div q^2 = \tfrac{64}{27} \div (\tfrac{38}{27})^2,$$

in which a is eliminated. Proceeding then as usual, we get $3 \log \frac{4}{3} - 2 \log \frac{38}{27} = .0779765$. Since p is —, we look in the 2nd table, and find m bet'n .648 and .649. By proportion, $m = .64833392$; $3 - m = 2.35166608$. Log $p = .1249387$. Log $(3-m) = .37137566$. Difference $= \bar{1}.75356304$. Log $A' = \frac{1}{2}[\log p - \log(3-m)] = \bar{1}.87678152 + \log a$; $A' = .7529767a$. Log $B' = \log A' + \frac{1}{2} \log m$. $B' = .6062906a$; $2A' = 1.5059534a$. Hence the roots of the equation

$$y^3 - \tfrac{4}{3}a^2y - \tfrac{38}{27}a^3 = 0$$

are $-.7529767a - .6062906a\sqrt{-1}$, $-.7529767a + .6062906a\sqrt{-1}$, and $+1.5059534a$. The roots of the original equat's are found by adding $\frac{1}{3}a = .333a$ to each of these. We get $-.4196434a - .6062906a\sqrt{-1}$, $-.4196434a + .6062906a\sqrt{-1}$, and $+1.83928671a$.

Let us now test one of these roots, say the last given. By substitution.

$$\sqrt{(1.83928671a+a)} \div \sqrt{(.83928671a)} = 1.83928671a \div a,$$
$$\sqrt{(2.83928671)} \div \sqrt{(.83928671)} = 1.83928671.$$

∴ ½ log 2.83928671 — ½ log .83928671 should = log 1.83928671.

Now ½ log 2.83928671 = .22660925

nd ½ log .83928671 = $\bar{1}$.96195515

Diff. = .2646541

nd log 1.83928671 = .2646495

Log. error .0000046

In numbers, 1.8393063 — 1.8392867 = 0.0000196.

TABLE II.

$x^3 - px \pm q = 0.$

m	$(3-m)A^2=p$
.0	.8293038
.1	.7023486
.2	.5810516
.3	.4641448
.4	.3506039
.5	.2395774
.6	.1303336
.7	.0232256
.8	—1.9146631
.9	—1.8070907
1.0	—1.6989700
1.1	—1.5897622
1.2	—1.4789121
1.3	—1.3658311
1.4	—1.2498776
1.5	—1.1303339
1.6	—1.0063774
1.7	—2.8770436
1.8	—2.7411677
1.9	—2.5973221
2.0	—2.4436975
2.1	—2.2779441
2.2	—2.0969100
2.3	—3.8962063
2.4	—3.6694360
2.5	—3.4067139
2.6	—3.0915150
2.7	—4.6929004
2.8	—4.1414628
2.9	—5.2158108
3.0	—∞.

TABLE III.

$x^3 - px \pm q = 0.$

m	$(m-3)A^2=p$
3.0	—∞.
3.1	—5.1723722
3.2	—4.0545314
3.3	—4.5623668
3.4	—4.9172146
3.5	—3.1884250
3.6	—3.4068782
3.7	—3.5890383
3.8	—3.7447276
3.9	—3.8802753
4.0	—2.
4.1	—2.1069777
4.2	—2.2034770
4.3	—2.2912184
4.4	—2.3715364
4.5	—2.4454885
5.	—2.7447275
6.	—1.1391077
7.	—1.3979400
8.	—1.5863650
9.	—1.7323938
10.	—1.8504487
11.	—1.9488476
12.	0.0327809
13.	0.0556855
20.	1.4448481

Use of Tables.—In Table II, find m; then $(3-m) \times A^2 = p$; ∴ $\log A = \frac{1}{2}[\log p - \log(3-m)]$. $B^2 = mA^2$, ∴ $\log B = \frac{1}{2}\log m + \log A$.

In Table III, find m; then $(m-3)A^2 = p$; ∴ $\log A = \frac{1}{2}[\log p - \log(m-3)]$. $B^2 =$ as in T. II.

The accuracy of the work may be tested by the formula, $\log A' = \frac{1}{3}[\log q - \log(2+2m)] = \bar{1}.87678151$, the value obtained above being $\bar{1}.87678152$

The equation $y^3 - \frac{4}{3}a^2y + \frac{16}{27}a^3 = 0$ has 3 real roots and may be solved by the old table. In general, if the exponent of the literal part of the absolute term be $\frac{3}{2}$ the exponent of the like literal part of the term containing the 1st power of the unknown quantity, the present method will solve the equation. Thus such equations as $x^3 \pm \frac{3}{4}a^{2/3}x \pm \frac{5}{6}a = 0$, or as $x^3 \pm \frac{3}{5}a^{3\div7}x \pm \frac{8}{11}a^{9\div14} = 0$, are easily managed. Also such as

$$x^3 \pm \frac{\sqrt{11}.\sqrt[3]{13}}{\sqrt[5]{15}.\sqrt[7]{19}}x \pm \frac{\sqrt[11]{23}.\sqrt[17]{27}}{\sqrt[13]{25}.\sqrt[19]{29}} = 0$$

offer no difficulty.

Considering the table previously furnished (see p. 79, Vol. V) as No. I, we designate the foregoing tables as Nos. II, and III. No. II is calculated by the formula $3\log(3-m) - 2\log(2+2m)$; No. III, by the formula $3\log(m-3) - 2\log(2+2m)$. In No II, p is $-$; in No. III, p is $+$; and in both cases there are two imaginary roots.

ANOTHER SOLUTION OF PROB. 435 (SEE P. 94), BY R. J. ADCOCK.—Let m^2 = the number of points, arranged uniformly in any manner, on a unit of surface, then m = average number on a unit of length. The number of bases MN (see figure on p. 94) on chord AB equals the number of positions of two points on AB, $= \frac{1}{2}m^2AB^2 = \frac{1}{2}m^2c^2 = 2m^2r^2\sin^2\theta$. (Math. Monthly, No. 1, Vol. I.) Hence the total number of triangles with bases on AB is $2m^2r^2\sin^2\theta \times \pi m^2r^2 = 2\pi m^4r^4\sin^2\theta$.

While AB passes through the number of positions $mrd.\cos\theta$, the number of triangles on it will be $2\pi m^5r^5\sin^3\theta d\theta$. And it will be in all its positions while θ changes from 0 to π. Hence.

$$\int_0^\pi 2\pi m^5r^5\sin^3\theta d\theta = 2m^5r^5(-\tfrac{1}{3}\sin^2\theta\cos\theta - \tfrac{2}{3}\cos\theta) + C = \tfrac{8}{3}\pi m^5r^5$$
$$= \text{number of triangles.}$$

The number of constant bases $MN = y$, on AB is $m(c-y)$, and their sum is $m(c-y)y$. Hence the sum of all bases on AB is

$$\int_0^c m^2(c-y)ydy = \tfrac{1}{6}m^2c^3 = \tfrac{4}{3}m^2r^3\sin^3\theta.$$

Now $\frac{1}{2}m$ multiplied by the square of perpendicular to AB from any point in the circumference equals sum of altitudes in that perpendicular. Therefore the sum, of all altitudes of each triangle having base on AB, $\frac{1}{2}(m^2\div\pi)$ multiplied by the sum of the two volumes described about AB as an axis by the two segments of the circle made by chord AB; that is

$m^2r^3(\pi\cos\theta+2\sin\theta-2\theta\cos\theta-\frac{2}{3}\sin^3\theta)=$ sum of altitudes for each base on AB. Hence $\frac{2}{3}m^4r^6(\pi\cos\theta+2\sin\theta-2\theta\cos\theta-\frac{2}{3}\sin^3\theta)\sin^3\theta=$ sum of triangles on AB. And

$$\int_0^{\frac{1}{2}\pi}\tfrac{4}{3}m^5r^7(\pi\sin^4\theta\cos\theta+2\sin^5\theta-2\theta\sin^4\theta\cos\theta-\tfrac{2}{3}\sin^7\theta)d\theta$$

$$=\tfrac{2}{3}m^5r^7\left\{\tfrac{1}{5}\pi\sin^5\theta-\frac{12}{5}\left(\tfrac{1}{5}\sin^4\theta+\frac{1.4}{3.5}\sin^2\theta+\frac{1.2.4}{1.3.5}\right)\cos\theta-\tfrac{2}{5}\theta\sin^5\theta\right.$$

$$\left.+\frac{2}{3}\left(\tfrac{1}{7}\sin^6\theta+\frac{1.6}{5.7}\sin^4\theta+\frac{1.4.6}{3.5.7}\sin^2\theta+\frac{1.2.4.6}{1.3.5.7}\right)\cos\theta\right\}+C$$

$$=\tfrac{2}{3}m^5r^7.\frac{1024}{525}.\ \text{ Dividing by } \tfrac{8}{3}\pi m^5r^5 \text{ gives } \frac{256r^2}{525\pi}$$

for the average area, the same as found by Mr. Seitz in his corrected result. [See *Errata*, p. 128.]

RECONSIDERATION OF SOLUTION OF PROB. 239. (*P.* 48, *VI.*)

BY CHAS. H. KUMMELL, U. S. COAST SURVEY, WASH., D. C.

THERE were three solutions of this problem published, the first one furnished by me and supported by the Editor,* the second by Mr. Adcock and the third by Prof. P. E. Chase. A thorough investigation of the law of error in two dimensions, which I am making at present, has however convinced me that Prof. Chase's solution is the only correct one. In this solution, formulæ are used which Sir John Herschel developed in his Lecture on target shooting. The proof he gives, though I now admit it to be perfectly correct, failed to convince me of my error. Recently I thought of investigating the matter from a new point of view. I regarded shooting as compounded of two independent operations, viz., sighting and leveling, and I assume that errors in sighting, i. e., deviations x from the vertical y-axis and errors in leveling, i. e., deviations y from the x-axis each follow the ordinary law of error, so that if $\varepsilon_x=$ mean error of sighting and $\varepsilon_y=$ mean error of leveling, then

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*It will be seen, by referring to p. 50, Vol. VI, that our "support" of Mr. Kummell's solution was *conditional.* We there stated, and still assert, that the solution is correct "if the equation $y=ce^{-h^2x^2}$ represents the relation between an error and its probability."

It is obvious, however, that, in target shooting, the equation does not represent that relation; for it is apparent, from a mere statement of the case, that the center of the target is not as likely to be hit, by any single shot, as a contiguous concentric circle; and hence a very small deviation from the center, of a single shot, is not the most probable.—Editor.

$$\frac{dx}{\varepsilon_x\sqrt{(2\pi)}}e^{-(x^2\div 2\varepsilon_x^2)} = \text{probability of error } \pm x \text{ in sighting,} \quad (1_x)$$

$$\frac{dy}{\varepsilon_y\sqrt{(2\pi)}}e^{-(y^2\div 2\varepsilon_y^2)} = \text{“ “ “ } \pm y \text{ in leveling,} \quad (1_y)$$

consequently $\frac{dx\,dy}{2\varepsilon_x\,\varepsilon_y\pi}e^{-(x^2\div 2\varepsilon_x^2)-(y^2\div 2\varepsilon_y^2)}$ = probability of hitting the point (x, y). (2)

I shall not consider here the general case any further, but assume, which is practically sufficient, $\varepsilon_x = \varepsilon_y = \varepsilon$. Transforming (2) to polar coordinates by assuming $x = r\cos a$ and $y = r\sin a$, when $dxdy$ must be replaced by $r\,dr\,da$, we have

$$\frac{r\,dr\,da}{2\varepsilon^2\pi}e^{-\frac{r^2}{2\varepsilon^2}} = \text{probability of hitting the point } (r, a), \quad (3)$$

and

$$\frac{r\,dr}{\varepsilon^2}e^{-\frac{r^2}{2\varepsilon^2}} = \text{prob. of shooting a dist. } r \text{ from center.} \quad (4)$$

This probability has a maximum, viz., if

$$0 = \frac{1}{\varepsilon^2} - \frac{r^2}{\varepsilon^4}, \therefore r = \varepsilon, \quad (5)$$

which means, that if we divide a target record in (infinitesimal) rings of eq'l width, then the one, whose radius = ε, will contain the greatest number of hits. This is then the most probable shot.

Integrating (4) from the center to the distance r, we have, if n = total number of shots and n_r = the number of hits on circle, radius r,

$$\frac{n_r}{n} = 1 - e^{-\frac{r^2}{2\varepsilon^2}}; \therefore \varepsilon = \frac{r}{\sqrt{2l[n\div(n-n_r)]}}. \quad (6)$$

We have then $n_\varepsilon \div n = 1 - e^{-\frac{1}{2}}$, or $n_\varepsilon = 0.395\ldots n = \frac{2}{5}n$ nearly. (7)

If then we count $\frac{2}{5}n$ shots nearest the center, then the farthest of these will be the most probable shot nearly; and it is obvious that this is the most accurate value of it we can expect to obtain by simply counting shots, because shots at this distance will be nearer together when revolved on the same line than anywhere within or without.

Herschel employs another quantity for comparing skill in shooting, viz., the radius of a circle which should receive half the number of shots. This circle I call the even chance circle, and if ρ denotes its radius we have in (6)

$$\tfrac{1}{2} = e^{-(\rho^2 \div 2\varepsilon^2)}; \therefore \rho = \varepsilon\sqrt{(2l2)} = 1.177..\varepsilon \quad (8)$$

Eliminating ε from (6) and (8) we have

$$\left(\frac{1}{2}\right)^{\rho^2} = \left(\frac{n-n_r}{n}\right)^{r^2}; \quad (9)$$

$$\therefore \rho = r\sqrt{\left(\frac{l\,\frac{n-n_r}{n}}{l\,\frac{1}{2}}\right)} = r\sqrt{\left(\frac{l\,\frac{n}{n-n_r}}{l\,2}\right)}. \tag{10}$$

This agrees with Herschel's formulæ which Prof. Chase employs.

I abstain from developing the more accurate formulæ for determining these constants from the sum of the squares of the distances, or from the sum of the distances (the string), also in the case of stray shots, and give in conclusion the most probable shots of the marksmen A and B. They are,

For marksman A, $\varepsilon_a = \dfrac{5}{\sqrt{(2l\,\frac{100}{64})}} = 5.292.$

For marksman B, $\varepsilon_b = \dfrac{10}{3\sqrt{(2l\,\frac{100}{36})}} = 4.664.$

PROBLEM 436.—"Integrate the equation

$$x^m y^n(aydx + bxdy) = x^{m'}y^{n'}(a'ydx + b'xdy)."$$

SOLUTION BY PROF. W. W. BEMAN.—Evidently $x^{-m-1}y^{-n-1}$ is an integrating factor of the first member, whence an integral of $x^m y^n(aydx+bxdy) = 0$, is $\log(x^a y^b) = c$, or $x^a y^b = C$.

The general form of integrating factor of the first member is, then,

$$x^{-m-1}y^{-n-1}\,\varphi(x^a y^b).$$

In the same way we may obtain a general integrating factor for the second member, $\quad x^{-m'-1}y^{-n'-1}\,\psi(x^a y^b)$.

That these two may be equal, we must have

$$x^{m'+1}y^{n'+1}\,\varphi(x^a y^b) = x^{m+1}\,y^{n+1}\,\psi(x^a y^b).$$

Let $\varphi(x^a y^b) = (x^a y^b)^r$, $\psi(x^a y^b) = (x^a y^b)^s$, r and s being indeterminate. Then

$$x^{m'+1}\,y^{n'+1}\,(x^a y^b)^r = x^{m+1}\,y^{n+1}\,(x^a y^b)^s.$$

$$\therefore\ m'+ar = m+a's, \quad n'+br = n+b's;$$

$$r = \frac{a'(n-n')-b'(m-m')}{a'b-ab'}, \quad s = \frac{a(n-n')-b(m-m')}{a'b-ab'}.$$

The integrating factor becomes

$$x^{ar-m-1}\,y^{br-n-1} = x^{a's-m'-1}\,y^{b's-n-1};$$

$$\therefore\ x^{ar-1}\,y^{br-1}(aydx+bxdy) = x^{a's-1}\,y^{b's-1}(a'ydx+b'xdy).$$

Integrating, $\qquad \dfrac{1}{r}x^{ar}\,y^{br} = \dfrac{1}{s}x^{a's}\,y^{b's} + C.$

When $a'b = ab'$ the equation is immediately integrable.

This solution is based upon that of a special form of the equation given, found in Hoüel's Calcul Infinitésimal.

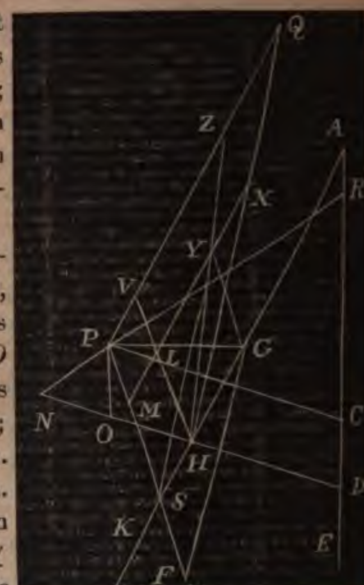
SOLUTIONS OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received as follows:

From Prof. W. P. Casey, 437, 438; Prof. H. T. Eddy, 440; Prof. A. Hall, 440; E. H. Moore, Jr., 438; Ernest G. Merritt, 438.

437. *By Prof. Casey.*—"AE, AK are two indefinite given straight lines, C and H given points in them, and P a given point in their plane. Req'd to draw through P two straight lines, PB, PD, intersecting AE and AK in R and S, respectively, and containing a given angle RPS, so that $RC\times SH$ may be equal to a given magnitude."

SOLUTION BY THE PROPOSER.



Let AE, AK be two indefinite straight lines given in position; C, H, given points in them, and P a giv'n point in their plane; and let PR, PS be drawn meeting them in R, S, so that $RC\times HS$ may be $=$ a given space, A^2; and the $\angle RPS =$ a given angle.

Analysis.—Join PC; it is given in position. Through P, H, draw PO, DHO, parallel to AE, PC, respectively; $\therefore$ O is a given point. Produce RP to meet DO in N. Now $RC\times ON = PC\times CO$ and is therefore given, and $RC\times HS$ is given; therefore the ratio of ON to HS is given. Angle RPS is given, $\therefore$ $\angle NPS$ is given.

Make $\angle OPG = \angle NPS$, then PG is in position and G is a given point. Make $\angle PGF = PON =$ a given angle, therefore GF is in position, and triangles PON and PGF are similar, and $ON:GF :: PO:PG$, i. e., in a given ratio, and the ratio of $ON:HS$ is given, $\therefore$ the ratio of $GF:HS$ is given.

Through H draw HQ parallel to GF, $\therefore$ HQ is in position. Draw SY parallel to HQ, and GY parallel to PS, meeting SY in Y. Through Y draw XYM parallel to AK, and through P draw PQ parallel to AK, $\therefore$ PQ is in position, and Q is a given point. Join HY and produce it to meet PQ in Z and draw HL parallel to PS. Then $GF = SY = HX$, and

$HS=XY$; and as the ratio of $GF:HS$ is given, $\therefore$ the ratio of HX to XY is given; hence $HQ:QZ$ is given and HQ is given, $\therefore$ QZ is given and Q is a given point, $\therefore$ Z is a given point and and PZ is a given line. But $QZ\times YL=XY\times ZV$, and $YL=GH$, a given line, also $XY=HS=VP$, $\therefore ZV\times VP=QZ\times GH$, a given space, and ZP is given, $\therefore$ V is a given point and VH is in position, hence PS is in position and so is PR, and the problem is solved geometrically.

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438. *By Prof. F. H. Loud.*—"Two equiangular parallelograms, $OACB$ and $OA'C'B'$ are so placed that the equal angles AOB and $A'OB'$ coincide. The sides of the former figure are constant, those of the latter are variable, subject to the condition $A'O+OB':AO+OB$ :: *area* $OA'C'B'$: *area* $OACB$. $A'C'$ meets BC in D, and $B'C'$ meets AC in E. Show that DE passes through a fixed point, and determine the point."

SOLUTION BY ERNEST G. MERRITT, CORNELL UNIV., ITHACA, N. Y.

Put $OB=a$ $OA=b$, coordinates of the point C, and let the coordinates of C' be $OB'=x'$ and $OA'=y'$.

Then, from the conditions of the problem,

$$x'+y':a+b=x'y'\sin AOB:ab\sin AOB;$$

$$\therefore (a+b)x'y'=ab(x'+y'), \tag{1}$$

whence

$$y'=\frac{abx'}{(a+b)x'-ab}. \tag{2}$$

The coordinates of D are (a, y'), and of E, (x', b). Substituting in the equation of a line through two points (x_1, y_1) and (x_2, y_2)

$$\left[y-y_1=(x-x_1)\frac{y_2-y_1}{x_2-x_1}\right],$$

we obtain $y-b=\left(\frac{x-x'}{a-x'}\right)\left(\frac{abx'}{(a+b)x'-ab}-b\right)$, which may be written

$$y-b=(x-x')\frac{\frac{ab}{a+b}-b}{\frac{ab}{a+b}-x'}. \tag{3}$$

Equation (3) represents a line passing through E and the fixed point $[ab\div(a+b),\ ab\div(a+b)]$, but it is also the eq'n of the line DE. Henc the line DE passes through the point P whose coordinates are $ab\div(a+b)$ and $ab\div(a+b)$.

COR. From eq'n (1) it is seen that the locus of C is an hyperbola whose center is at P and whose asymptotes are parallel to OB and OA.

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439. *By Prof. Nicholson.*—"Solve geometrically the following:

On a line whose length is a are two points x distance apart; what is the average value of x?"

No solution of this problem has been received. It may however readily be solved as follows:

Let x be any distance $Ab=bc$, estimated from A ($AB=a$). Then is $(a-x)x=$ area $bBb'c=$ the sum of the distances of the two points, which being taken for all values of x, from 0 to a, will represent the volume of a triangular pyramid whose base is the area of the triangle BCD (D being vertically above B), and altitude, $BA=a$, and is, therefore, $\frac{1}{6}a^3$. This divided by $\frac{1}{2}a^2$, the whole num. of positions, gives $\frac{1}{3}a$ for the required average.

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440 *Selected.*—"A lamina is bounded on two sides by two similar ellipses, the ratio of the axes in each being m, and on the other two sides by two similar hyperbolas, the ratio of the axes in each being n. These four curves have their principal diameters along the coordinate axes. Prove that the product of inertia about the coordinate axes is

$$\frac{(\alpha^2-\alpha'^2)\,(\beta^2-\beta'^2)}{4(m^2+n^2)}$$

where $\alpha\alpha'$, $\beta\beta'$ are the semi-major axes of the curves." (Routh's Rigid Dynamics.)

SOLUTION BY PROF. HENRY T. EDDY, PH. D.

$F=\int\int xydxdy$. $\therefore$ by Tod. Int. Cal., Art. 239,

$$F=\pm\int\int\frac{xydudv}{\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}}.$$

But $u = x^2 + m^2y^2$, $\quad v = x^2 - n^2y^2$; therefore

$$F = \pm \int_{a'^2}^{a^2}\int_{\beta'^2}^{\beta^2} \frac{xydudv}{4xy(m^2+n^2)} = \frac{(a^2-a'^2)(\beta^2-\beta'^2)}{4(m^2+n^2)}.$$

SOLUTION BY PROF. ASAPH HALL.

The ratio of the axes of the curves being given, we may write the eq'ns

$$x^2 + m^2y^2 = u; \qquad x^2 - n^2y^2 = v.$$

The product of inertia is given by the integral $\int xydxdy$, and we have to transform this to the variables u and v. The partial derivatives are,

$$\frac{dx}{du} = \frac{n^2}{2x(m^2+n^2)}; \qquad \frac{dx}{dv} = \frac{m^2}{2x(m^2+n^2)};$$

$$\frac{dy}{du} = \frac{1}{2y(m^2+n^2)}; \qquad \frac{dy}{dv} = \frac{-1}{2y(m^2+n^2)}.$$

Forming the known determinant for the transformation we have,

$$\int xy\,dxdy = \frac{1}{4(m^2+n^2)}\int du\,dv = \frac{(a^2-a'^2)\,(\beta^2+\beta'^2)}{4(m^2+n^2)};$$

since the limits of u are a^2 and a'^2, and of v, β^2 and β'^2.

Remarks on "New Rule for Cube Root."—We published on page 98, No. 3, what purports to be a new Rule for Cube Root, and were not aware, at the time, that substantially the same rule had been published before; and we have no doubt the author believed it to be new, and original with him.

The same Rule, in effect, may be found at page 32, Vol. I of the *Mathematical Monthly*, published in Nov., 1859, at Cambridge, Mass. The editor (J. D. Runkle) there says, "In the Nouvelles Mathe'matiques for January, 1858, we find the following method for extracting the cube root of numbers, which ought, on account of its easy application, to be generally used. The editor remarks, in the April number, that the method had previously been given in a work entitled *Calcul pratiques*, in which it is claimed as new. The reader will find the same procees, entitled a new method, in the American edition of Young's Algebra, published as long ago as 1832. It may also be found in some of our arithmetics; and many teachers undoubtedly already know and use it."

PROBLEMS.

441. *By Wm. Hoover, A. M., Dayton, Ohio.*—A cone revolves around its axis with a known angular velocity. The altitude begins to diminish

and the vertical angle to increase, the volume being constant. Show that the angular velocity is proportional to the altitude.

442. *By Prof. Casey.*—ABN is a given circle, D, F and O are given points in the same plane. It is required to describe a circle passing through D and F and intersecting the given circle in the points G, H, so that the triangle GOH may be of a given magnitude.

443. *By O. H. Merrill.*—In cutting the maximum rectangular parallelopipedon from a frustum of a cone, five pieces are cut off. Find the volume of each of these pieces.

444. *Selected by Prof. H. T. Eddy.*—Given the five equations,

$$\begin{aligned} x_1^2+x_2^2+x_3^2 &= 3\beta^2, \\ y_1^2+y_2^2+y_3^2 &= 3\alpha^2, \\ x_1y_1+x_2y_2+x_3y_3 &= 0, \\ x_1+x_2+x_3 &= 0, \\ y_1+y_2+y_3 &= 0. \end{aligned}$$

Eliminate x_2y_2 x_3y_3, and show that

$$\alpha^2x_1^2+\beta^2y_1^2 = 2\alpha^2\beta^2.$$

(Routh's Dynamics, 4th Edition, Article 38.)

PUBLICATIONS RECEIVED.

Annual Report of the Chief Signal Officer to the Secretary of War for the fiscal year ending June 30, 1881. 8vo. 981 pages, with 69 maps. Washington: 1881.

Transactions of the Wisconsin Academy of Science, Arts, and Letters. Vol. V. 1877–1881. Madison, Wisconsin. 1882.

ERRATA.

On page 169, line 21 (Vol. VI), in the exponent of c, for w read ω.
" " 138, line 13 from bottom (Vol. IX), for b^3, read b_3.
" " 85, lines 10, 11, 12, 14 and 15, read for exponents of x in the last eq'n of the several lines, respectively, 2, 3, n, 2, n.
" " 95, " 6, 8, and 10, divide each fraction before $\int$ by 2.
" " ", line 7, insert y before dy.
" " ", " 10, for $512r^2 \div 525\pi$, read $256r^2 \div 525\pi$.
" " 102, " 3, for y_3, read y^3.
" " 115, " 23, for $(2—2m)$, read $(2+2m)$.
" " 116, " 4, for $= \infty$, read $= -\infty$.
" " 118, " 15 from bottom, for $\frac{4}{3}\frac{8}{7}a$, read $\frac{4}{3}\frac{8}{7}a^2$.
" " 119, at head of Table III, for $-px$, read $+px$.

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CIRCULAR COORDINATES.

BY PROF. WILLIAM WOOLSEY JOHNSON.

1. BY an extension of the idea of the "position ratio" of a point, referred to two fundamentel points A and B, any point in a plane may be determined by the complex ratio of its position ratio. Put

$$\frac{AP}{BP} = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta) = x + iy; \qquad (1)$$

then, if r and r' are the lengths of the lines AP and BP, the usual interpretation of the ratio of directed lines gives

$$\rho = \frac{r}{r'}, \qquad \text{and} \qquad \theta = BPA.$$

The real quantities ρ and θ may be taken as a system of coordinates analogous to the ordinary polar system; and x and y may be taken as a system of coordinates bearing to this last system the same relations [see eq. (1)],

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

that connect the ordinary rectangular and polar coordinates.

2. The locus of $\theta = \alpha$ is obviously a circle passing through A and B, in one segment of which (say the upper segment in Fig. 1) the angle BPA, considered as reckoned from PB towards PA, is α, and in the other segm't ρ must be taken as negative and $\theta = \alpha$, or else ρ is positive and $\theta = \pi + \alpha$. The locus of $\rho = a$ is also a circle, being the locus for which the ratio $r \div r'$ is constant, that is, the circle described on CD as a diameter, where C and D cut AB harmonically in the ratio ρ. These circles cut at right angles; for, if O is the centre of the latter, $OA.OB = OC^2 = OP^2$, hence OP is tangent to the circle BAP. Thus, in the ρ θ system, a point P is determined as the intersection of two circles which cut at right angles, the second point of intersection P', being distinguished from P either by regarding ρ as negative or by adding 180° to θ.

3. If we draw a tangent to the θ-circle at A, θ is the inclination of this tangent to the direction BA produced. θ is indeterminate at each of the points B and A; $\theta = 0$ represents the infinite circle through $B\,A$; that is, $\theta = 0$ not only on the straight line BA, but at every infinitely distant p't. $\rho = 0$ and ∞ respectively at A and B, the points where θ is indeterminate. $\rho = 1$ is the equation of the straight line bisecting BA at right angles, and, since this is to be regarded as an infinite circle, $\rho = 1$ at every infinitely distant point We should however in this system speak of *the point* and not *the line at infinity*, and the coordinates of this point are $\rho = 1$, $\theta = 0$.

4. If the fundamental point B be removed to infinity, the θ-circle becomes a straight line making an angle θ with the axis BA, and the ρ-circle ultimately has A for its centre; ρ becomes infinitesimal, but multiplying it by the constant infinite factor r', we have the ordinary polar coordinates.

5. Proceeding now to the $x\,y$ coordinates, the locus of x = a constant is also a circle; for, if the circle whose diameter is BA cuts r' in R, $RP = r \cos\theta$; hence if

$$x = \rho \cos\theta = \frac{r \cos\theta}{r'} = \frac{RP}{BP} \tag{2}$$

be constant, the line BP bears a constant ratio to BR, and P describes a locus similar to that of R, in other words, a circle whose diameter is BX, where X is that point of BA whose position ratio is the given value of x, since at X $x = \rho = AX \div BX$. Since

$$1 - x = \frac{BX - AX}{BX} = \frac{BA}{BX},$$

the diameter of this circle is

$$BX = \frac{BA}{1 - x} = \frac{c}{1 - x}. \tag{3}$$

The locus of $y =$ a constant is also a circle; for let

$$y = \rho \sin \theta = \frac{r \sin \theta}{r'} = \frac{AR}{BP} \tag{4}$$

be constant; then since AR is perpendicular to BP, a point R' on BR such that $BR' = AR$ will describe a circle equal to BRA and touching BA; and, BP having a constant ratio to BR', P will describe a similar curve. If this circle cut the perpendicular to BA through B in Y, eq'n (4) gives

$$y = \frac{AB}{BY},$$

hence the diameter of this circle is

$$BY = \frac{BA}{y} = \frac{c}{y}. \tag{5}$$

The x- and y-circles cut at right angles; thus the x y coordinates determine a point as the variable intersection of two circles which cut also in the fixed point B. When B is removed to infinity, the arcs PX and RA become straight lines perpendicular to BA, and PB, a straight line parallel to BA; the values of x and y vanish, but being multiplied by the infinite factor r', become RP and AR [equations (2) and (4)] which are now ordinary rectangular coordinates.

6. Negative values of x correspond to points within the circle $x = 0$, or $\theta = 90°$, whose diameter is AB. $x = 1$ is the equation of the perpendicular BY, but $x = 1$ also at every infinitely distant point; the coordinates of the "point at infinity" are $x = 1, y = 0$.

7. If we put $1 - x = x_1$, equations (3) and (5) become

$$x_1 = \frac{c}{BX}, \qquad y = \frac{c}{BY}, \tag{6}$$

and the coordinates (x_1, y) are the reciprocals of the diameters of the circles. Dropping the suffix, $x = 0$ and $y = 0$ now represent the rectangular axes intersecting at B, but their intersection is not the point (0, 0), since the p't (x, y) is defined as the variable intersection of the circles, which is now the point at infinity.

8. It is evident that P is the foot of a perpendicular from B upon XY; x and y, the reciprocals of the intercepts upon the axes, are Dr. Booth's tangential coordinates of this line, and they are the ordinary rectangular coordinates of the pole of this line with respect to B, which is the inverse point to P, B being the centre of inversion. In fact, if x', y' are the rectangular coordinates of P we have $x'.BX = BP^2 = x'^2 + y'^2$, and, comparing with (6),

$$x = \frac{cx'}{x'^2 + y'^2}, \text{ and similarly } y = \frac{cy'}{x'^2 + y'^2}, \tag{8}$$

which are also the relations between the coordinates of inverse points. Thus the circular coordinates of P are the rectangular coordinates of the inverse point.

9. It follows that the equation of a curve in circular coordinates is the same as that of its inverse in rectangular coordinates. Thus the equation of the first degree in circular coordinates may be written in the form

$$mx + ny + c = 0;$$

the corresponding rectangular equation is

$$x'^2 + y'^2 + mx' + ny' = 0,$$

showing that the locus is a circle passing through B, and making BA an angle whose tangent is $-(n \div m)$. On the other hand, the equation

$$x^2 + y^2 + mx + ny = 0$$

in circular coordinates is the general equation of the sraight line not passing through B; while the general equation of the circle retains the same form as in rectangular coordinates.

10. Let us now suppose that the position of P is determined not simply by its position ratio, but by the ratio of its position ratio to that of a third fixed point Q referred to A and B, that is, by the anharmonic ratio

$$\frac{P\ A}{B\ Q} = \frac{AP}{BP} : \frac{AQ}{BQ}.$$

Denoting as before the position ratio of P by $\rho e^{i\theta}$, and denoting that of Q by $\rho_0 e^{i\theta_0}$, we have for the value of this expression,

$$\frac{P\ A}{B\ Q} = \frac{\rho}{\rho_0} e^{i(\theta-\theta_0)} = \frac{x + iy}{x_0 + iy_0},$$

and if we put

$$\frac{P\ A}{B\ Q} = Re^{i\theta} = X + iY,$$

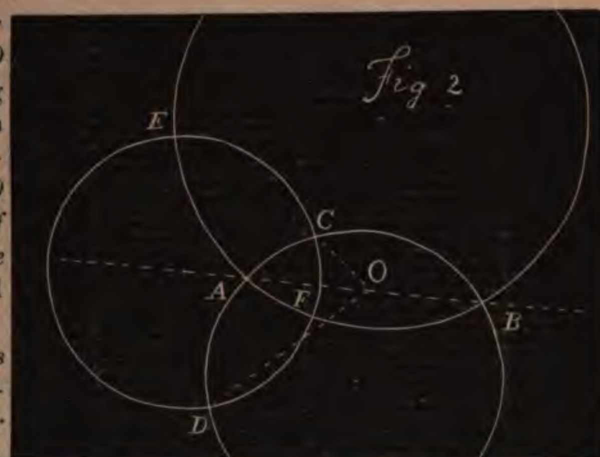
it is evident that the locus of $R =$ constant and of $\theta =$ constant are circles of the ρ- and θ-systems respectively, but R is a fixed multiple of the ratio of the lengths of AP and BP, and θ is the angle between the tangents at A of the circles APB and AQB.

Since $$X+iY = \frac{x + iy}{x_0 + iy_0} = \frac{xx_0+yy_0}{x_0^2 + y_0^2} = i\frac{yx_0-xy_0}{x_0^2 + y_0^2},$$

the locus of $X=$ a constant is, by § 9, a circle passing through B and touching at that point a fixed line which makes with BA the angle whose tang't is $-(x_0 \div y_0)$. In like manner, the locus of $Y =$ a constant is a circle which touches at B a line inclined to BA at an angle whose tangent is $y_0 \div x_0$; that is, a line perpendicular to that touched by the X-circles. Furthermore, a linear relation between X and Y is equivalent to a linear relation

etween x and y and represents a circle passing through B. In fact, the stem $X\ Y$ bears to the system $x\ y$ the same relations that connect two ornary rectangular systems with the same origin, to which systems indeed ey reduce when B is removed to infinity.

11. Four points on the same circle have a real anharmonic ratio. In rticular the anharmonic ratio of the section of AB by PP' in Fig. 1 is -1. The points P, P' are therefore harmonic conjugates with respect to , B, and reciprocally A, B are harmonic conjugates with respect to P, P'; at is, *when two circles cut orthogonally, the points in which a diameter of e cuts the circumference of the other are harmonic conjugates with respect to e points of intersection.*



12. The harmonic rcle of points AB, CD as also the following roperty analogous to a roperty of the harmonic range; viz., *if O the middle point of B, OC and OD make ual angles with OA* nd $OC.OD = OA^2$.

13. *If three circles t each other orthogonally*, as in Fig. 2, *their oints of intersection rm three pairs of points AB, CD and EF each of which is a pair of harmonic conjugates with respect to either of the other pairs;* for the centre of ch circle is on the radial axis of the other two.

In the same figure the position ratios of C and E, with respect to AB, are ual in modulus and their arguments differ by 90°, hence their ratio is i $-i$; the six anharmonic ratios of four points thus situated being

$$i, \quad -i, \quad 1+i, \quad \frac{1-i}{2}, \quad \frac{1+i}{2}, \quad 1-i.$$

14. The real anharmonic ratio of four points on the circumference of a rcle is the same as the anharmonic ratio of the pencil formed by joining e points to any point of the circumference, a property by which the anharmonic ratio of four points on a circumference is generally defined. But e anharmonic ratio of the concyclic points is not the same as that of the encil formed by joining the points to *any* point as in the case of the rectinear range; but it is to be noticed that when the radius of the circle

becomes infinite, any finite point may be regarded as at an infinitesimal distance from the circumference relatively to the infinite radius.

15. There is a special case of the complex anharmonic ratio which is worthy of notice, in which, as in the case of the harmonic ratio, there are less than six *different* values of the anharmonic ratio. In the harmonic ratio, there are in fact three different values, viz, -1, 2, and $\frac{1}{2}$, the first of these numbers being its own reciprocal, the next being (in the nomenclature of my articles, ANALYST Vol. IX, p. 185, and Vol. X, p. 76) its own conjugate, and the third its own complement. The only other case in which there are but three different values is that of 1, 0, and ∞, which occurs when two of the four points are coincident. In the case in question however there are but two different values, namely $\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$, each of which is at once the reciprocal, complement and conjugate of the other. If the fundamental points A, B and C are given, two points P and P' may be found such that

$$\begin{matrix} P & A \\ B & C \end{matrix} \quad \text{and} \quad \begin{matrix} P' & A \\ B & C \end{matrix}$$

shall have these two values, and then each of the anharmonic ratios of the points $ABCP$ or $ABCP'$ will have one of the two values, so that the interchange of any two points of the four has the same effect. Since for these values, which are $e^{\pm\frac{1}{3}i\pi}$, $R=1$ (see § 10) $\rho = \rho_0$; hence, denoting the sides of the triangle by a, b, c and the distances from P to the vertices A, B, C by a', b' and c', we have $aa' = bb' = cc'$; and, since $\theta = \pm\frac{1}{3}\pi$, the difference between the angles BCA and BPA (reckoned in the same direction), QAP and QBP, etc., is in each case equal to $60°$.

If ABC is an isosceles triangle, P and P' are on the bisector of the angle opposite the base. If one of the angles of the triangle is $60°$ or $120°$, one of the points is on the side opposite this angle; but if the triangle is equilateral this point is at infinity, the other being the centre of the triangle.

ON THE DIVISIBILITY OR NON-DIVISIBILITY OF NUMBEBS BY SEVEN.

BY ALEXANDER EVANS, ESQ., ELKTON, MARYLAND.

THE test for divisibility by seven has been considered in relation to the number of digits composing the dividend: As "for two or three figures; for three or four figures; for five figures; for five, six, seven or more figures."

A writer, after so treating the subject, says, "the various tokens of seven are not so difficult to remember as at first may appear".

In the examples of the processes of Arithmetic and Algebra, by the Society for the Diffusion of Useful Knowledge, the author, at page two, says, division by seven is "according to no rule sufficiently simple to be useful."

The writer of this article submits the following rule:

Multiply the right hand, or units digit, by nine, and deduct this product from the remaining digits, treat the remainder in the same way, multiplying its units digit by nine and deducting the product from the remaining digits, until a number is arrived at which is evidently divisible or not divisible by seven; according as the one or the other occurs the original number is, or is not divisible by seven.

EXAMPLES.

1.

$$\begin{array}{rl} 2121 & \\ \underline{9} & = 1\times 9 \\ 203 & \\ \underline{27} & = 3\times 9 \\ 7 & \end{array}$$

Hence 2121 is divisible by 7.

2.

$$\begin{array}{rl} 13433 & \\ \underline{27} & = 3\times 9 \\ 1316 & \\ \underline{54} & = 6\times 9 \\ 77 & \end{array}$$

Hence 13433 is divisible by 7.

This process, though only given as a rule, and not for its practical usefulness, is often easier when "working in the head" than the direct method.

3.

$$\begin{array}{rl} 99009 & \\ \underline{81} & = 9\times 9 \\ 9819 & \\ \underline{81} & = 9\times 9 \\ \underline{900} & \\ 9 & \end{array}$$

Hence 99009 is not divisible by 7.

If 12 be used as a multiplier instead of 9, then the above rule becomes a test for divisibility by 11. Also, 22 is the multiplier and test for divisibilisy by 13; and 39 is the multiplier and test for divisibility by 17.

It will be observed that the *least* test multiplier has not always been used; this is because the least does not always lead quickest to the result.

For divisibility by every prime number, test multipliers may be found. A list extending as far as 43 is given below.

For divisibility	by	7,	test	multipliers,	2;	9;	16,	&c.
"	"	" 11,	"	"	1;	12;	23,	&c.
"	"	" 13,	"	"	9;	22;	35,	&c.
"	"	" 17,	"	"	5;	22;	39,	&c.
"	"	" 19,	"	"	17;	36;	55,	&c.
"	"	" 23,	"	"	16;	39;	62,	&c.
"	"	" 29,	"	"	26;	55;	84,	&c.
"	"	" 37,	"	"	11;	48;		&c.
"	"	" 43,	"	"	30;	73;		&c.

For an art. on this subject by Prof. Brooks, see ANALYST, Vol. II, p. 129.

DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

BY WILLIAM T. JORDAN, WATERVILLE, MAINE.

To differentiate
$$y = e^x. \tag{1}$$
$$y + dy = e^{x+dx} = e^x . e^{dx}. \tag{2}$$
Now $e^0 = 1$, and as dx is infinitesimal, e^{dx} will be but an infinitesimal greater than e^0, and as x is our variable, e^{dx} will be written $1 + dx$; $\therefore$
$$e^{dx} = 1 + dx. \tag{3}$$
Substituting this value of e^{dx} in (2) we have
$$y + dy = e^x(1 + dx) = e^x + e^x dx. \tag{4}$$
Subtracting (1) from (4), we have
$$dy = e^x\, dx. \tag{5}$$
To differentiate $y = a^x$ with reference to the Napierian system, we have
$$a = e^{\log_e a}. \tag{6}$$
Substituting $e^{\log_e a}$ for a from (6), we have
$$y = e^{\log_e ax}. \tag{7}$$
Differentiating as above we have
$$dy = e^{\log_e ax} \log_e a dx; \tag{8}$$
but $a = e^{\log_e a}$, $\therefore$
$$dy = a^x \log_e a dx. \tag{9}$$
To differentiate $y = a^x$ (10) with regard to any system. We have, if s be the base of the system, $a = e^{(\log_s a) \div m_s}$. Substituting this value of a in (10), we have
$$y = e^{(\log_s ax) \div m_s}. \tag{11}$$
Defferentiating as before, we have
$$dy = e^{(\log_s ax) \div m_s} (\log_s a dx) \div m_s. \tag{12}$$
Now $a = e^{(\log_s a) \div m_s}$; $\therefore$
$$dy = \frac{a^x \log_s a dx}{m_s}. \tag{13}$$
To differentiate the logarithm of a variable, let $y = e^x$, therefore
$$x = \log_e y;\ \therefore dx = d(\log_e y). \tag{14}$$
Now $dy = e^x dx$, $\therefore$ from (14) $dy = y dx$; $\therefore dx = dy \div y$, but $dx = d(\log_e y)$,
$$\therefore d(\log_e y) = \frac{dy}{y}. \tag{15}$$
Let s be the base of any system, then $\log_s y = m_s(\log_e y)$, $\therefore d(\log_s y) = m_s \times (\log_e y)$, $\therefore d(\log_s y) = m_s(dy \div y)$, the differential of a logarithm in any system.

INTEGRATION BY AUXILIARY INTEGRALS.

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

[Continued from page 114.]

WE have seen that all (or nearly all) the integrals contained in the larger collections may be found with far less labor than by the methods of the books when simply employing the ordinary primitive integrals as auxiliaries, in our present sense of the word. When the integration of a function, $dv = f(x).dx$, is required we take for auxiliary such primitive function $du = \varphi(x).dx$ as may appear most suitable, then we form the expression $dv \div du$ in terms of u and integrate, thus finding v in terms of u. Now, our new primitive integrals were devised for the purpose of enlarging the range of integrable functions and to facilitate the integration of functions already otherwise integrable. This latter remark applies also to that class of functions known as "binomial" which we have thus far only slightly touched upon and which by means of our method become integrable without the laborious process of rationalization and decomposition into partial fractions.

When a binomial is proposed for integration it is convenient to employ different auxiliaries, varying with the nature of the constant quantities occurring in the proposed function. For illustration, let it be required to integrate

$$dv = \frac{(2x^5-1)^{\frac{1}{2}}}{x^{11}}dx.$$

Here it is convenient to take the auxiliary

$$du = \frac{dx}{x\sqrt{(2x^5-1)}}; \therefore u = \tfrac{1}{5}\text{arc sin}\left(\frac{x^5-1}{x^5}\right);$$

$$\therefore \frac{1}{x^5} = 1 - \sin 5u; \cos 5u = \frac{1}{x^5}\sqrt{(2x^5-1)};$$

$$\therefore \frac{dv}{du} = \frac{2x^5-1}{x^{10}} = \cos^2 5u;$$

$$v = \int \cos^2 5u\, du$$

the integration of which needs no further remark. Again, let it be required to integrate

$$dv = \frac{dx}{x^3(x^4-1)^{\frac{5}{4}}}.$$

Here we take for auxiliary

$$du = \frac{dx}{x\sqrt{(x^4-1)}}; \therefore u = \tfrac{1}{2}\text{arc tg}\sqrt{(x^4-1)}; \therefore x = \sec 2u,$$

$$\therefore v = \int \frac{\cos^3 2u}{\sin^2 2u} du,$$

which is integrable without any difficulty.

If we wished to extend the ordinary reference-tables of integrals, so as to give new forms, one mode of so doing would be to write the same integrals contained in the tables in terms of our own variables by substituting our variables for x. One example of this kind will suffice. Take the well known integral

$$v = \int \frac{du}{a + b.\cos u} = \frac{1}{\sqrt{(a^2-b^2)}} \operatorname{arc\,cos} \left(\frac{b+a\cos u}{a+b\cos u}\right). \quad (a^2>b^2.)$$

Now, putting $u = -\operatorname{arc\,sin}\sqrt{[(1-x^2)\div(1+x^2)]}$,

$$du = \sqrt{2}.\frac{dx}{(1+x^2)\sqrt{(1-x^4)}}; \quad \cos u = x\sqrt{\frac{2}{1+x^2}};$$

$$dv = \frac{\sqrt{2}.dx}{(1+x^2)\sqrt{(1-x^4)}\left[a+bx\sqrt{\frac{2}{1+x^2}}\right]};$$

$$\therefore \int \frac{\sqrt{2}.dx}{[a\sqrt{(1+x^2)}+bx\sqrt{2}]\sqrt{(1-x^4)}} = \frac{1}{\sqrt{(a^2-b^2)}} \times \operatorname{arc\,cos}\left[\frac{b\sqrt{(1+x^2)}+ax\sqrt{2}}{a\sqrt{(1+x^2)}+bx\sqrt{2}}\right].$$

A great variety of new forms of integrals is obtained by the substitution of such forms as $k \operatorname{tg} \varphi$, $1 + k \operatorname{tg} \varphi$, $\sqrt{(1+k \operatorname{tg} \varphi)}$, &c. for the variable x in those integrals belonging to the goniometric types. Thus take the ordinary primitive form

$$y = \int \frac{dx}{\sqrt{(1-x^2)}} = \operatorname{arc\,sin} x; \; \therefore x = \sin y,$$

and make $x = k \operatorname{tg} \varphi$, then

$$y = \int \frac{k.d\varphi}{\cos^2\varphi\sqrt{(1-k^2.\operatorname{tg}^2\varphi)}} = \int \frac{2k.d\varphi}{\sqrt{[(1-k^2)+2\cos 2\varphi+(k^2+1).\cos^2 2\varphi]}}.$$

Putting $\cos 2\varphi = t$; $\therefore$ $2d\varphi = -[dt\div\sqrt{(1-t^2)}]$, we have

$$y = -\int \frac{k.dt}{\sqrt{(1+t^2)}\sqrt{[(1-k^2)+2t+(k^2+1)t^2]}} = \operatorname{arc\,sin}(k \operatorname{tg} \varphi)$$
$$= \operatorname{arc\,sin} \left\{ k\sqrt{[(1-t)\div(1+t)]} \right\}.$$

This formula may, of course, be used as a primitive or auxiliary, and it is easy to develop by means of it a great number of new integrals. The simplest method again would be to express in terms of t any integral known in terms of y.

Treating in the same way any of our primitive forms we arrive at the corresponding new forms which in their turn may be employed as auxiliaries. As an example of the kind, take

$$dy = \frac{dx}{(2+x)\sqrt{(1+x)}}; \therefore y = 2\,\text{arc tg}\sqrt{(1+x)}; \therefore 2+x = \frac{1}{\cos^2\frac{1}{2}y};$$

$$\therefore \cos\tfrac{1}{2}y = \frac{1}{\sqrt{(2+x)}}; \therefore \sin\tfrac{1}{2}y = \sqrt{\frac{1+x}{2+x}}.$$

Putting $x = k\,\text{tg}\,\varphi$,

$$= \frac{k\,d\varphi}{\cos^2\varphi(2+k\,\text{tg}\,\varphi)\sqrt{(1+k\,\text{tg}\,\varphi)}} = \frac{k\,d\varphi}{(2\cos\varphi+k\sin\varphi)\sqrt{(\cos^2+k\sin\varphi\cos\varphi)}}$$

d this, putting $\cos\varphi = t$, gives

$$\frac{-k\,dt}{[2t+k\sqrt{(1-t^2)}]\sqrt{(1-t^2)}[t^2+kt\sqrt{(1-t^2)}]^{\frac{1}{2}}} = 2\,\text{arc tg}\left(k\frac{\sqrt{(1-t^2)}}{t}+1\right)^{\frac{1}{2}}$$

ıich equation may be used as an auxiliary.

Another fruitful method of finding new integrals is the combination of r elementary forms, so as to express the one in terms of the other. As example take

$$v = \int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = -\frac{1}{\sqrt{2}}\,\text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}}$$

d compare it with

$$u = \int\frac{2x\,dx}{\sqrt{(1-x^2)}\sqrt{(1+x^2)}} = \text{arc sin}\,x^2; \therefore x^2 = \sin u;$$

$$\therefore \frac{dv}{du} = \frac{1}{2x\sqrt{(1+x^2)}} = \frac{1}{2\sqrt{(\sin u)}\sqrt{(1+\sin u)}};$$

$$\therefore v = \int\frac{du}{2\sqrt{(\sin u)}\sqrt{(1+\sin u)}} = -\sqrt{2}.\text{arc sin}\left(\frac{1-\sin u}{1+\sin u}\right)^{\frac{1}{2}}$$

d comparing this again with

$$u = \sqrt{2}\int\frac{dt}{(1+t^2)\sqrt{(1-t^2)}} = -\,\text{arc sin}\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}}$$

$$\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}} = -\sin u; \therefore \sqrt{(\sin u)} = i\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{4}},\ \text{we find}$$

$$v = \frac{1}{\sqrt{2}}\int\frac{dt}{(1+t^2)\sqrt{(1-t^2)}.i\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{4}}\left[1-\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}$$

$$v = \frac{-i}{\sqrt{2}}\int\frac{dt}{(1-t^2)^{\frac{3}{4}}(1+t^2)^{\frac{1}{4}}[\sqrt{(1+t^2)}-\sqrt{(1-t^2)}]^{\frac{1}{2}}}$$

$$= -\sqrt{2}.\text{arc sin}\left[\frac{(1+t^2)^{\frac{1}{4}}-(1-t^2)^{\frac{1}{4}}}{(1+t^2)^{\frac{1}{4}}+(1-t^2)^{\frac{1}{4}}}\right]^{\frac{1}{2}}.$$

A great number of new integrals are found by the differentiation of such uations as

$$y = \text{arc sin}\,(ax^{\frac{1}{4}}+bx^{\frac{1}{2}}),$$
$$y = \text{arc sin}\,(ax^{-\frac{1}{4}}+bx^{\frac{1}{2}}),$$
$$y = \text{arc sin}\,(ax^{-\frac{1}{4}}+bx^{\frac{1}{4}}+cx^{\frac{1}{2}}),$$

&c. &c. &c.

d then using the integrals so found as auxiliaries.

LAW OF RANDOM ERRORS.

BY R. J. ADCOCK, ROSEVILLE, ILLINOIS.

LET the unknown error in position of one measured determination of a point in space be δ, then the point sought may be anywhere on the surface of the sphere whose centre is the observed point and radius δ. Therefore, by the definition of probability, the probability that any point on this surface is the one sought is $\frac{1}{4\pi m\delta^2}$, where m is the number of points on a unit of surface.

Since then each random error gives a number of liable positions proportional to its square, therefore the probability that any one of the total number of liable points, resulting from n measured determinations of a point in space, is the one sought, is

$$y = \frac{1}{4\pi m S(\delta^2)},$$

where $S(\delta^2)$ is the sum of the squares of the random errors belonging to the n points given by measurement or observation. Hence of all points resulting from the n unknown errors, that which has the greatest probability is the one which makes $S(\delta^2)$ a maximum, that is, it is at the centre of gravity of the given points, and its probability is

$$p = \frac{1}{4\pi m S(\delta_1{}^2)},$$

where $S(\delta_1{}^2)$ is the sums of the squares of the distances from the given p'ts to their centre of gravity. Hence

$$y = p\frac{S(\delta_1{}^2)}{S(\delta^2)} = \frac{pS(\delta_1{}^2)}{S(\delta_1{}^2) + nx^2} = \frac{p}{1 + \dfrac{nx^2}{S(\delta_1{}^2)}},$$

where x, by problem 200 (ANALYST, Vol. V, page 91), is the distance from any one of the $4\pi mS(\delta^2)$ points to the centre of gravity of the n observed points, and y is the probability, frequency or density of errors at magnitude x. Hence

$$\int_0^x y\,dx = z = p\sqrt{\frac{S(\delta_1{}^2)}{n}}\tan^{-1}\sqrt{\frac{nx^2}{S(\delta_1{}^2)}},$$

where z is the number of errors included by x. Hence

$$n = p\sqrt{\frac{S(\delta_1{}^2)}{n}}\tan^{-1}\sqrt{\frac{nl^2}{S(\delta_1{}^2)}},$$

where l is the magnitude of the error which includes the whole number n.

$$\therefore \frac{z}{n} = \frac{\tan^{-1}\sqrt{\dfrac{nx^2}{S(\delta_1^2)}}}{\tan^{-1}\sqrt{\dfrac{nl^2}{S(\delta_1^2)}}} = \frac{\tan^{-1}\sqrt{\dfrac{nx^2}{S(\delta_1^2)}}}{2\tan^{-1}\frac{1}{2}\sqrt{2}},$$

where $z \div n$ is the probability that an error shall not exceed x. It being proved at page 189, Vol. VII, that the constant

$$\tan\tfrac{1}{2}\tan^{-1}\left(\frac{nl^2}{S(\delta_1^2)}\right)^{\frac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

Hence

$$x = \sqrt{\frac{S(\delta_1^2)}{n}} = \tan\left(\frac{2z}{n}\tan^{-1}\tfrac{1}{2}\sqrt{2}\right).$$

And for $z \div n = \frac{1}{2}$, the probable error is

$$x_1 = \tfrac{1}{2}\sqrt{2}\sqrt{\frac{S(\delta_1^2)}{n}}.$$

A line or surface is in its most probable position when the sum of the squares of the normals upon it, from the given positions, is a minimum, which normals are the errors.

In problem 239 (ANALYST, Vol. VI, p. 49), for A, $z \div n = \frac{36}{100}$, and his angular error $x = \tan^{-1}\frac{5}{3600} = \tan^{-1}\frac{1}{720}$. Hence

$$\frac{\text{A's prob. er.}}{\text{B's prob. er.}} = \sqrt{\frac{S(\delta_1^2)}{S(\delta_1'^2)}} = \frac{\tan\left(\frac{32}{25}\tan^{-1}\frac{1}{2}\sqrt{2}\right)}{\tan\left(\frac{18}{25}\tan^{-1}\frac{1}{2}\sqrt{2}\right)} \times \frac{\tan^{-1}\frac{1}{720}}{\tan^{-1}\frac{1}{540}} = 1.45.$$

SOLUTION OF A PROBLEM.

BY MARCUS BAKER, DIRECT'R OF U. S. MAGNETIC OBS., LOS ANGELES, CAL.

Problem:— In a plane triangle ABC there is given $a+b$, c and m, a perpendicular to BC drawn from B to AC, to solve the triangle.

Solution. From triangle ACN we have

$$b^2 = (a+y)^2 + c^2 - y^2 = a^2 + 2ay + c^2,$$

whence $\quad 2ay = b^2 - a^2 - c^2 = (a+b)^2 - 2a(a+b) - c^2,$

or
$$y = \frac{(a+b)^2 - c^2}{2a} - (a+b) = \frac{k}{a} - n, \tag{1}$$

where $2k = (a+b)^2 - c^2$ and $n = a+b$.

From the similar triangles CMB and CAN

$$\frac{a}{m}=\frac{a+y}{\sqrt{(c^2-y^2)}},\text{ or } a^2c^2-a^2y^2=a^2m^2+2am^2y+m^2y^2,$$

whence $$(m^2-c^2)a^2+(m^2+a^2)y^2+2am^2y=0. \quad (2)$$

Substituting the value of y from (1) we have

$$(m^2-c^2)a^2+(m^2+a^2)\left(\frac{k}{a}-n\right)^2+2am^2\left(\frac{k}{a}-n\right)=0,$$

or $$(m^2-c^2)a^4+(m^2+a^2)(k^2-2akn+a^2n^2)+2a^2m^2(k-an)=0,$$

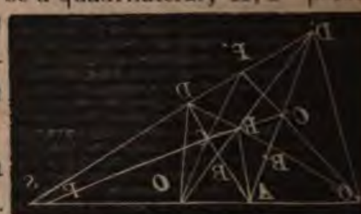
which by expansion and reduction becomes

$$(m^2+n^2-c^2)a^4-2n(k+m^2)a^3+[m^2(n^2+2k)+k^2]a^2-2knm^2a+k^2m^2=0,$$

from which to find a.

NOTE.— This problem has been known in schools under the following form. A tree of known height n, standing on a side hill, was broken over by the wind, and while still clinging to the stump its top touched the ground at a distnce c from the foot of the stump. The perpendicular distance from the foot of the tree to the broken over part was measured and found to be m; required the height of the stump.

DEMONSTRATION OF THE THEOREM OF APOLLONIUS AND ITS RECIPROCAL BY W. E. HEAL.—1. Let $ODD'O'$ be a quadrilateral; A, P' points taken in opposite sides. The diagonals of the quadrilaterals $ODD'O'$, $ODP'A$, $O'D'P'A$ intersect in p'ts that lie in a strai't line, CRC'.



2. In any quadrilateral $AB'RS$ let a p't B be joined to two opposite vertices, A, R. The lines DD', OC, $O'C'$ joining the intersection of opposite sides of the original quadrilateral and of the two derived quadr'ls meet in a point, P'.

To prove 1, produce $D'D$, $O'O$ to meet in P. The lines $ABCD$, $PDP'D'$ cutting the pencil AOD give $[ABCD]^* = [PD'P'D]$. The lines $PDP'D'$ cutting the pencil $AO'D'$ give $[PD'P'D]=[AD'C'B']$; $\therefore$ $[ABCD]=[AD'C'B']$. The points A, B, C, D; A, B', C', D' having the same anhar. ratio and one point, A, common, the lines BD', CC', DB' joining the other corr. points meet in a point R; $\therefore$ C, R, C' are in the same straight line.

To prove 2, join R, A. The lines $ABCD$, $AB'C'D$ cutting the pencil ARD give $[ABCD]=[AD'C'B']$. The pencils $O'OD$, $OO'D'$ having the same anh. ratio and one ray OO' common, the intersections D, P', D' of the other corr. rays lie in a straight line. That is DD', OC, $O'C'$ meet in a point.

*The notation $[ABCD]$ denotes the anharmonic ratio of A, B, C, D.

AN EASY METHOD OF COPUTING LOGARITHMS TO MANY DECIMAL PLACES.

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

THOSE who have had occasion actually to compute logarithms to 10, 12 or more places of decimals will have found it a matter of considerable labor when any of the formulæ given in the books was employed, a task moreover wherein errors are easily made. The following very smple method may therefore prove of advantage in making these calculations. In all the books treating of the subject of logarithms, so far as my knowledge extends, it is stated that the fundamental formula

$$\lg(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

is not adapted to numerical computations except when x is less than unity, because for values of x greater than 1 the series is divergent. Yet this very series serves our purpose best when observing that $\lg k = \frac{1}{n}\lg(k^n)$ and that $\lg(ab) = \lg a + \lg b$. It will presently be seen that for our purposes it is well first to compute the log. of a few of the first prime numbers, 2, 3, 5, 7.

I begin therefore with lg 2. Since $\lg 2 = \frac{1}{n}\lg(2^n)$ we may operate thus: $\lg 2 = \frac{1}{3}\lg 8$, but

$$\lg 8 = \lg[10(1-\tfrac{2}{10})] = \lg 10 + \lg(1-\tfrac{2}{10}).$$

Developing $\lg(1-\frac{2}{10})$ into a series and reducing at once to common log. we have

$$\lg 8 = 1 + M[-(\tfrac{2}{10}) - \tfrac{1}{2}(\tfrac{2}{10})^2 - \tfrac{1}{3}(\tfrac{2}{10})^3 - \ldots]$$

M denoting the well known modulus, viz.; $M = .434924\ldots$

But we may employ any other power of 2 if such suits our purpose, for instance the 10th. Thus, $\lg 2 = \frac{1}{10}\lg 1024$, but

$$\lg(1024) = \lg[1000(1+\tfrac{24}{1000})] = 3 + M[.024 - \tfrac{1}{2}(.024)^2 + \tfrac{1}{3}(024)^3 - \ldots$$

To show the working of this method I will compute lg 2 from this series to 12 places of decimals.

The series in parentheses gives:

+ .024 000 000 000	— .000 288 000 000
.000 004 608 000	.000 000 082 944
.000 000 001 592	.000 000 000 000 031
+ .024 004 609 592	.000 288 082 944
— .000 288 082 944	
.023 716 526 648	

This is the result of adding the five first terms of the series, for we see that the 6th and following terms have no influence upon the 12th decimal.

This last number must be multiplied by said modulus M. We find therefore,

.023	716	526	648	
.423	294	481	903	
.009	486	610	659	2
	711	495	799	2
	94	866	106	4
	4	743	305	2
	2	134	486	8
		94	866	0
		9	486	4
		1	896	8
			23	7
			20	7
.010	299	956	650	4

Now adding the number 3 and dividing by 10 we have

$$\log 2 = .301029995665,$$

true to 12 decimal places.

To compute log. 3 we may also proceed in various ways. If convenient we may now make use of log 2 already ascertained. Thus we may say $4 \times 3^3 = 108$, so that $\log 108 = 2 \lg 2 + 3 \lg 3$. But

$$\log 108 = \log[100(1+\tfrac{8}{100})] = 2+M[\tfrac{8}{100}-\tfrac{1}{2}(\tfrac{8}{100})^2+\tfrac{1}{3}(\tfrac{8}{100})^3-\ldots]$$

which gives a rapidly converging series. Or we may operate thus:

$$\log 3 = \tfrac{1}{2}\lg 9 = \tfrac{1}{2}\lg[10(1-\tfrac{1}{10})] = \tfrac{1}{2}+\tfrac{1}{2}M[-\tfrac{1}{10}-\tfrac{1}{2}(\tfrac{1}{10})^2-\tfrac{1}{3}(\tfrac{1}{10})^3-..]$$

a series computable with but very little labor.

To compute log 7 we may operate thus:

$$\log 98 = 2\lg 7 + \lg 2$$

$$\log 98 = \log[100(1-\tfrac{2}{100})] = 2+M[-.02-\tfrac{1}{2}(.02)^2-\tfrac{1}{3}(.02)^3-\ldots].$$

It will of course generally be advantageous to choose such power or product that its value is near 10, 100, 1000 &c.

Now let it be required to calculate the log. of some prime number betw'n 10 and 100, say of 57. Taking $57 \times 18 = 1026$,

$$\log 1026 = 2\log 3 + \lg 2 + \log 57$$

$$\log 1026 = \log[1000(1+\tfrac{26}{1000})] = 3+M[\tfrac{26}{1000}-\tfrac{1}{2}(\tfrac{26}{1000})^2+\tfrac{1}{3}(\tfrac{26}{1000})^3-.]$$

a rapidly converging series.

It is quite as easy to calculate the log. of the larger prime numbers. Let log 2503 be required. Multiplying by 4 we have 10012.

$$\log 10012 = 2\lg 2 + \log 2503$$

$$\log 10012 = \log[10000(1+\tfrac{12}{100000})] = 4+M[.00012-\tfrac{1}{2}(.00012)^2+..].$$

This series converges so rapidly that three terms of it give the log true to 12 places of decimals.

ON THE EFFECT OF THE EARTH'S ROTATION ON BODIES MOVING ON ITS SURFACE.

BY THE EDITOR.

THAT any meridian plane of a revolving sphere will rotate, with respect to a fixed plane, around a normal axis, which pierces the sphere at any point of its surface, with a uniform angular velocity equal to the angular velocity of the revolving sphere multiplied by the sine of the latitude of the point at which the normal axis pierces the surface of the sphere, is a geometrical truth which is wholly independent of any consideration of force whatever. (For a demonstration of this proposition see American Journal of Science for 1852, Vol. XIII, p. 212.)

As a consequence of the inertia of matter, a mass in motion, constrained to move about a center, by a constant force directed to that center, will develop a centrifugal force which will be directly proportional to the square of the velocity and inversely proportional to the radius of the circle.

If we put R for the earth's equatorial radius and V for its equatorial velocity per hour (sidereal time), we have for its equatorial centrifugal force

$$\frac{V^2}{R} = \frac{g}{289};\ V = \frac{2\pi R}{24}. \tag{1}$$

If a velocity v be impressed upon a body, at any point P in latitude λ of the earth's surface, the body will, under the impulse of the impressed force and the earth's gravitating force, move in a fixed plane passing through the center of the earth and therefore intersecting its surface in a great circle of which the point P will be the extremity of a diameter.

Now, in accordance with the above proposition, a meridian plane will, in consequence of the earth's rotation, revolve, with respect to the fixed plane through P, about a normal axis at P, with a uniform angular velocity.

Hence, instead of supposing the moving body to remain in the fixed plane, if we suppose it to be deflected from its path by the rotation of a meridian plane around the normal at P, it will be, at every point of its path, forced to move with a uniform velocity at right angles to a tangent at that point, and hence will be compelled to describe a circle, and will, by the above proposition, return to P in the time occupied by the meridian plane in making one rotation about the normal at P, viz., 24 hours divided by $\sin\lambda$; therefore the circumference of the circle in which the body is forced to move is $24v \div \sin\lambda$, and its radius is $24v \div 2\pi\sin\lambda$, and because the deflecting or centripetal force f acting on the body is to the earth's equatorial centrifugal force directly as the squares of their velocities and inversely as their radii;

$$\therefore f : \tfrac{1}{289}g :: \frac{v^2}{24v \div 2\pi \sin\lambda} : \frac{V^2}{R};$$

$$\text{or } f : \tfrac{1}{289}g :: \frac{2\pi v \sin\lambda}{24} : \frac{4\pi^2 R}{(24)^2}; \text{ whence}$$

$$f = \frac{\frac{1}{289}g\, 24v \sin\lambda}{2\pi R}. \tag{2}$$

Equation (2) represents the total deflecting force at P which results from the rotation of the tangent plane around the normal, and is entirely independent of the direction in which the body moves, provided the centrifugal force of the moving body corresponds with that of the earth's surface at the point P; but this is only the case when the initial motion of the body is in the meridian of P.

Let β denote the angle between the meridian plane and the initial direction in which the body moves, estimated from the south toward the west, then is the centrifugal force of the earth's surface at P to the centrifugal force in the same orbit due the velocity v as $V^2 \cos^2\lambda$ is to $v^2 \sin^2\beta$, or

$$\tfrac{1}{289}g \cos\lambda : F :: V^2 \cos^2\lambda : v^2 \sin^2\beta,$$

from which we find

$$F = \frac{\frac{1}{289}g\, v^2 \sin^2\beta \cos\lambda}{V^2 \cos^2\lambda} = \frac{\frac{1}{289}g\, v^2 \sin^2\beta}{V^2 \cos\lambda}.$$

Resolving F into horizontal and vertical components at P, we find for its horizontal component

$$F' = \frac{\frac{1}{289}g\,(24)^2 v^2 \sin\lambda \sin^2\beta}{4\pi R^2 \cos\lambda} = \frac{24v \sin^2\beta}{2\pi R \cos\lambda} \times \frac{\frac{1}{289}g\, 24v \sin\lambda}{2\pi R}. \tag{3}$$

Adding equations (2) and (3) we get for the total deflecting force at P

$$f + F' = \left(1 + \frac{24v \sin^2\beta}{2\pi R \cos\lambda}\right) \times \frac{\frac{1}{289}g\, 24v \sin\lambda}{2\pi R}. \tag{4}$$

If we suppose the point P to be in latitude 87° 45′ and v to represent a westward velocity of 40 miles per hour, the body, though moving at the rate of 40 miles per hour relative to the earth's surface, will be in a state of absolute rest, and will therefore develop no centrifugal force about the earth's axis, and consequently it will tend to a state of equilibrium by moving toward the pole, or to the right, with a force repesented by the earth's centrifugal force at that latitude resolved in the direction of the tangent plane, i. e., with a force $= \frac{1}{289}g \cos\lambda \sin\lambda$; this result is obtained independently of any consideration of the rotation of the meridian about a normal axis at P, and from which there results precisely the same deflecting force at that point, as

shown by equation (2). Hence equation (4) is true when $v = V\cos\lambda$, and is therefore true for all values of v and β.

If the moving body be a pendulum suspended at P in latitude λ, its centrifugal force about the earth's axis will correspond with that of the point P, and v in equation (3) will become zero, so that in whatever direction a pendulum may be vibrated, its deflecting force will be completely represented by f in equation (2); for the centrifugal force produced by the motion of the pendulum about its point of suspension is always in the plane of that motion and therefore has no tendency to deflect the pendulum out of that plane; but the velocity v' with which the pendulum passes the point P, if referred to the earth's axis as its center of motion, may be substituted for v in equation (2) and hence f in equation (2) represents the whole of the deflecting force, or reaction, which a pendulum vibrating with a maximum velocity v' would exert on the rotating meridian through P.

As the term F', equation (3), results from the difference between the centrifugal force of the moving body and that of the point P, about the earth's axis, if the moving body be a projectile this term will vanish, but in its stead we shall then have the term

$$F'' = \frac{v\cos\lambda\cos\beta}{\sin\lambda} \times \frac{\frac{1}{289}g\,24v\sin\lambda}{2\pi R}, \tag{5}$$

so that, at the end of any time t, the deflecting force will be

$$f + F'' = \left(1 + \frac{v\cos\lambda\cos\beta}{\sin\lambda}\right) \times \frac{\frac{1}{289}g\,24v\sin\lambda}{2\pi R}. \tag{6}$$

Because the second term in the parentheses in equation (4) vanishes when $\sin\beta = 0$ and the corresponding term in equation (6) vanishes when $\cos\beta = 0$, it follows that, for a projectile, the deflecting force is a maximum when the direction of the motion is in the plane of the meridian, while for a body moving on the earth's surface, as a locomotive on a straight track, the deflecting force is a maximum when the track is at right angles with the meridian through the origin of the motion.

In a paper on this subject published in Van Nostrand's *Engineering Magazine* for June, 1883, p. 468, Mr. Randolph says truly that, "While the earth was in a sufficiently fluid state its materials adjusted themselves into the form of an oblate spheroid . . . which is exactly that necessary to counteract the tendency of a detached mass to move on such surface by virtue of the centrifugal force"; but he fails to note that the centrifugal force we have to deal with is not $V\cos\lambda$, to which the earth's figure is adapted, but $V^2\cos^2\lambda \pm v^2\sin^2\beta$, the substitution of which introduces the term F' as above.

AURORAL AND MAGNETIC PERIODS.

[Republished from the Scientific American of November 12, 1870.]

MESSRS. EDITORS:—In the SCIENTIFIC AMERICAN of Oct. 22, you quote from Prof. Langley the statement, that the magnetic needle moves responsive to the great changes that transpire in the sun; and that our winter sky is lit up by auroras more frequently when the solar action is most violent. The fact, he says, is certain, though the cause is still wholly unknown to science.

We have Prof. Langley for authority, therefore, that the *fact* is establish'd, that auroras depend upon, or are in some way influenced by, physical disturbances in the sun, and that the magnetic needle is also, directly or indirectly, influenced by the same cause. But how this influence is produced, he says, is wholly unknown to science. I may be permitted therefore to attempt a possible explanation, or answer to the question, how do physical changes in the sun produce auroras, and influence the magnetic needle on the earth?

Light and heat are sensations believed to be produced by vibrations of a material substance, the luminiferous ether; and from the known phenomena of light it is inferred that the ether possesses elasticity and inertia but not gravity. We may assume therefore the omnipresence of an inert, elastic and non-gravitating fluid, which will necessarily be less dense within and immediately around revolving bodies than at a distance from them; for, being inert, it will, from centrifugal force, recede from the center of rotation; there will therefore be a continual tendency to the formation of an ethereal vacuum along the axis of rotation, so that equilibrium along the axis of rotation can only be maintained by an in-flowing current of ether from the polar regions of the revolving body. We have, therefore, within and about revolving bodies, not only the phenomena of light and heat from ethereal vibratory motion, but also ethereal motion of translation; and this motion of translation will always be outward about the equator and middle latitudes and inward about the poles. As the earth has a tolerably rapid motion on its axis, we might, *a priori*, be led to expect some tangible indication of such ethereal motion of translation. Have we any such indication? I answer, we have in the phenomena of magnetism and the auroras.

In order that ethereal motion of translation may produce the phenomena of magnetism we assume that, though the vibratory motion of the ether is reflected or absorbed by all opaque bodies, yet in its motion of translation it passes freely through most bodies, but the molecules of a few bodies, such as

iron and steel, may be so arranged that the body is impervious to the ether in one direction and yet will allow its free transit in a direction at right angles with the impervious axis. Let such a body be freely suspended, and, as a vane indicates the direction of a current of air, it will indicate the direction of an ethereal current by its impervious axis assuming a position at right angles with the direction of such current. We have here, therefore, a possible physical cause sufficient to produce the observed phenomena of magnetism.

As obstructions to a current of air will cause vibrations sufficient to produce sound, so obstructions to a current of ether should produce light; and as we have, at all times, an in-flowing current of ether in polar regions, the interference, which it encounters in its passage through the atmosphere, should, and probably does, produce the polar lights, or auroras.

Let it be granted that we have assigned the true cause of magnetism and the polar lights, should we expect these phenomena to indicate great physical disturbance in the sun? In attempting to answer this question I will assume that the ether, though imponderable, is the ultimate state or condition of ponderable matter, and that ponderability, or gravity, is a result of its elastic pressure and vibratory motion, and that it is continually being absorbed, or converted into ponderable matter by the sun. Hence, in the great chemical changes which transpire in the sun and which are manifested by the immense cyclones that are visible, even to the naked eye, at the ditance of 95,000,000 miles, I conclude that a vast amount of ether is absorbed by the sun, the result of which must be a motion of translation of the ether in the surrounding space toward the sun; and as the elastic force of the ether is supposed to be in equilibrium with its centrifugal force while the sun is in its normal condition, this increased current toward the sun will diminish its elastic force about the earth, and the centrifugal force will for a time, and until the equilibrium is restored, increase the tendency to a vacuum along the axis and, consequently, increase the polar currents, and we should have the phenomena of auroras and increased magnetic force.

J. E. HENDRICKS.

Proposition. BY R. J. ADCOCK.—The resultant in any given direction of the mutual attraction between a unit mass concentrated at a given point and a volume of density ∂,

$$= \partial k_1 \int_b^a S dx,$$

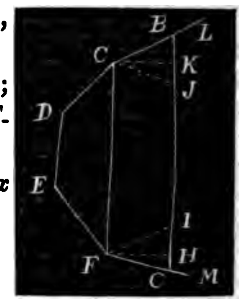
where $x =$ distance from, and $S =$ solid angle subtended at the given point by, a plane secton of the volume perpendicular to the given direction.

DIVIDING LAND.

BY PROF. L. G. BARBOUR, RICHMOND, KENTUCKY.

THE following method is so obvious that it must be in some of the text books, though I have not met with it. The need of it has arisen in actual practice.

The area $CDEF$ lacks (say) 20 acres of the required amount. Find the bearing and distance of FC. Since CL and FM are boundary lines, their bearings and distances are known. We desire now to get the length of CK or its equal FH, perpendicular to CF, so that $BCFG$ shall be 20 acres and BG parallel to CF. Angle $BCK = BCF - 90°$, and $GFH = CFG - 90°$.



Let $CK = FH = x$; tan $BCK = t$; tan $GFH = t'$; $\therefore$ $BK = tx$, $GH = t'x$, area $BCK = \frac{1}{2}x^2t$, area $GFH = \frac{1}{2}x^2t'$, area $BCFG = CF \times CK + \frac{1}{2}(t+t')x^2$.

Let $CF = d$; area $BCFG = \mathcal{A}$; $\therefore$ $(t+t')x^2 + 2dx = 2\mathcal{A}$, whence

$$x = \frac{-d + \sqrt{[2\mathcal{A}(t+t') + d^2]}}{t+t'}.$$

CB and FG may diverge, as in the figure, or converge, as CJ and FI, or be parallel. If they converge, the algebraic sum of $t + t'$ will be negative, and we shall have

$$x = \frac{d - \sqrt{[-2\mathcal{A}(t+t') + d^2]}}{t+t'}.$$

If CB and FG are parallel, $t + t' = 0$, and x is indeterminate. But in this case the added area is a parallelogram, $= CF \times CK = \mathcal{A}$; from which CK is at once known.

In the 2nd equation, there is a limit to the value of $\mathcal{A}$; for if $2\mathcal{A}(t + t') > d^2$ the radical becomes imaginary. The maximum possible value of $\mathcal{A}$ is $d^2 \div 2(t+t')$. Then $x = d \div (t+t')$.

Example.—Let $d = 50$ chains $BCK = 5°$; $GFH = 6°$; $\mathcal{A} = 20$ acres $= 200$ square chains; then is

$$x = \frac{-50 + \sqrt{[400(.08749 + .10510) + 2500]}}{.08749 + .10510} = 3.97 \text{ rods.}$$

Again, let the boundary lines converge, $t = .08749$, $t' = .10510$, $d = 50$ and $\mathcal{A} = 200$ square chains, as before; then

$$x = \frac{50 - \sqrt{[-400(.08749 + .10510) + 2500]}}{.08749 + .10510} = 4.031 \text{ rods.}$$

Whence we find $BG = 50.76458$; $IJ = 49.2354$; $CB = x \div \sin BCK$.

DEMONSTRATION OF A PROPOSITION.

BY P. F. MANGE, ALAMOS, SONORA, MEXICO.

Proposition. The area of any triangle is equal to the radius of the circumscribing circle multiplied by half the perimeter of the triangle formed by joining the feet of the perpendiculars drawn from the angles of the given triangle to the opposite sides.

Let ABC be the given triangle, O the centre of the circumscribed circle, and call its radius R. Also, let $A'B'C'$ be the triangle formed by joining the feet of the perpendiculars.

Join AO, BO and CO; prolong CO to meet the circumference in P, and also AO to meet a perpendicular from B in G. Draw BH and BG respectively perpendicular to CP and AG; draw CS also perpendicular to AG at the point S, and join BP, HB', $A'S$, $B'G$, HA', GC', and $C'S$.

We have angle $BPC = BAC$, and right angle $PBC = AA'C$, therefore $PCB = ACA'$. Now as angles BHC, $BA'C$, and $BB'C$ are all right angles, it follows that the points B, H, A', B', and C are all in the semicircumference of a circle whose diameter is BC, consequently the arcs BH and $A'B'$ are equal; $\therefore$ their chords are also equal; hence $BH = A'B'$. By the same course of reasoning it may be shown that $BG = B'C'$ and $CS = A'C'$.

Now the area of triangle ABC is evidently equal to the sum of the areas of the triangles BOC, BOA and AOC; but we know that the area of trian. $BOC = \frac{1}{2}BH.CO = \frac{1}{2}A'B'.R$,

" " " $BOA = \frac{1}{2}BG.AO = \frac{1}{2}B'C'.R$,

" " " $AOC = \frac{1}{2}CS.AO = \frac{1}{2}A'C'.R$. Hence by addition we get

" " " $ABC = \frac{1}{2}(A'B' + B'C' + A'C')R$. Q. E. D.

Sch. I. Denoting the sides of the triangle ABC by a, b, c, and those of the pedal triangle by a', b', c'; their respective areas by K and K' and the radius of the inscribed circle by r, we have, by the above demonstration,

$$K = \tfrac{1}{2}(a'+b'+c')R; \text{ but we know that } K = \tfrac{1}{2}(a+b+c)r;$$

$$\therefore \frac{(a'+b'+c')R}{2} = \frac{(a+b+c)r}{2}; \;\therefore \frac{a'+b'+c'}{a+b+c} = \frac{r}{R}.$$

Sch. II. Denoting the radius of the circle inscribed in the pedal triangle by r' we have as above $K = \frac{1}{2}(a'+b'+c')R$ and $K' = \frac{1}{2}(a'+b'+c')r'$;

$$\therefore \frac{K'}{K} = \frac{r'}{R}.$$

432. *By R. J. Adcock.*—Show that the quadrant of the ellipse equals

$$a\int_0^1\left(\frac{1-e^2x^2}{1-x^2}\right)^{\frac{1}{2}}dx = \tfrac{1}{2}\pi\cos\theta\Big[1+(\tfrac{1}{2}\tan\theta)^2-\frac{1}{3}\Big(\frac{1.3}{2.4}\tan^2\theta\Big)^2$$
$$+\frac{1}{5}\Big(\frac{1.3.5}{2.4.6}\tan^3\theta\Big)^2-\frac{1}{7}\Big(\frac{1.3.5.7}{2.4.6.8}\tan^4\theta\Big)^2+\ \&c.\Big],$$

where $a=$ semi transverse axis, b semi conjugate, $e^2=1-(b^2\div a^2)$, $\tan^2\theta = e^2\div(1-e^2)$.

SOLUTION BY THE PROPOSER.

While one foot of the isosceles triangle CBD, $CB=BD=\frac{1}{2}(a+b)$, moves along the axis CD of x, the other foot remaining fixed at the origin C, any p't G in the side BD describes an ellipse of which the semi axes are a and b, $CB=\frac{1}{2}(a+b)$, $DG=b$.

Let the angle $CBD=2\theta$, then, in the equation $a^2y^2+b^2x^2=a^2b^2$,

$$x=\tfrac{1}{2}(a+b)\cos\theta+\tfrac{1}{2}(a-b)\cos\theta=a\cos\theta,\ y=b\sin\theta.$$
$$\sqrt{(dx^2+dy^2)}=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}\,d\theta=\sqrt{(b^2+c^2\sin^2\theta)}\,d\theta,$$
$$=b\Big(1+\frac{c^2}{b^2}\sin^2\theta\Big)^{\frac{1}{2}}d\theta;\ c^2=a^2-b^2.$$
$$\int_0^{\frac{1}{2}\pi}\sqrt{(dx^2+dy^2)}=\int_0^{\frac{1}{2}\pi}b\Big(1+\frac{c^2}{2b^2}\sin^2\theta-\frac{1}{2.4}\frac{c^4}{b^4}\sin^4\theta+\ \&c.\Big)d\theta$$
$$=\tfrac{1}{2}\pi b\ \Big(1+(\tfrac{1}{2})^2\frac{c^2}{b^2}\Big)-\frac{1}{3}\Big(\frac{1.3}{2.4}\frac{c^2}{b^2}\Big)^2+\frac{1}{5}\Big(\frac{1.3.5}{2.4.6}\frac{c^3}{b^3}\Big)^2-\&c.$$

which is the length of the quadrant after supplying the factor a in the second member.

[We may prove that the locus of G is an ellipse as follows:—From the similar triangles ABC, ABD and EGD we have

$$DG:DE::BG:AE;\ \text{whence}\ AE=\frac{DE\times BG}{DG};$$
$$DG:DE::CB:CA;\quad \text{“}\quad CA=\frac{DE\times CB}{DG}.$$

Putting x, y for coordinates of G and substituting the given values for DG, CB and BG we get

$$x=CA+AE=\frac{\frac{1}{2}(a+b)\sqrt{(b^2-y^2)}+\frac{1}{2}(a-b)\sqrt{(b^2-y^2)}}{b},$$

whence $b^2x^2+a^2y=a^2b^2$, which is the equation of an ellipse.—Ed.]

THE MULTISECTION OF ANGLES.

BY GEO. H. JOHNSON, B. S., CORNELL UNIVERSITY.

THE problem of the multisection of angles is afforded a general solution by a plane curve called the Chordel. This curve was discovered about three years ago by J. Buren Miller, B. S., of Rutgers College, and a geometrical discussion of the curve was published in Van Nostrand's Engineering Magazine for March, 1880. Mr. Miller defines the Chordel to be a plane curve, generated by one of the points of intersection of a series of right lines which shall intersect in such a manner that each line will contain two points of intersection at a given distance apart, the lines moving so that they shall constantly be in the same plane, their points of intersection equally distant from a fixed point in the plane, and one of their points of intersection constantly remaining on a fixed line in the plane. From this definit'n it is evident that the curve may be constructed mechanically as follows:

Take a ruler of any convenient length which is divided by hinges into n equal parts. At every joint, and at the extremities of the ruler, attach cords of equal length. Pass the cords around a pin or any fixed point. In the same plane draw any line for a directrix, and let a pencil point be attached to the ruler at one extremity or at any joint. Then move the ruler in the plane so that the other extremity of the ruler will constantly touch the directrix and the cords be taut. Every joint will describe a Chordel. The distance between two joints is called an element of the curve; the fixed p't is called the focus.

From the foregoing definition it is evident that Prof. Nicholson's "Polyode" described in the March ANALYST, and Dr. Hillhouse's curve given in the November ANALYST, are only cases of the Chordel; that is, they are Chordels of two elements.

ANOTHER SOLUTION OF PROB. 434. BY PROF. E. B. SEITZ.—"Find a number, the mantissa of the logarithm of which equals the number."

By reference to a table of logarithms, we find that $\log .1371 = \bar{1}.137037$, and $\log .1372 = \bar{1}.137354$; therefore let $\log(.1371+x) = \bar{1}.1371+x$. But $\log(.1371+x) = \log .1371 + \log\left(1+\frac{x}{.1371}\right) = \bar{1}.137037 + .434294.\frac{x}{.1371}$
All the powers of x above the first being very small are omitted. Therefore

$$\bar{1}.137037 + .434294 \times \frac{x}{.1371} = \bar{1}.1371 + x,$$

whence $x = .000029$. Therefore the required number is .137129.

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

Solutions of problems in No. 4 have been received as follows:

From Prof. A. B. Evans, 444; W. E. Heal, 444; Henry Heaton, 444; Prof. A. Hall, 444; William Hoover, 441; E. H. Moore, Jr., 441, 444; P. Richardson, 442; Prof. O. Root, Jr., 444; Prof, E. B. Seitz, 444; Thomas Spencer, 441.

441. *By Wm. Hoover, A. M., Dayton, Ohio.*—"A cone revolves around its axis with a known angular velocity. The altitude begins to diminish and the vertical angle to increase, the volume being constant. Show that the angular velocity is proportional to the altitude."

SOLUTION BY THE PROPOSER.

Let ω, ω_1 be the known and required angular velocities, r, r_1 the corresponding radii of base, h, h_1 the altitudes, k, k_1 the radii of gyration, and m the constant mass.

The moment of the momentum being constant,

$$\tfrac{1}{2}mk^2\omega = \tfrac{1}{2}mk_1^2\omega_1;$$

but

$$k^2 = \tfrac{3}{10}r^2 = \frac{9m}{10\pi h}, \quad k_1^2 = \tfrac{3}{10}r_1^2 = \frac{9m}{10\pi h_1}.$$

Substituting,

$$\omega_1 = \frac{h_1}{h}\,\omega.$$

SOLUTION BY THOMAS SPENCER, SOUTH MERIDEN, CONN.

As there are no external forces, the moment of momentum of the cone about its axis will be constant. Let us call this M, then we have

$$M = m\int_0^h\int_0^{\frac{ax}{h}}\int_0^{2\pi}\omega r^3 dx dr d\theta = \tfrac{1}{10}m\pi\omega a^4 h,$$

where m is the mass of a unit of volume, $h =$ the height AC, a the radius of the base, $x = AB$, ra = radius of gyration, $\theta =$ the vertical angle and ω the given angular velocity.

But the volume of the cone is constant;

$$\therefore \frac{\pi a^2 h}{3} = V, \text{ a constant; which gives}$$

$$a^4 = \frac{9V^2}{\pi^2 h^2}.$$

Substituting this value in the above, we have

$$\omega = \frac{10M\pi}{9mV^2} \cdot h.$$

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442. *By Prof. Casey.*—"ABN is a given circle, D, F and O are given points in the same plane. It is required to describe a circle passing through D and F and intersecting the given circle in the points G, H, so that the triangle GOH may be of a given magnitude."

SOLUTION BY P. RICHARDSON, BROOKLYN, N. Y.

Let C be the center of the given circle ABN; D, F and O the given points. Bisect DF in M; with radius MD and center M describe a circle. Join CM and draw the radical axis, LK, of the circles ABN and DF. Produce D-F to meet the radical axis LK in K, then K is a given point; join KO, it is a given line. Draw SH, parallel to OK and at a given distance from it, intersecting the given circle in H. Join KH and produce it to meet the circle ABN in G. Now O, H and G are given points, hence the sides of the triangle OHG are given in magnitude. With center P describe a circle through DFH and let KH produced meet this circle in G_1. It now remains to show that G and G_1 are coincident points.

Because K is a point in the radical axis LK, hence $DK \times FK = HK \times GK$; but DF is the radical axis of circles DF and DFH, hence we have $DK \times FK = HK \times G'K$; therefore G and G' are the same point.

COR. If a tangent KR be drawn to circle ABN the circle through D, F and R will be tangent to ABN.

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443. *By O. H. Merrill.*—"In cutting the maximum rectangular parallelopipedon from a frustum of a cone, five pieces are cut off. Find the volume of each of these pieces."

No solution received.

[Putting a for the side of a square inscribed in the lower base, b for the side of a square inscribed in the upper base, h for the height of the frustum, and x for the height of the maximum parallelopipedon, we find for the volume V_1, correponding to the height x,

$$V_1 = \frac{a^2h^2x - 2ahcx^2 + c^2x^3}{h^2},$$

where $c = a—b$. Differentiating for a maximum and solving for x we find

$$x_1 = \frac{ah}{3(a—b)}.$$

The height of the maximum parallelcpipedon will therefore be x_1 and the area of its horizontal faces will be $\frac{4}{9}a^2$, and therefore its volume, V_1, $= \frac{4}{9}a^2x_1$. This subtracted from the volume of the frustum V'' whose h'ht is x_1 will give the volume, $2V_2+2V_3$, of the four slabs cut off by the four vertical planes which coincide with the four vertical sides of the maximum parallelopipedon. If the vertical sections are made simultaneously, the volume cut off will consist of eight pieces, four of which will be respectively eq'l to V_3, and the other four pieces, $4V_4$, will also be similar and equal, each to each; their vertices will be points, their altitude x_1, and the sum of the areas of their bases may be expressed in values of a, b, h and π and is therefore a known quantity; call this sum m, then $V_4 = \frac{1}{12}mx_1$, and $2V_2 = 2V_3+\frac{1}{3}mx_1$. The volumes of the pieces cut off will therefore be :—

1. A frustum V', whose height is $h - x_1$ and the radii of whose bases are r and $\frac{2}{3}R$, r and R being the radii of the bases of the given frustum.

2 and 3. Two equal slabs, each $= V_2 = \frac{1}{4}(V''- V_1- \frac{1}{3}mx_1)+ \frac{1}{6}mx_1$; where V'' is a frustum whose height is x_1 and radii of bases R and $\frac{2}{3}R$.

4 and 5. Two equal slabs, each $= V_3 = \frac{1}{4}(V''- V_1 -\frac{1}{3}mx_1)$.

If b is equal to, or greater than, $\frac{2}{3}a$, $V' = 0$, and V'', in the values of 2, 3, 4 and 5, becomes V, the volume of the given frustum.]

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444. *Selected by Prof. H. T. Eddy.*—"Given the five equations,

$$\begin{aligned} x_1^2+x_2^2+x_3^2 &= 3\beta^2, \\ y_1^2+y_2^2+y_3^2 &= 3a^2, \\ x_1y_1+x_2y_2+x_3y_3 &= 0, \\ x_1+x_2+x_3 &= 0, \\ y_1+y_2+y_3 &= 0. \end{aligned}$$

Eliminate x_2y_2 x_3y_3, and show that

$$a^2x_1^2+\beta^2y_1^2 = 2a^2\beta^2.$$

(Routh's Dynamics, 4th Edition, Article 38.)"

SOLUTION BY PROF. ASAPH. HALL.

Problem 444 is a simple piece of elimination. The values of x_3 and y_3 from the two last equations give

$$x_1^2 + x_2^2 + x_1x_2 = \tfrac{3}{2}\beta^2,$$
$$y_1^2 + y_2^2 + y_1y_2 = \tfrac{3}{2}a^2,$$
$$2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1 = 0.$$

The two equations give

$$x_2 = -\tfrac{1}{2}x_1^2 \pm \sqrt{(\tfrac{3}{2}\beta^2 - \tfrac{3}{4}x_1^2)},$$
$$y_2 = -\tfrac{1}{2}y_1^2 \pm \sqrt{(\tfrac{3}{2}a^2 - \tfrac{3}{4}y_1^2)}.$$

Substituting these values of x_2, y_2 in the third equation and reducing we have

$$a^2x_1^2 + \beta^2y_1^2 = 2a^2\beta^2.$$

SOLUTION BY PROF. ASHER B. EVANS.

Put $x_2 = nx_1$ and $y_2 = my_1$; then, from equations fourth and fifth,

$$x_3 = -(n+1)x_1, \qquad y_3 = -(m+1)y_1.$$

By aid of these values of x_2 y_2 x_3 y_3 we may reduce equation third to

$$x_1y_1(2+2mn+m+n) = 0.$$

Place the factor $(2+2mn+m+n)$ equal to zero, then

$$m = -\frac{n+2}{2n+1},$$
$$-(m+1) = \frac{1-n}{2n+1}.$$

These values enable us to reduce equations first and second to the forms

$$2x_1^2(1 + n + n^2) = 3\beta^2,$$
$$2y_1^2(1 + n + n^2) = a^2(2n + 1)^2.$$
$$\therefore 2a^2x_1^2(1+n+n^2) = 3a^2\beta^2,$$
$$2\beta^2y_1^2(1+n+n^2) = a^2\beta^2(2n+1)^2.$$

By addition

$$2(a^2x_1^2+\beta^2y_1^2)(1+n+n^2) = 4a^2\beta^2(1+n+n^2).$$
$$\therefore a^2x_1^2 + \beta^2y_1^2 = 2a^2\beta^2.$$

SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

From equations (1), (2), (4) and (5), we see that x_1, x_2, x_3 are the roots of a cubic equation in which the coefficient of x^2 is zero, and that of x is

$-\frac{3}{2}\beta^2$; and that y_1, y_2, y_3 are the roots of another in which the coefficient of y^2 is zero and that of y is $-\frac{3}{2}a^2$. Hence if

$$x_1 = \sqrt{2}.\beta\cos\varphi,$$
$$x_2 = \sqrt{2}.\beta\cos(\varphi+\tfrac{2}{3}\pi),$$
$$x_3 = \sqrt{2}.\beta\cos(\varphi+\tfrac{4}{3}\pi);$$
$$y_1 = \sqrt{2}.a\cos\theta,$$
$$y_2 = \sqrt{2}.a\cos(\theta+\tfrac{2}{3}\pi),$$
$$y_3 = \sqrt{2}.a\cos(\theta+\tfrac{4}{3}\pi).$$

Equation (3) of the problem then becomes

$$2a\beta[\cos\theta\cos\varphi+\cos(\theta+\tfrac{2}{3}\pi)\cos(\varphi+\tfrac{2}{3}\pi)+\cos(\theta+\tfrac{4}{3}\pi)\cos(\varphi+\tfrac{4}{3}\pi)]=0.$$

But twice the product of the cosines of two arcs is equal to the cosine of the sum of the two arcs plus the cosine of their difference. Hence

$$3\cos(\theta-\varphi)+\cos(\theta+\varphi)+\cos(\theta+\varphi+\tfrac{4}{3}\pi)+\cos(\theta+\varphi+\tfrac{2}{3}\pi)=0.$$

But $\cos(\theta+\varphi)+\cos(\theta+\varphi+\frac{2}{3}\pi)+\cos(\theta+\varphi+\frac{4}{3}\pi)=0$ whatever the values of θ and φ; $\therefore \cos(\theta-\varphi)=0$, and $\theta-\varphi=\frac{1}{2}\pi$. Hence

$$\cos^2\theta+\cos^2\varphi=1.$$

But $\cos\theta = \dfrac{y_1}{\sqrt{2}.a}$ and $\cos\varphi = \dfrac{x_1}{\sqrt{2}.\beta}$. Therefore

$$\frac{y_1^2}{2a^2}+\frac{x_1^2}{2\beta^2}=1;$$
$$\therefore a^2x_1^2+\beta^2y_1^2=2a^2\beta^2.$$

Since $\theta-\varphi=\frac{1}{2}\pi$, $\cos^2(\theta+\frac{2}{3}\pi)+\cos^2(\varphi+\frac{2}{3}\pi)=1$. Hence

$$a^2x_2^2+\beta^2y_2^2=2a^2\beta^2,$$

and in a similar manner it may be shown that

$$a^2x_3^2+\beta^2y_3^2=2a^2\beta^2.$$

CORRESPONDENCE.

Editor Analyst:

I was somewhat surprised that you should think that your readers would not take much interest in what I regard the most curious mathematical discoveries ever made, and which have enabled me to calculate the logarithms of all the prime numbers under 100 000 true to 40 decimal places, and all under 3000 true to 64 places; and all the Napierian logarithms under 1300 true to 76 places. Also to make four sets of tables by either of which I can find the logarithm of any number whatever, true to 65 decimal places and

the Napierian logarithms true to 76 places. And I have determined the modulus of the Napierian, and also of Briggs' system of logarithms true to more than 100 places.

I wish to call your attention to the logarithms on one of the sheets which I send enclosed, in which you will see that the several numbers and their corresponding logarithms are composed of the same figures, the index of the log. representing the first figure of the number and the following figures in each exactly corresponding.

If you choose to publish these numbers with the offer of $10 to any one who will add 10 more figures to either of them that will correspond to the logarithm of the same, I will be pleased to have you do so; and as soon as I have tested the truth of the additions I will forward the money. I inclose two or three papers that give some slight clue to my method of finding logarithms, and close by sending you the logarithms of 2 and 3 to 91 places each.

Log. 2 = .301029995663981195213738894724493026768189881462108541310427461127108189274424509486927251 9.

Log. 3 = .477121254719662437295027903255115309200128864190695864829865640305229152783661123042968355 6.

D. M. Knapen.

Castleton, Vermont.

Announcement.—We regret to have to inform our readers that we have concluded to discontinue the publication of the Analyst on the completion of Vol. X. This determination has not been induced by any lack of interest in the publication manifested by our subscribers and contributors, most of whom have generously stood by us and assisted us during the whole of the ten years life of our publication, but wholly on account of our declining health. In taking leave of our contributors and subscribers, we do not propose to waste words in any attempt to apologize for the many defects in our production, but will only say that we are fully sensible of, and regret, their existence, but did the best we could, under the circumstances, to avoid them.

We trust that, notwithstanding its defects, the Analyst will be found to contain many papers of much interest and permanent value, which have been contributed by some of America's ablest mathematicians and astronomers.

No. 6 of Vol. X, the concluding number, will be issued about the first of November, and will be devoted mainly to a general index of the ten Vol's published. In this index, besides correcting the errors and supplying the unintentional omissions in the published indexes, the names of contributors of questions and solutions will also be inserted.

Any subscriber, for any Vol. of the ANALYST, except Vols. I and II, who may have failed to receive, or may have subsequently lost, any No. of such Vol., and who may desire to have a complete file, can have the missing Nos. supplied without charge if he will notify us in time to mail such missing Nos. with No. 6.

We have a few (about 20) complete sets of the ANALYST which we will send to any address, free of postage, if ordered before Jan. 1, 1884, for $15. per set; and any volume, except I and II, will be sent singly for $1.50.

J. E. HENDRICKS.

PUBLICATIONS RECEIVED.

System and Tables of Life Insurance. A Treatise developed from the Experience and Records of Thirty American Life Offices, under the direction of a Committee of Actuaries. By LEVI W. MEECH, *Actuary in Charge.* Royal Octavo of 551 pages, published at $10 per copy under direction of the Actuary in Charge, Norwich, Conn.

"There will, we are sure, be but one feeling with respect to the manner in which this laborious work has been carried through. And the volume, of which, even after so extended a consideration, we take leave with reluctance, will ever remain an imperishable monument to the originality, the thoroughness, and the high scientific attainments of the actuaries to whose charge its preparation was entrusted."—From the London Insurance Record.

ERRATA.

On page 85, last line, for $\sqrt{(x^n - 1)}$, read $\sqrt{(x^n + 1)}$.
" " 88, line 10, for $2a+c$, read $2ax+c$.
" " 136, " 2, from bottom, insert d after $\times$.
" " 140, " 17, for maximum, read minimum.
" " 141, " 7, for second "$=$", read $\times$.
" " 142, " 9, from bottom, insert $AB'C'D'$ before "cutting".
" " " transpose P and S in Fig.

THE ANALYST.

VOL. X. NOVEMBER, 1883. No. 6.

DESCRIPTION OF A NEW ELLIPSOGRAPH.

BY THE EDITOR.

THE figure at page 152, illustrating Mr. Adcock's solution of problem 432, suggests a method of constructing an ellipsograph which, so far as we know, is new, and which, on account of the facility with which it can be used and the simplicity of its construction, seems to possess some advantage over the various forms of the instrument we have examined.

The instrument here described consists of six parts:—

(1). A horizontal bar, with a vertical hole through its center and a groove along the middle of its under surface, in which a dovetail pin, in the loose end of a cursor, may slide freely from end to end.

(2). A vertical axis, having a milled head suitable for rotating with the thumb and fingers, and a wrist to fit the vertical hole in the horizontal bar; the axis must extend beyond the under surface of the bar and terminate in a square tenon, and between the tenon and the under surface of the bar one half the axis must be cut away so that the loose end of the cursor may pass the axis, and a groove must be cut in the axis extending upward from the under surface of the bar so that when the loose end of the cursor passes the axis the groove in the axis will be continuous with the groove in the bar; thus permitting the loose end of the cursor to pass the axis without interference.

(3). A radius, with a square mortise in one end in which the tenon at the extremity of the axis is to be inserted, and a round hole at the other end in which the wrist of a connecting pin may revolve.

(4). A cursor, with a square mortise in one end, in which a tenon at the upper end of the connecting pin is to be inserted, and a dovetail pin at the other end, made to slide smoothly along the groove in the bar.

(5). A connecting pin, with a wrist to fit the round hole in the end of the radius and a tenon at its upper end made to fit the mortise in the end of the cursor, and having a square head extending downward, below the radius, and perforated, parallel with the cursor, to permit a movable horizontal pen or pencil holder to slide back and forth parallel with the cursor.

(6). A pencil holder, constructed so as to slide back and forth through the perforation in the head of the connecting pin, parallel with the cursor.

Besides the six parts above described there should be a tension screw in the side of the bar, extending into a shallow groove cut around the axis, to retain the axis in position, and a clamp screw in the head of the connecting pin to clamp the pencil holder when the describing point is brought into the required position; and under each end of the bar (1) there should be fastened a block, or support, extending downward a little farther than the head of the connecting pin, so that the head of the connecting pin may move over the paper without friction.

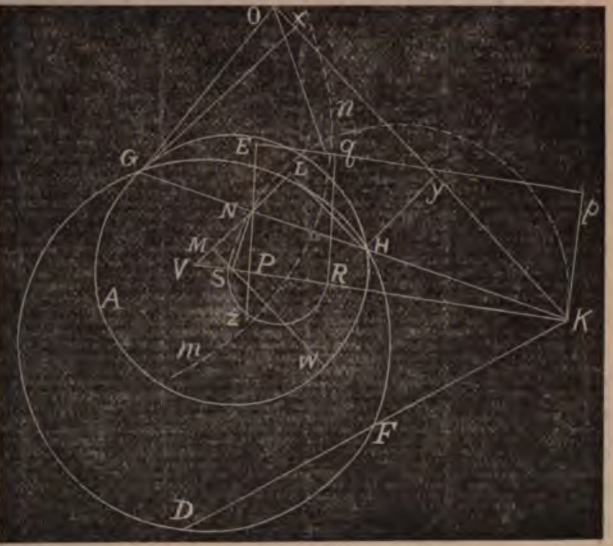
The instrument may be made of any desired dimensions, provided only that the length of the radius, from the center of the axis to the center of the connecting pin must be equal to the length of the cursor from the center of the connecting pin to the center of the dovetail pin, and the bar (1) must be not less than double the sum of the lengths of the radius and cursor.

The major axes of ellipses, described by the same instrument, may be varied by making several sets of holes for the connecting pin through the radius and cursor, and the minor axis is varied by moving the pencil holder in the head of the connecting pin.

To draw a given ellipse; insert the connecting pin through corresponding holes in the radius and cursor so that the length of each shall be equal to one fourth the sum of the axes of the given ellipse; place, and clamp, the pencil holder so that the distance of the describing point from the center of the connecting pin shall equal one fourth the difference of the given axes; place the bar upon the paper so that the groove is over the desired position of the major axis and hold it in position with one hand, then, with the other hand, turn the axis through one complete revolut'n and the describing point will trace out the given ellipse.

SOLUTION OF PROBLEM 442. (*SEE PAGE* 155.)

BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.



Let A be the given circle, D, F and O the given points and $DFHG$ the required circle passing through the given points D, F and intersecting the giv'n circle in the p'ts H, G, so that the triangle GHO may = a given magnitude.

Analysis.—If a circle be described passing through D, F and the center S, of the giv. circle, their common chord will be a line in position and will pass through the point K, $\therefore$ K is a given point.

Join OK, it is in position; draw Gx, Hy perpendiculars to it. Then as the triangle GHO is given and is $=\frac{1}{2}OK(Gx-Hy)$, $\therefore$ $Gx-Hy$ is given.

Draw SN perpendicular to GH, join SK and draw NP perpendicular to it, draw HL and SW parallel to OK, $\therefore$ SW is in position. Draw NM perpendicular to SW and produce it to meet KS produced in V: then is $NL = \frac{1}{2}(Gx-Hy)$ and is $\therefore$ given, and the triangles NLH, NSM are similar, $\therefore$ $HN : NS :: NL : SM$, and $HN^2 : NS^2 :: NL^2 : SM^2$.

Make $SK\times SR = SH^2$, $\therefore$ as SK and SH are given lines, SR is a given line and R is a given point. Now $SK\times SP = SN^2$, and $SK\times SR = SH^2$; $\therefore$ $SK\times PR = HN^2$, and $HN^2 : NS^2 :: SK\times PR : SK\times SP :: PR : SP$. Hence $NL^2 : SM^2 :: PR : SP$, and as VK, SW are in position, $\therefore$ the $\angle$ VSM is a given $\angle$, and $\angle$ SMV a right angle; $\therefore$ the triangle VMS has all of its angles given, and therefore the ratio of SM to SV is given, and as $SM^2 : SV^2$ so make $NL^2 : PE^2$. As NL is a given line, $\therefore$ PE is given, and as it is perpendicular to SK, $\therefore$ the line Ep, parallel to SK, is in position. Now $NL^2 : SM^2 :: PE^2 : SV^2$, hence $PR : PS :: PE^2 : SV^2$. Upon SR describe the semicircle SZR, it is given in position; produce EP to

meet it in Z, then $PR : PS :: PZ^2 : PS^2$, hence $PZ^2 : PS^2 :: PE^2 : SV^2$, or $PZ^2 : PE^2 :: PS^2 : SV^2$; $\therefore$ $PZ : PE :: PS : SV$, and by composition we have $PZ : ZE :: PS : PV$, $\therefore$ $PZ^2 : ZE^2 :: PS^2 : PV^2$; but the triangle VNP has all its angles given, being similar to the triangle VMS, hence $VP^2 = m \times PN^2$, $\therefore$ $PZ^2 : ZE^2 :: PS^2 : m \times PN^2$, or $PZ^2 : ZE^2 :: PS^2 : m \times PS \times PK :: PS : m \times PK :: PS \times PR : m \times PR \times PK$, but $PZ^2 = PS \times PR$, $\therefore$ $ZE^2 = m \times PR \times PK$. Draw Rq, Kp perpendiculars to SK and meeting Ep in q and p, $\therefore$ the points p and q are given, and $m \times pE \times Eq = ZE^2$, $\therefore$ $pE \times qE : EZ^2$ in a given ratio of $1 : m$; hence the locus of the point Z is a hyperbola mn in position, and the semicirele is in position, $\therefore$ the point Z is fixed and the perpendicular ZPE is in position, and so is the semicircle SNK; $\therefore$ the point N is fixed and the line KN is in position, $\therefore$ the points H, G are fixed and the circle $DFGH$ is given in position.

The synthesis of this problem is not long, and will be easily seen from the analysis.

NOTE ON PROBLEM 443.—Prof. Seitz has called our attention to the fact that problem 443 is identical with problem 183, his solution of which was published at pages 27 and 28 of Vol. V.

As the problem had accidentally been placed with the unpublished problems, after its insertion in Vol. V, the fact of its having been published was not remembered when it was inserted in Vol. X, nor when the method of solution, published at p. 156, was sketched.

Prof. Seitz has also pointed out that the equation $V_4 = \frac{1}{12} m x_1$, at page 156, is not exact, because, when the equation is exact, the edges of the pieces V_4 are straight lines, whereas, in this case, they are arcs of a hyperbola. This objection is valid, and the equation should have been written,

$$V_4 = \int_0^{x_1} \varphi(x) dx,$$

where $\varphi(x)$ is the value of m at the altitude x above the lower base of the frustum. But as this method possesses no advantage over that pursued by Prof. Seitz, the reader is referred to the solution of problem 184 at pp. 27-28 of Vol. V for a solution of the problem in detail.

Since the above was put in type we have received from Professor J. M. Greenwood, of Kansas City, Mo., the following letter announcing the death of Professor Seitz, which we take the liberty to publish, as a brief tribute to his virtue and ability, by one who knew him personally.

Kansas City, Mo., Oct. 11, 1883.

Dear Sir:—

The brilliant mathematician, Enoch B. Seitz, died at Kirksville, Mo., on the 8th inst., of Typhoid Fever, after a protracted illness of five weeks. He went to the "Normal" on the day the session opened, but was unable to take charge of his class the next day.

His death causes unusual regret among the thousands of students, teachers, and citizens of this state, who admired him not only on account of his transcendent powers as a mathematician, but as a model of excellence in his daily life.

He was about 34, I think, though I speak from memory only.—The rising star set ere it reached the meridian.

J. M. Greenwood.

We have never had the pleasure of meeting Mr. Seitz and our earliest knowledge of him dates with the commencement of the Analyst, since which the pages of the Analyst bear witness to his constant and valuable correspondence, from which we have long regarded him as possessed of extraordinary mathematical ability and precision of thought. And though we have been favored with the correspondence of many able mathematicians, we believe that, in acuteness of perception, and in conciseness and elegance of style, Mr. Seitz would rank with the ablest. We had anticipated valuable results from his labors, and believe that, had his life been spared, his industry and ability would have materially assisted in enlarging the boundaries of exact science.

The readers of the Analyst are indebted to Mr. Seitz not only for the many elegant solutions by him that have been published, but also for the "Index to Contributors of Solutions of Problems", which was furnished by him, voluntarily and unsolicited, and must have been about the last work that he was permitted to do, as it bears date Sept. 5.

As announced in the Sept. ANALYST, this No. will terminate the publication, under its present management. We had hoped to be able, in this issue, to answer the many inquiries that have been made as to its probable continuance, by a definite announcement of a publication to take its place, under favorable auspices and an able management, as several gentlemen of acknowledged ability, and well and favorably known by all mathematicians and astronomers, both in this country and in Europe, have expressed a willingness to assume the labor and responsibility of continuing the publication; but, as the arrangements for its continuance appear to be still incomplete, we have not been authorized to make a definite announcement. We feel confident, however, from the correspondence we have had on the subject, that the work will not be abandoned but will be placed on a permanent basis, under a management that will insure its usefulness and success. And we earnestly solicit, for the new publication, the active assistance and patronage of *all* our readers.

In conclusion, we desire to tender our sincerest thanks to our patrons and contributors for their continued support during the ten years life of the ANALYST, and especially for their kind words and manifestations of interest in our personal welfare.

EDITOR.

PUBLICATIONS RECEIVED.

Essentials of Geometry. By ALFRED H. WELCH, A. M. 267 pages 8vo. Chicago: S. C. Griggs and Company. 1883.

We have many good text books on Geometry, but it would be difficult to find one containing all the meritorious features of this book. The author's style is clear, concise, and logical, and the exercises and suggestions that are interspersed through the work will be found of much value to the student. Besides, the publishers have greatly enhanced the value of the book by a typography and diagrams that could not be excelled.

Steam Heating: An exposition of the American practice of Warming Buildings by Steam. By ROBERT BRIGGS. 108 pages 18mo. New York: D. Van Nostrand. 1883. (No. 68, Van Nostrand's Science Series.)

Chemical Problems; With brief statement of the Principles involved. By JAMES C. FOYE, A M., PH. D. 141 pages 18mo. New York: D. Van Nostrand. 1883. (No. 69, Van Nostrand's Science Series.)

Results of some experiments made to determine the variations in length of certain bars at the temperature of melting ice; by R. S. WOODWARD, E. WHEELER, A. R. FLINT and W. VOIGT [From the American Journal of Science, Vol. XXV, June, 1883.]

Nouveau calcul des Mouvements Elliptiques, par EDOUARD SANG. Turin: 1879.

In a letter dated at Edinburgh, Scotland, August 17, 1883, Professor Sang writes: "In No. 3 of the ANALYST, there is a paper on Kepler's Problem. I take the liberty of sending you a copy of a solution which, by help of the appropriate tables, gives the true from the mean, or the mean from the true anomaly with one or two minutes labour. * * *"

Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac. Vol. I, part VI. Washington: 1882.

Professional Papers of the Signal Service:

No. VIII. Part I. *The Motions of Fluids and Solids on the Earth's Surface,* by *Professor* WILLIAM FERREL. Reprinted, with Notes by FRANK WALDO. Washington: 1882.

No. XII. *Popular Essays on the Movements of the Atmosphere.* By PROFESSOR WILLIAM FERREL. Washington: 1882.

No. IX. *Charts and Tables, showing Geographical Distribution of Rainfall in the United States.* By H. H. C. DUNWOODY. Washington: 1883.

No. XI. *Meteorological and Physical Observations on the East Coast of America,* by ORRAY TAFT SHERMAN. Washington: 1883.

ERRATA.

On page 63, line 11, for" By Prof. M. L. Comstock", read Selected by Prof. M. L. Comstock.
" " 136, lines 14 and 30, "(6)" and "(14) should each be one line higher on the page,
" " 152, last line, for y read y^2.
representing respectively $y = a^x$ and $y = e^x$.
" " 157, " 8 and 9, for $-\frac{1}{2}x_1^2$ and $-\frac{1}{2}y_1^2$, read $-\frac{1}{2}x_1$ and $-\frac{1}{2}y_1$.

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DEX TO CONTRIBUTORS OF SOLUTIONS OF PROBLEMS.

*This notation signifies that 29 solutions were furnished and 10 published.

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